INDUSTRIAL
MATHEMATICS
INSTITUTE

2000:03

Explicit construction of framelets

A. Petukhov

IMI
Preprint Series

Department of Mathematics
University of South Carolina

Explicit construction of framelets

Alexander Petukhov

February 29, 2000

Abstract

We study *tight wavelet frames* and *bi-frames*, associated with given scaling functions, which are obtained with *the unitary and mixed extension principles.* All possible solutions of the corresponding matrix equations are found. It is proved that the problem of the extension may be always solved with two framelets. In particular, if masks of the scaling functions are polynomials (rational functions), then the corresponding framelets with polynomial (rational) masks can be found.

AMS subject classification: 42C15
Key Words: frames, tight frames, multiresolution analysis, wavelets

1 Introduction

The main goal of our paper is to present an explicit construction of an arbitrary *wavelet frames* (or *framelets*), generated by a refinable function. We shall consider only functions of one variable in the space $\mathbb{L}^2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

As usual we denote by $\hat{f}(\omega)$ Fourier transform of the function $f(x) \in \mathbb{L}^2(\mathbb{R})$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\, dx.$$

Suppose a real-valued function $\varphi \in \mathbb{L}^2(\mathbb{R})$ satisfies the following conditions:
(a) $\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega)$, where m_0 is essentially bounded 2π-periodic function;
(b) $\lim_{\omega\to 0} \hat{\varphi}(\omega) = (2\pi)^{-1/2}$;
then the function φ is called *refinable* or *scaling*, m_0 is called *a mask* of φ, and the relation in item (a) is called *a refinement equation.*

In spite of the fact that in most practically important cases the refinement function can be easily reconstructed by its mask, the problem of existence of a scaling function, satisfying a refinement equation with the given mask is not completely solved. Here we shall not discuss the problem of recovering the function φ by its mask. So in what follows the notion of a refinable function is primary for us and a mask is only attribute of a refinable function.

Every refinable function generates *multiresolution analysis* (MRA) of the space $\mathbb{L}^2(\mathbb{R})$, i.e., a nested sequence

$$\cdots \subset V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^j \subset \dots$$

of closed linear subspaces of $\mathbb{L}^2(\mathbb{R})$ such that
(a) $\cap_{j\in\mathbb{Z}} V^j = \varnothing$;
(b) $\overline{\cup_{j\in\mathbb{Z}} V^j} = \mathbb{L}^2(\mathbb{R})$;
(c) $f(x) \in V^j \Leftrightarrow f(2x) \in V^{j+1}$.
To obtain the MRA we just have to take as above V^j the closure of the linear span of the functions $\{\varphi(2^j x - n)\}_{n\in\mathbb{Z}}$. Fulfillment of item (a) and (b) for the obtained spaces V^j was proved in [1]. Property (c) is evident.

The most popular approach to the design of orthogonal and bi-orthogonal wavelets is based on construction of MRA of the space $\mathbb{L}^2(\mathbb{R})$, generated with a given refinable function. S.Mallat [7] shown that if the system $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ constitutes Riesz basis of the space V^0, then there exists a refinable function $\phi \in V^0$ with a mask m_ϕ such that the functions $\{\phi(x-n)\}_{n\in\mathbb{Z}}$ form an orthonormal basis of V^0. If we denote by W^j the orthogonal complement of the space V^j in the space V^{j+1}, then the function ψ (which is called *a wavelet*), defined by the relation

$$\hat{\psi}(2\omega) := m_\psi(\omega)\hat{\phi}(\omega),$$

where $m_\psi(\omega) = \overline{e^{i\omega} m_\phi(\omega+\pi)}$, generates orthonormal basis $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ of the space W^0. Thus, the system

$$\left\{2^{k/2}\psi(2^k x - n)\right\}_{n,k\in\mathbb{Z}} \tag{1}$$

constitutes an orthonormal basis of the space $\mathbb{L}^2(\mathbb{R})$.

We see that if we have a refinable function, generating Riesz basis, then we have explicit formulae for the wavelets, associated with this functions. It gives a simple method for constructing wavelets. Generally speaking, any orthonormal basis of $\mathbb{L}^2(\mathbb{R})$ of the form (1) is called a wavelet system. However, wavelet construction based on multiresolution has advantage from the point of view effectiveness of computational algorithms, because it leads to pyramidal scheme of wavelet decomposition and reconstruction (sf. [3]).

It is well-known that the problem of finding orthonormal wavelet bases, generated by a scaling function, can be reduced to solving the matrix equation

$$M(\omega)M^*(\omega) = I, \tag{2}$$

where

$$M(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) \end{pmatrix},$$

$m_0(\omega), m_1(\omega)$ are essentially bounded functions $m_0(-\omega) = \overline{m_0(\omega)}$, i.e., Fourier series of these functions have real coefficients. It is known (see [3]) that for any scaling function $\varphi(x)$ and associated wavelet $\psi(x)$, generating an orthogonal wavelet basis, the corresponding masks $m_0(\omega), m_1(\omega)$ satisfy (2). Any refinable function φ, whose mask m_0 is solution to (2), generates a tight frame (see [6] for the case when m_0 is polynomial, the general case was proved in [2]).

We cannot independently look for functions m_0 and m_1. In fact, usually we find a solution of the equation

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1, \tag{3}$$

then all possible functions m_1 can be represented in the form

$$m_1(\omega) = \alpha(\omega)\overline{e^{i\omega} m_0(\omega+\pi)}, \tag{4}$$

were $\alpha(\omega)$ is an arbitrary π-periodic function, satisfying $|\alpha(\omega)| = 1$, $\alpha(-\omega) = \overline{\alpha(\omega)}$.

Now suppose we have an arbitrary refinable function $\varphi(\omega)$ with the mask m_0 which does not satisfy (3). Then the set $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ does not constitute an orthonormal basis of V^0. If this set forms a Riesz basis, then we can use orthogonalization, proposed by S.Mallat. However, in this case, when the function φ has a compact support, usually this property fails for the orthogonalized basis. This argues for construction other systems keeping compactness of support. It will be shown in Section 4 that tight frame of wavelets leads to one of the possible compactly supported syastems.

We note that sometimes the orthogonalization can be conducted even if our set is not a Riesz basis. The simplest example gives a refinable function

$$\varphi(x) = \begin{cases} 1/2, & |x| \le 1; \\ 0, & |x| > 1; \end{cases}$$

with the mask $m_0(\omega) = \cos 2\omega$. In this case the MRA coincides with the Haar's MRA. Thus, the function

$$\varphi(x) = \begin{cases} 1, & 0 \le x \le 1; \\ 0, & x > 1 \text{ or } x < 0; \end{cases}$$

is a natural orthogonalization.

Nevertheless, it is easy to design a refinable function such that its MRA does not allow orthogonalization. Indeed, let us introduce a refinable function $\varphi(x) = \sin \pi a x / \pi x$, where $0 < a < 1$. It generates the space V^0 which consists of those functions of $\mathbb{L}^2(\mathbb{R})$ with Fourier transform supported on $[-a\pi, a\pi]$. Thus, for any function $f \in V^0$ the function $\sum_{k\in\mathbb{Z}} |\hat{f}(\omega + 2k\pi)|^2$ vanishes on the set $[-\pi,\pi] \setminus [-a\pi, a\pi]$. Hence, its integer translates do not form an orthonormal bases (see [3]). In this case the traditional procedure of constructing orthonormal wavelet basis cannot be applied. We note that by the same reason even a bi-orthogonal pair with this MRA cannot be constructed.

The last example gives one more reason to consider wavelet frames. Let us recall that a frame in a Hilbert space $\mathcal{H}$ is a family of its elements $\{f_k\}_{k\in\mathbb{Z}}$ such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B\|f\|^2,$$

where optimal A and B are called frame constants. If $A = B$, the frame is called *a tight frame*. In the case when a tight frame has unit frame constants (for example, if it is an orthonormal basis) for any function $f \in \mathbb{L}^2(\mathbb{R})$ the expansion

$$f = \sum_{n\in\mathbb{Z}} \langle f_n, f\rangle f_n \tag{5}$$

is valid. To construct the expansion in an arbitrary frame we need a dual frame $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$ such that

$$\langle f_n, \tilde{f}_k\rangle = \delta_{n,k} := \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

In this case

$$f = \sum_{n\in\mathbb{Z}} \langle \tilde{f}_n, f\rangle f_n. \tag{6}$$

The frame $\left\{\{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}}\right\}_{l=1}^n$, where $\psi^l_{j,k}(x) = 2^{j/2}\psi_l(2^j x - k)$, generated by translates and dilations of finite number of functions, is called *an affine or wavelet frame.*

The both relations (5) and (6) require computation of the inner products, i.e., integrals. It means that even if the function ψ or the dual function $\tilde{\psi}$ have a short support, the computation of the inner products for big negative j has a high computational cost. The problem of decreasing the computational cost of decomposition and reconstruction algorithms is solved in the theory of orthogonal wavelet bases with the help of the pyramidal scheme. The main idea of this algorithm is to find a good approximation of the function f from the space V^n in the form

$$f(x) \approx \sum_{k\in\mathbb{Z}} c_{n,k}\varphi_{n,k}(x), \text{ where } \varphi_{n,k}(x) := 2^{n/2}\varphi(2^n x - k).$$

Then the computation of the expansion

$$\sum_{k\in\mathbb{Z}} c_{n,k}\varphi_{n,k}(x) = \sum_{l<n}\sum_{k\in\mathbb{Z}} d_{l,k}\psi_{l,k}(x)$$

can be implemented only with discrete convolutions by the recursion formulae

$$c_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{h}_{k-2l}, \qquad d_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{g}_{k-2l},$$

and

$$c_{j+1,l} = \sum_{k\in\mathbb{Z}} c_{j,k}h_{l-k} + \sum_{k\in\mathbb{Z}} c_{j,k}g_{l-k},$$

where h_k and g_k are coefficients of the expansions

$$m_0(\omega) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k e^{-ik\omega}, \qquad m_1(\omega)\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} g_k e^{-ik\omega}.$$

In the case when the mask m_0 of a refinable function φ does not satisfy (3) we cannot construct an orthonormal bases of V^1 of the form $\{\varphi(x-k), \psi(x-k)\}$. However, we can hope that there exists a collection of several framelets $\psi^1, \psi^2, \dots, \psi^n \in V^1$, satisfying the following conditions:

1) functions $\left\{\{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}}\right\}_{l=1}^n$ form a tight frame of the space $\mathbb{L}^2(\mathbb{R})$;

2) for algorithms of decompositions and reconstruction the recurrent formulae

$$\langle \varphi_{j,k}, f\rangle = c_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{h}_{k-2l}, \qquad \langle \psi^g_{j,k}, f\rangle = d^q_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{g}^q_{k-2l}, \quad 1 \le q \le n, \tag{7}$$

and

$$c_{j+1,l}=\sum_{k\in\mathbb{Z}}c_{j,k}h_{l-k}+\sum_{q=1}^{n}\sum_{k\in\mathbb{Z}}d^q_{j,k}g^q_{l-k}, \tag{8}$$

where g^q_k are coefficients of the expansions $m_q(\omega)=2^{-1/2}\sum_{k\in\mathbb{Z}}g^q_k e^{-ik\omega}$, take place.

The goal of Section 2 is to show that this problem can be solved with at most two framelets and to present explicit formulae for masks of the framelets. In Sections 3 and 4 we prove that in the case when $m_0(\omega)$ is either a rational function or a polynomial we can choose $m_1(\omega)$, $m_2(\omega)$ as rational functions or polynomials respectively.

In Section 5 generalization of the results from Sections 2 – 4 for wavelet bi-frames are considered.

2 General Framelets

Let φ be a refinable function with a mask m_0, $\hat{\psi}^k(\omega)=m_k(\omega/2)\hat{\varphi}(\omega/2)\in V^1$, where each mask m_k is a 2π-periodic and essentially bounded function for $k=1,2,\ldots,n$. It is well-known that for constructing practically important tight frames the matrix

$$\mathcal{M}(\omega)=\begin{pmatrix} m_0(\omega) & m_1(\omega) & \ldots & m_n(\omega)\\ m_0(\omega+\pi) & m_1(\omega+\pi) & \ldots & m_n(\omega+\pi)\end{pmatrix}.$$

plays an important role.

It is easy to see that the equality

$$\mathcal{M}(\omega)\mathcal{M}^*(\omega)=I \tag{9}$$

is equivalent to (7) and (8).

It turns out that (9) also implies the tightness of the corresponding frame.

Theorem 1. *If (9) holds, then the functions $\{\psi^k\}_{k=1}^n$ generate a tight frame of $\mathbb{L}^2(\mathbb{R})$.*

Remark. For $n=1$ this theorem was proved in [2]. For an arbitrary n it was proved in [9] under some additional decay assumption for $\hat{\varphi}$. In [9] Theorem 1 was called *the unitary extension principle.*

We split the proof of Theorem 1 into several lemmas.

Lemma 1. *Let the masks $\{m_k\}_{k=0}^n$ satisfy (9), then for any ω*

$$|m_l(\omega)|^2+|m_l(\omega+\pi)|^2\le 1,\quad l=0,1,\ldots,n. \tag{10}$$

Proof. Obviously, without loss of generality it suffices to prove inequality (10) only for $l=0$. Let us rewrite relation (9) in the form

$$\mathbb{M}(\omega):=\mathcal{M}_\psi(\omega)\mathcal{M}^*_\psi(\omega)=\begin{pmatrix} 1-|m_0(\omega)|^2 & -m_0(\omega)\overline{m_0(\omega+\pi)}\\ -\overline{m_0(\omega)}m_0(\omega+\pi) & 1-|m_0(\omega+\pi)|^2\end{pmatrix}, \tag{11}$$

where

$$\mathcal{M}_\psi(\omega) = \begin{pmatrix} m_1(\omega) & m_2(\omega) & \dots & m_n(\omega) \\ m_1(\omega+\pi) & m_2(\omega+\pi) & \dots & m_n(\omega+\pi) \end{pmatrix}.$$

The Hermitian matrix $\mathbb{M}(\omega)$ has eigen-values

$$\lambda_1(\omega) \equiv 1, \qquad \lambda_2(\omega) = 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2.$$

By definition (11), $\mathbb{M}(\omega)$ is a positive definite matrix. Hence, $\lambda_2(\omega) \geq 0$. It implies (1). $\square$

Lemma 2. *If $\Phi \in \mathbb{L}^2(\mathbb{R})$ is a refinable function with a mask $m(\omega)$ satisfying the condition*

$$|m(\omega)|^2 + |m(\omega+\pi)|^2 \leq 1 \ a.e., \tag{12}$$

then $S_j := \sum_{k\in\mathbb{Z}} |\langle f, \Phi_{j,k}\rangle|^2 < \infty$ for any function $f \in \mathbb{L}^2(\mathbb{R})$ and

$$(i) \quad \lim_{j\to\infty} S_j = \|f\|^2; \qquad (ii) \quad \lim_{j\to-\infty} S_j = 0,$$

where $\Phi_{j,k} = 2^{j/2}\Phi(2^j x - k)$.

Proof. First, we prove that

$$\sum_{k\in\mathbb{Z}} |\hat{\Phi}(x+2\pi k)|^2 \leq \frac{1}{2\pi}. \tag{13}$$

We note that due to (12) and the continuity $\hat{\Phi}(\omega)$ at $\omega = 0$ we have $|\hat{\Phi}(\omega)| \leq (2\pi)^{-1/2}$ a.e. Thus, for any positive $l \in \mathbb{Z}$ we obtain

$$\begin{aligned}
\sum_{k=-2^l}^{2^l-1} |\hat{\Phi}(\omega+2\pi k)|^2 &= \sum_{k=-2^l}^{2^l-1} \prod_{n=1}^{l+1} |m(2^{-n}(\omega+2\pi k))|^2 |\hat{\Phi}(2^{-n}(\omega+2\pi k))|^2 \\
&\leq \frac{1}{2\pi} \sum_{k=-2^l}^{2^l-1} \prod_{n=1}^{l+1} |m(2^{-n}(\omega+2\pi k))|^2 \\
&\leq \frac{1}{2\pi} \sum_{k=0}^{2^l-1} \left(\prod_{n=1}^{l+1} |m(2^{-n}(\omega+2\pi k))|^2 + \prod_{n=1}^{l+1} |m(2^{-n}(\omega+2\pi(k-2^l)))|^2 \right) \\
&\leq \frac{1}{2\pi} \sum_{k=0}^{2^l-1} \prod_{n=1}^{l} |m(2^{-n}(\omega+2\pi k))|^2 \\
&\leq \frac{1}{2\pi} \sum_{k=0}^{2^{l-1}-1} \left(\prod_{n=1}^{l} |m(2^{-n}(\omega+2\pi k))|^2 + \prod_{n=1}^{l} |m(2^{-n}(\omega+2\pi(k+2^{l-1})))|^2 \right) \\
&\leq \frac{1}{2\pi} \sum_{k=0}^{2^{l-1}-1} \prod_{n=1}^{l-1} |m(2^{-n}(\omega+2\pi k))|^2 \leq \cdots \leq \frac{1}{2\pi}.
\end{aligned}$$

Applying Plancherel and Parseval formulae, we have

$$\sum_{k\in\mathbb{Z}} |\langle f,\Phi_{j,k}\rangle|^2 = 2\pi 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{\Phi}(2^{-j}\omega)}e^{i2^{-j}\omega k}\,d\omega\right|^2$$

$$= 2\pi 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-\pi 2^j}^{\pi 2^j}\left(\sum_{n\in\mathbb{Z}}\hat{f}(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right)e^{i2^{-j}\omega k}\,d\omega\right|^2$$

$$= (2\pi)^2\int_{-\pi 2^j}^{\pi 2^j}\left|\sum_{n\in\mathbb{Z}}\left(\hat{f}(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right)\right|^2 d\omega = (2\pi\,\|F_j\|)^2, \quad (14)$$

where $F_j(\omega) = \sum_{n\in\mathbb{Z}}\left(\hat{f}(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right)$. Let us introduce the following sequences of functions

$$\hat{g}_j(\omega) = \begin{cases}\hat{f}(\omega), & |\omega| < 2^j\pi;\\ 0, & |\omega| \geq 2^j\pi;\end{cases}, \quad h_j = f - g_j, \quad j = 0,1,2,\ldots,$$

$$G_j(\omega) = \sum_{n\in\mathbb{Z}}\left(\hat{g}_j(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right),$$

$$H_j(\omega) = \sum_{n\in\mathbb{Z}}\left(\hat{h}_j(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right).$$

It is clear that on the one hand $\|G_j\| \to (2\pi)^{-1/2}\|f\|$ as $j\to\infty$ and on the other hand, in view of (13),

$$\|H_j\|^2 = \int_{-\pi 2^j}^{\pi 2^j}\left|\sum_{n\in\mathbb{Z}}\left(\hat{h}_j(\omega+2\pi 2^j n)\overline{\hat{\Phi}(2^{-j}(\omega+2\pi 2^j n))}\right)\right|^2 d\omega$$

$$\leq \int_{-\pi 2^j}^{\pi 2^j}\sum_{n\in\mathbb{Z}}\left|\hat{h}_j(\omega+2\pi 2^j n)\right|^2\sum_{n\in\mathbb{Z}}\left|\Phi(2^{-j}\omega+2\pi n)\right|^2 d\omega$$

$$\leq \frac{1}{2\pi}\int_{-\pi 2^j}^{\pi 2^j}\sum_{n\in\mathbb{Z}}\left|\hat{h}_j(\omega+2\pi 2^j n)\right|^2 d\omega = \frac{1}{2\pi}\|\hat{h}_j\|^2 \to 0, \text{ as } j\to\infty. \quad (15)$$

Thus, since

$$\|G_j\| - \|H_j\| \leq \|F_j\| = \|G_j + H_j\| \leq \|G_j\| + \|H_j\|,$$

it follows from (14) and (15) that

$$\sum_{k\in\mathbb{Z}}|\langle f,\Phi_{j,k}\rangle|^2 = (2\pi\|F_j\|)^2 \to 2\pi\|\hat{f}\|^2 = \|f\|^2, \text{ as } j\to+\infty.$$

Thus, relation (i) is proved.

Now we shall prove (ii). Let us denote by χ_R the indicator function of a segment $[-R,R]$ and by f_R the function $f\chi_R$. We fix an arbitrary $\varepsilon > 0$ and choose $R > 0$ such that $\|f(1-\chi_R)\| < \varepsilon$.

Since

$$\sum_{k\in\mathbb{Z}} |\langle f, \Phi_{j,k}\rangle|^2 \le 2\sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + 2\sum_{k\in\mathbb{Z}} |\langle f - f_R, \Phi_{j,k}\rangle|^2$$
$$\le 2\sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + \|f - f_R\|/\pi \le 2\sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + \varepsilon/\pi,$$

we need only to prove that

$$\lim_{j\to-\infty} \sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 = 0.$$

If we assume that $2^j R \le 1/2$, then the last relation follows from the chain of inequalities

$$\sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 = \sum_{k\in\mathbb{Z}} \left(\int_{|x|\le R} f(x)\Phi_{j,k}(x)\,dx\right)^2 \le \|f\|^2 \sum_{k\in\mathbb{Z}} \int_{|x|\le R} \Phi_{j,k}^2(x)\,dx$$
$$= \|f\|^2 \sum_{k\in\mathbb{Z}} \int_{|x+k|\le 2^j R} \Phi^2(x)\,dx = \|f\|^2 \int_{\cup_{k\in\mathbb{Z}}[-2^jR+k,2^jR+k]} \Phi^2(x)\,dx \longrightarrow 0 \text{ as } j \to -\infty$$

□

Lemma 3. *If (9) holds, then for any $f \in \mathbb{L}^2(\mathbb{R})$ and $J \in \mathbb{Z}$*

$$\sum_{k=1}^{n} \sum_{j,l\in\mathbb{Z}} |\langle f, \psi_{j,l}^k\rangle|^2 = \sum_{l\in\mathbb{Z}} |\langle f, \varphi_{J,l}\rangle|^2 + \sum_{k=1}^{n} \sum_{j\ge J} \sum_{l\in\mathbb{Z}} |\langle f, \psi_{j,l}^k\rangle|^2 < \infty.$$

Proof. It follows from (9) that

$$|m_0(\omega)|^2 + |m_1(\omega)|^2 + \cdots + |m_k(\omega)|^2 = 1,$$
$$m_0(\omega)\overline{m_0(\omega+\pi)} + m_1(\omega)\overline{m_1(\omega+\pi)} + \cdots + m_n(\omega)\overline{m_n(\omega+\pi)} = 0.$$

So by analogy with (14) for any $L \in \mathbb{Z}$

$$\sum_{l\in\mathbb{Z}} |\langle f, \varphi_{L,l}\rangle|^2 + \sum_{k=1}^{n} \sum_{l\in\mathbb{Z}} |\langle f, \psi_{L,l}^k\rangle|^2$$
$$= (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^L l)\overline{\hat{\varphi}(2^{-L}(\omega + 2\pi 2^L l))}\right)\right|^2 d\omega$$
$$+ (2\pi)^2 \sum_{k=1}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^L l)\overline{\hat{\psi}^k(2^{-L}(\omega + 2\pi 2^L l))}\right)\right|^2 d\omega$$
$$= (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^L l)\overline{m_k(2^{-L-1}(\omega + 2\pi 2^L l))\hat{\varphi}(2^{-L-1}(\omega + 2\pi 2^L l))}\right)\right|^2 d\omega$$

$$= (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \underbrace{\sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^{L+1} l) \overline{\hat{\varphi}(2^{-L-1}\omega + 2\pi l)} \right)}_{\Delta_1} \overline{m_k(2^{-L-1}\omega)} \right|^2 d\omega$$

$$+ (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \underbrace{\sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^{L+1} l + 2\pi 2^L) \overline{\hat{\varphi}(2^{-L-1}\omega + 2\pi l + \pi)} \right)}_{\Delta_2} \overline{m_k(2^{-L-1}\omega + \pi)} \right|^2 d\omega$$

$$+ (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \Delta_1 \overline{m_k(2^{-L-1}\omega)} \overline{\Delta_2} m_k(2^{-L-1}\omega + \pi) d\omega$$

$$+ (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \Delta_2 \overline{m_k(2^{-L-1}\omega + \pi)} \overline{\Delta_1} m_k(2^{-L-1}\omega) d\omega$$

$$= (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^{L+1} l) \overline{\hat{\varphi}(2^{-L-1}\omega + 2\pi l)} \right) \right|^2 d\omega$$

$$+ (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^{L+1} l + 2\pi 2^L) \overline{\hat{\varphi}(2^{-L-1}\omega + 2\pi l + \pi)} \right) \right|^2 d\omega$$

$$= (2\pi)^2 \int_{-\pi 2^{L+1}}^{\pi 2^{L+1}} \left| \sum_{l\in\mathbb{Z}} \left(\hat{f}(\omega + 2\pi 2^{L+1} l) \overline{\hat{\varphi}(2^{-L-1}\omega + 2\pi l)} \right) \right|^2 d\omega$$

$$= \sum_{l\in\mathbb{Z}} |\langle f, \varphi_{L+1,l} \rangle|^2 < \infty.$$

Using Lemma 2, we obtain the statement of Lemma 3. $\square$

Now Theorem 1 is an easy consequence of Lemmas 1 – 3.

Thus, the problem of constructing tight frames, generated by a refinable function, can be reduced to finding m_k, satisfying (9). Now we shall describe all possible solutions to (9).

Let the mask m_0 satisfy (12). Unit eigen-vectors of the matrix $\mathbb{M}(\omega)$ can be represented in the form

$$\vec{\mathbf{v}}_1(\omega) = \begin{pmatrix} \overline{\left(\dfrac{e^{i\omega} m_0(\omega+\pi)}{B(\omega)} \right)} \\ -\overline{\left(\dfrac{e^{i\omega} m_0(\omega)}{B(\omega)} \right)} \end{pmatrix}, \qquad \vec{\mathbf{v}}_2(\omega) = \begin{pmatrix} \dfrac{m_0(\omega)}{B(\omega)} \\ \dfrac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix}, \qquad B(\omega) \neq 0,$$

where $B(\omega)$ is an arbitrary π-periodic measurable functions, satisfying $|B(\omega)|^2 = |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$ a. e. For definiteness, we can take here the positive root of the right-hand expression. For those ω when $m_0(\omega) = m_0(\omega+\pi) = 0$ the matrix $\mathbb{M}(\omega)$ becomes the identity matrix, so any vector is its eigen vector. In this case we put $\vec{\mathbf{v}}_1(\omega) = (1,0)^T$, $\vec{\mathbf{v}}_2(\omega) = (0,1)^T$.

Thus, we have

$$\mathbb{M}(\omega) = P(\omega)\Lambda(\omega)P^*(\omega), \tag{16}$$

where

$$P(\omega) = \begin{pmatrix} \overline{\left(\dfrac{e^{i\omega} m_0(\omega+\pi)}{B(\omega)}\right)} & \dfrac{m_0(\omega)}{B(\omega)} \\ -\overline{\left(\dfrac{e^{i\omega} m_0(\omega)}{B(\omega)}\right)} & \dfrac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix}, \quad \Lambda(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2 \end{pmatrix}.$$

We note that eigen vectors are determined up to multiplicaion by a scalar function of absolute value 1 a.e. We have chosen normalization convenient for the further considerations.

Theorem 2. *Let a 2π-periodic function $m_0(\omega)$ satisfy (12). Then there exists a pair of 2π-periodic measurable functions m_1, m_2 which satisfy (9) for $n = 2$. Any solution of (9) can be represented in the form of the first row of the matrix*

$$\widetilde{\mathcal{M}}(\omega) = P(\omega)\sqrt{\Lambda(\omega)}Q(\omega),$$

where $Q(w)$ is an arbitrary unitary (a.e.) matrix with π-periodic measurable components.

Proof. The matrix $\mathcal{M}_\psi$ can be represented in the form of its singular decomposition

$$\mathcal{M}_\psi(\omega) = \mathcal{P}(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega),$$

where $\mathcal{P}, \mathcal{Q}$ are unitary matrices, $\mathcal{D}(\omega)$ is a non-negative diagonal matrix. These representations may differ by multiplication of columns of the matrix $\mathcal{P}$ by functions $\alpha_1(\omega)$, $\alpha_2(\omega)$, $|\alpha_1(\omega)| = |\alpha_2(\omega)| \equiv 1$ and simultaneous multiplication of rows of the matrix $\mathcal{Q}$ by $\alpha_1^{-1}(\omega)$ and $\alpha_2^{-1}(\omega)$. Thus, in view of (11), (16) without loss of generality we can suppose $\mathcal{P} \equiv P$, $\mathcal{D} \equiv \sqrt{\Lambda}$.

Let us prove that we can take any a. e. unitary matrix with π-periodic elements as above $Q(\omega) = \mathcal{Q}(\omega)$. In fact, our choice is restricted to such matrices.

For any 2×2 matrix Z we denote by Z^R the matrix with the transposed rows. On the one hand we have

$$\mathcal{M}_\psi(\omega+\pi) = P(\omega+\pi)\mathcal{D}(\omega+\pi)\mathcal{Q}(\omega+\pi) = P^R(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega+\pi),$$

on the other hand

$$\mathcal{M}_\psi^R = (P(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega))^R = P^R(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega).$$

Since $\mathcal{M}_\psi^R(\omega) = \mathcal{M}_\psi(\omega+\pi)$, it means that $\mathcal{Q}(\omega+\pi) = \mathcal{Q}(\omega)$ at least for those ω and $\omega+\pi$ for which $\lambda_2(\omega) = \lambda_2(\omega+\pi) \neq 0$. If $\lambda_2(\omega) = \lambda_2(\omega+\pi) = 0$, then $\mathcal{M}_\psi(\omega)$ does not depend on the choice of the second row of the matrix $\mathcal{Q}$, so we can take an arbitrary values of $\mathcal{Q}(\omega+\pi)$ and $\mathcal{Q}(\omega)$. In particular, we can suppose $\mathcal{Q}(\omega+\pi) = \mathcal{Q}(\omega)$. $\square$

Remark. To describe all possible solution to (9) for an arbitrary n we have to take an arbitrary $n \times n$ unitary matrix Q with π-periodic elements and $2 \times n$ matrix D' which is extension of the matrix $\sqrt{\Lambda}$ by mean of filling all new columns with zeros.

3 Framelets with rational masks

For numerical implementation framelets with rational and polynomial masks are the most suitable. Under the assumptions of Section 2 we require additionally that $m_0(\omega)$ is a rational 2π-periodic function with real coefficients, i.e., m_0 is a ratio of trigonometric polynomials with real coefficients. It is well-known that in spite of the fact that such functions have infinitely many non-zero Fourier coefficients, implementation of numerical algorithms for this case can be economically designed with, so-called, recursive filters.

The only difference of the case of a rational mask and the general case is that we have to extract square root with more care. If $m_0(\omega)$ is a rational function, then $\mathcal{B}(\omega) = |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$ and $\mathcal{A}(\omega) = 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2$ are rational non-negative functions. So according to Riesz lemma, we can take such rational π-periodic functions $A(\omega)$ and $B(\omega)$ that $|A(\omega)|^2 = \mathcal{A}(\omega)$, $|B(\omega)|^2 = \mathcal{B}(\omega)$. Thus, we have proved the following statement.

Theorem 3. *Let a 2π-periodic rational function $m_0(\omega)$ satisfy (12). Then there exists a pair of 2π-periodic rational functions m_1, m_2 which satisfy (9). Any such rational solution to (9) can be represented in the form of the first row of the matrix*

$$\widetilde{\mathcal{M}}(\omega) = P(\omega)D(\omega)Q(\omega), \tag{17}$$

where $Q(w)$ is an arbitrary unitary rational matrix with π-periodic rational components,

$$D(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & A(\omega) \end{pmatrix}.$$

4 Framelets with polynomial masks

The subject of this section is framelets generated by compactly supported refinable functions. Such functions has polynomial masks. They are the most simple from the point of view of numerical implementation. Our main goal is to prove the existence of compactly supported framelets for this case.

Here the degree of trigonometric polynomial $\sum_{j=l}^{k} a_k e^{ijx}$, where $a_l \neq 0$ and $a_k \neq 0$, is defined to be $k-l$.

We denote by $\mathcal{L}$ a set of all Laurent polynomials with real coefficients, and by $\mathcal{L}_n$ a set of Laurent polynomial with real coefficients of degree at most n, i.e.,

$$\mathcal{L}_n := \left\{ \sum_{j=l}^{k} a_j z^j \;\middle|\; l,k \in \mathbb{Z};\ a_j \in \mathbb{R};\ 0 \le k-l \le n \right\}.$$

Theorem 4. *Let a trigonometric polynomial $m_0(\omega)$ of degree n satisfy (12). Then there exists a pair of trigonometric polynomials m_1, m_2 of the degree at most n which satisfy (9).*

Proof. In fact, we cannot control the choice of the matrices $P(\omega)$ and $D(\omega)$ in (17). So we need to choose a unitary rational π-periodic matrix $Q(\omega)$ such that $\mathcal{M}_\psi(\omega)$ consists of trigonometric polynomials.

Let us use the change of variable $z = e^{i\omega}$ in (17). In what follows we consider the Laurent polynomials $h(e^{i\omega}) = m_0(\omega)$, $b(e^{2i\omega}) = B(\omega)$, $a(e^{2i\omega}) = A(\omega)$.

After the change of variable the matrix $P(\omega)$ becomes

$$H(z) = \begin{pmatrix} \dfrac{\frac{1}{z}h\left(-\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{h(z)}{b(z^2)} \\ -\dfrac{\frac{1}{z}h\left(\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{h(-z)}{b(z^2)} \end{pmatrix}.$$

We put the last representation of the matrix $H(z)$ through procedure of reduction. If polynomials $h(z)$, $h(-z)$, $b(z^2)$ have a common factor $z - z_0$, we cancel the corresponding fractions in the second column of $H(z)$ by $z^2 - z_0^2$ and the first column we cancel by $1/z^2 - z_0^2$. After all possible cancellations we obtain the same matrix $H'(z) = H(z)$ but its elements are expressed in terms of new functions $h'(z)$ and $b'(z)$. It is clear that $b'(z^2)b'(1/z'^2) = h'(z)h'(1/z) + h'(-z)h'(-1/z)$ and numerators of the matrix $H(z)$ do not vanish simultaneously. Indeed, since the determinant of $H(z)$ is equal to $1/z$, if for some z_0 we have $h(z_0) = h(-z_0) = h(1/z_0) = h(-1/z_0) = 0$, then either $b(z_0^2) = 0$ or $b(1/z_0^2) = 0$. It means that the reduction of $H(z)$ can be continued. We note that because the coefficients of $h(z)$ and $b(z)$ are real, the polynomials $h'(z)$ and $b'(z)$ also have real coefficients.

After takin the z-transform the elements $q_{11}(z^2)$, $q_{12}(z^2)$, $q_{21}(z^2)$, $q_{22}(z^2)$ of the matrix $Q(\omega)$ satisfy the relations

$$q_{22}(z) = q_{11}(1/z)z^N, \quad q_{12}(z) = -q_{21}(1/z)z^N, \quad N \in \mathbb{Z}.$$

Here, without loss of generality, we may suppose $N = 0$, because any other choice leads to the integer shift of one of the basic framelets.

To reduce poles of the matrix $H'(z)$ after multiplication by $Q(\omega)$ we suppose that

$$q_{11}(z) = \frac{g_1(z)}{b'(z)}, \qquad q_{21}(z) = \frac{g_2(z)}{b'(1/z)},$$

where g_1, g_2 are Laurent polynomials.

Let $\mathcal{R} = \{\pm z_1^{\pm 1}, \pm z_2^{\pm 1}, \ldots, \pm z_n^{\pm 1}\}$ be the set of all different roots of polynomial $b'(z^2)b'(1/z^2)$. We denote by k_j the multiplicity of the root z_j. It is clear that all four roots $\pm z_j^{\pm 1}$ have the same multiplicity. So the degree of polynomial $b'(z^2)b'(1/z^2)$ is equal to $4\sum k_j = 4k$, where k is the degree of polynomial b'.

To prove the theorem we need to find polynomials g_1, g_2 which satisfy equations

$$\frac{1}{z}h'\left(-\frac{1}{z}\right)g_1(z^2) + a(z^2)h'(z)g_2(z^2) = b'(z^2)b'(1/z^2)f_1(z); \tag{18}$$

$$-\frac{1}{z}h'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) + a(z^2)h'(z)g_1\left(\frac{1}{z^2}\right) = b'(z^2)b'(1/z^2)f_2(z); \tag{19}$$

$$-\frac{1}{z}h'\left(\frac{1}{z}\right)g_1(z^2)+a(z^2)h'(-z)g_2(z^2)=b'(z^2)b'(1/z^2)f_3(z); \tag{20}$$

$$\frac{1}{z}h'\left(\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right)+a(z^2)h'(-z)g_1\left(\frac{1}{z^2}\right)=b'(z^2)b'(1/z^2)f_4(z). \tag{21}$$

where $f_1, f_2, f_3, f_4 \in \mathcal{L}$. Moreover, we need satisfy the condition of the unitarity of the matrix $Q(\omega)$. Hence,

$$g_1(z)g_1(1/z)+g_2(z)g_2(1/z)=b'(z)b'(1/z). \tag{22}$$

Now we leave aside equation (22) and prove existence of polynomials $g_1, g_2 \in \mathcal{L}_k$, satisfying (18) – (21). Let us fix the lowest and highest powers of the polynomials g_1 and g_2 and suppose that their degree is equal to k. We have $2k+2$ unknown coefficients.

First we show that there exist polynomials g_1 and g_2, satisfying equations (18) – (21) at points of the set $\mathcal{R}$. As it usually is in the case of a root $\tilde{z}$ of multiplicity $\tilde{k}$, we require that not only (18) – (21) turn to equality but also their derivatives of orders $1, 2, \ldots, \tilde{k}-1$ do.

Equations (18) – (21) give us $16k$ homogeneous linear equations for $2k+2$ unknown coeffitients of polynomials g_1 and g_2. We shall prove that at most $2k$ of them are linearly independent. The proof of this fact we conduct in 3 steps. Each of these steps are based on the following lemma.

Lemma 4. *Let $a_1(z)$, $a_2(z)$, $a_3(z)$, $a_4(z)$, $b_1(z)$, $b_2(z)$ $c_1(z)$, $c_2(z)$ be Laurent polynomials, $|a_1(z_0)|^2+|a_2(z_0)|^2 \neq 0$, l is a positive integer. If*

$$a_1(z)b_1(z)+a_2(z)b_2(z)=(z-z_0)^l c_1(z), \tag{23}$$

$$a_1(z)a_4(z)-a_2(z)a_3(z)=(z-z_0)^l c_2(z), \tag{24}$$

then we have

$$a_3(z)b_1(z)+a_4(z)b_2(z)=(z-z_0)^l c(z), \tag{25}$$

where $c(z) \in \mathcal{L}$.

Proof. Let us assume for definiteness that $a_1(z_0) \neq 0$. We express b_1 from (23) and a_4 from (24). Using the obtained representations, we have

$$\begin{aligned} &a_3(z)b_1(z)+a_4(z)b_2(z)= \\ &\qquad a_3(z)\frac{(z-z_0)^l c_1(z)-a_2(z)b_2(z)}{a_1(z)}+b_2(z)\frac{(z-z_0)^l c_2(z)+a_2(z)a_3(z)}{a_1(z)}= \\ &\qquad\qquad (z-z_0)^l\frac{a_3(z)c_1(z)+b_2(z)c_2(z)}{a_1(z)}=:(z-z_0)^l c(z). \end{aligned}$$

$\square$

In the first step we prove that for every $\tilde{z} \in \mathcal{R}$ only one equation of the pairs $\{(18), (20)\}$ and $\{(19), (21)\}$ should be retained. Indeed, on the one hand

$$\det \begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ -\frac{1}{z}h'\left(\frac{1}{z}\right) & a(z^2)h'(-z) \end{vmatrix} = \frac{1}{z}a(z^2)b'(z^2)b'(1/z^2) = (z-\tilde{z})^{\tilde{k}}c_1(z), \quad c_1(z) \in \mathcal{L},$$

on the other hand, since $a(z)a(1/z) = 1 - b(z)b(1/z)$, $a(\tilde{z}^2) \neq 0$ for any $\tilde{z} \in \mathcal{R}$. Hence, the last matrix at point $\tilde{z}$ has at least one non-zero element. We assume for definiteness that the first row contains non-zero element. Then by Lemma 4, if g_1 and g_2 at the point $\tilde{z}$ satisfy (18) with multiplicity $\tilde{k}$, then they also satisfy (20) at least with the same multiplicity. So at the point $\tilde{z}$ we can exclude equation (20) from consideration. In the same manner we eliminate one of equations (19) and (21).

In the second step we reject equations, corresponding to the roots $\tilde{z}$ and $1/\tilde{z}$. Now for two roots $\tilde{z}$ and $1/\tilde{z}$ we have $4\tilde{k}$ equations. It turns out that at most $2\tilde{k}$ of them are linearly independent. We show that we can keep only equations of the form (18) and (20). Indeed, let us assume that in the previous step we kept equation (18) for $\tilde{z} \in \mathcal{R}$ and equation (19) for $1/\tilde{z}$. Now we prove that linear equations generated by (19) for $1/\tilde{z}$ can be omitted. We apply the change of variable $z \mapsto 1/z$ to (19). Then the left-hand part of (19) becomes

$$a(1/z^2)h'(1/z)g_1(z^2) - zh'(-z)g_2(z^2). \tag{26}$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ a\left(\frac{1}{z^2}\right)h'\left(\frac{1}{z}\right) & -zh'(-z) \end{vmatrix} = b'(z^2)b'(1/z^2)(b''(z^2)b''(1/z^2)h'(z)h'(1/z) - 1),$$

where $b''(z) = b(z)/b'(z)$, is divisible by $(z-\tilde{z})^{\tilde{k}}$, so expression (26) is also divisible by $(z-\tilde{z})^{\tilde{k}}$ and the left-hand part of (19) is divisible by $(z - 1/\tilde{z})^{\tilde{k}}$.

Dependence of the equations, generated by (21), is obtained by the same reasons. Indeed, after transform $z \mapsto 1/z$ the left-hand part of (21) is equal to

$$a(1/z^2)h'(-1/z)g_1(z^2) + zh'(z)g_2(z^2).$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ a\left(\frac{1}{z^2}\right)h'(-1/z) & zh'(z) \end{vmatrix} = b(z^2)b(1/z^2)h'(z)h'(-1/z),$$

then the left-hand part of (21) is divisible by $(z - 1/\tilde{z})^{\tilde{k}}$.

In the third step we prove that equations, corresponding $\tilde{z}$ and $-\tilde{z}$, are linear dependent.

Let us assume that we have chosen equation (18) for the both roots $\pm\tilde{z}$. After substitution $z \mapsto -z$ the right-hand part of (18) is transformed to

$$-\frac{1}{z}h'\left(\frac{1}{z}\right)g_1(z^2) + a(z^2)h'(-z)g_2(z^2).$$

Since

$$\det \begin{vmatrix} \dfrac{1}{z}h'\left(-\dfrac{1}{z}\right) & a(z^2)h'(z) \\ -\dfrac{1}{z}h'\left(\dfrac{1}{z}\right) & a(z^2)h'(-z) \end{vmatrix} = \frac{1}{z}a(z^2)b'(z^2)b'(1/z^2)$$

is divisible by $(z-\tilde{z})^{\tilde{k}}$, then the equations for $-\tilde{z}$ are linear dependent of the equations for $\tilde{z}$.

In the case, when we take equation (18) for $\tilde{z}$ and equation (20) for $-\tilde{z}$, the corresponding linear equations coincides.

Thus, we have proved the existence of a pair of polynomials $g_1, g_2 \in \mathcal{L}_n$, satisfying equations (18) – (21) on all of $\mathcal{R}$. Although the polynomial $b'(z^2)b'(1/z^2)$ can have complex roots, it is easy to check that we can choose polynomials g_1, g_2 with real coefficients. Indeed, if z_0 is a root of $b'(z^2)b'(1/z^2)$, then $\bar{z}_0$ is also a root. Coefficients of the equations, corresponding these roots, differ in complex conjugation. So instead of them we can consider real equations, corresponding to real and imagine parts of the initial equations.

Thus, we have $2k$ homogeneous linear equation for $2k+2$ unknown values. Let us take any non-degenerate solution of the system. Now we prove that for the corresponding pair of polynomials g_1 and g_2 of order at most k, satisfying (18) – (21) and the relation

$$g_1^2(1) + g_2^2(1) = {b'}^2(1), \tag{27}$$

satisfies also the equation

$$g_1(z)g_1(1/z) + g_2(z)g_2(1/z) = b'(z)b'(1/z). \tag{28}$$

Indeed, let us assume for definiteness that $\tilde{z} \in \mathcal{R}$ and $|h'(-1/\tilde{z})|^2 + |a(z^2)h'(z)|^2 \neq 0$. By (19), we have

$$\det \begin{vmatrix} \dfrac{1}{z}h'\left(-\dfrac{1}{z}\right) & a(z^2)h'(z) \\ g_1\left(\dfrac{1}{z^2}\right) & g_2\left(\dfrac{1}{z^2}\right) \end{vmatrix} = \frac{1}{z}h'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) - a(z^2)h'(z)g_1\left(\frac{1}{z^2}\right) =$$

$$- b'(z^2)b'(1/z^2)f_2(z).$$

Thus, by Lemma 4 and from (18) the expression $g_1(z)g_1(1/z) + g_2(z)g_2(1/z)$ is divisible by $(z-\tilde{z})^{\tilde{k}}$. It means that polynomials in the left-hand and right-hand part of (28) have common $2k$ zeros. It remains to normalize the left-hand polynomial according to (27). The normalization is not possible only in the case when $g_1(1) = g_2(1) = 0$. However, it implies that the left-hand part of (28) has $2k+1$ zeros. It follows from this that $g_1(z)g_1(1/z) + g_2(z)g_2(1/z) \equiv 0$. Hence, $g_1(z) \equiv g_2(z) \equiv 0$. It contradicts the assumption that at least one of the polynomials g_1 and g_2 is non-degenerate.

$\square$

We note that there are infinitely many solutions g_1 and g_2 satisfying (27). It is not difficult to prove that there is a unique solution, satisfying the initial conditions

$$g_1(1) = a, \quad g_2(1) = b, \quad a^2 + b^2 = (b'(1))^2. \tag{29}$$

Indeed, let us introduce 2 real linear independent vectors $\vec{r}'$, $\vec{r}''$ of the dimension $2k+2$, composed of coefficients of the polynomials g_1', g_2' and g_1'', g_2'', satisfying (18) — (21). In this case the vectors $(g_1'(1), g_2'(1))$ and $(g_1''(1), g_2''(1))$ are linear independent. It follows from the fact that in the opposite case we can obtain a non-degenerated solution g_1, g_2 to (18) — (21) for which $g_1(1) = g_2(1) = 0$. However, we mentioned above that this is impossible. Thus, we can satisfy any initial condition (29).

Since differences between any two solutions, satisfying (29), take value 0 at the point $z = 1$, they are equal to 0 identically. Hence, there is only one pair of polynomials g_1, g_2, satisfying (29).

Problems of data representation require that the basic functions have approximately as many vanishing moments as order of smoothness which the data have. So to adapt the frame to the given data we need control the number of vanishing moments. Conditions (29) can be useful to control multiplicity of root of trigonometric polynomials $m_1(\omega)$ and $m_2(\omega)$ at the point $\omega = 0$. In turn the multiplicity determines the number of vanishing moments of the framelets.

5 Bi-framelets

Many authors explain the popularity of bi-orthogonal wavelets by the possibility to combine symmetry and compact support of the basic functions. However, the bi-orthogonality gives more flexibility for adaptation to problems arising in applications.

Here we consider only bi-frames generated by two refinable functions. More detail information on general construction of bi-framelets can be found in [5], [8], and [10].

Let $\varphi, \widetilde{\varphi} \in \mathbb{L}^2(\mathbb{R})$ be refinable functions with essentially bounded masks $m_0(\omega)$, $\widetilde{m}_0(\omega)$. It was shown in [5] and [10] that under appropriate choice of $m_0(\omega)$ and $\widetilde{m}_0(\omega)$, when $\hat{\varphi}$ and $\hat{\widetilde{\varphi}}$ satisfy some very mild decaying condition, if the masks of functions $\{\psi^k, \tilde{\psi}^k\}_{k=1}^n$ satisfy, so-called, *the mixed extension principle*

$$\mathcal{M}(\omega)\widetilde{\mathcal{M}}^*(\omega) = I, \tag{30}$$

where

$$\mathcal{M}(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \dots & m_n(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) & \dots & m_n(\omega+\pi) \end{pmatrix},$$

$$\widetilde{\mathcal{M}}(\omega) = \begin{pmatrix} \widetilde{m}_0(\omega) & \widetilde{m}_1(\omega) & \dots & \widetilde{m}_n(\omega) \\ \widetilde{m}_0(\omega+\pi) & \widetilde{m}_1(\omega+\pi) & \dots & \widetilde{m}_n(\omega+\pi) \end{pmatrix},$$

then the functions

$$\left\{ \left\{ 2^{j/2}\psi^l(2^j x - k) \right\}_{j,k\in\mathbb{Z}} \right\}_{l=1}^n, \quad \left\{ \left\{ 2^{j/2}\tilde{\psi}^l(2^j x - k) \right\}_{j,k\in\mathbb{Z}} \right\}_{l=1}^n$$

constitute dual bi-frame systems.

In what follows we shall not discuss conditions for $m_0(\omega)$ and $\widetilde{m}_0(\omega)$, under which we can construct dual pair of frames. We just find all solutions to (30) for the masks m_0, $\tilde{m}_0$ of refinable functions, satisfying inequality

$$|m_0(\omega)\overline{\widetilde{m}_0(\omega)} + m_0(\omega+\pi)\overline{\widetilde{m}_0(\omega+\pi)}| \geq \delta > 0 \text{ a. e.} \tag{31}$$

We introduce new matrices

$$\mathcal{M}_\psi(\omega) = \begin{pmatrix} m_1(\omega) & m_2(\omega) & \dots & m_n(\omega) \\ m_1(\omega+\pi) & m_2(\omega) & \dots & m_n(\omega+\pi) \end{pmatrix},$$

$$\widetilde{\mathcal{M}}_\psi(\omega) = \begin{pmatrix} \widetilde{m}_1(\omega) & \widetilde{m}_2(\omega) & \dots & \widetilde{m}_n(\omega) \\ \widetilde{m}_1(\omega+\pi) & \widetilde{m}_2(\omega+\pi) & \dots & \widetilde{m}_n(\omega+\pi) \end{pmatrix},$$

which contains only masks of bi-framelets, and give the general relations for their computation.

Theorem 5. *For any essentially bounded 2π-periodic functions m_0, $\widetilde{m}_0$, satisfying inequality (31), there exist 2π-periodic measurable functions m_1, m_2, $\widetilde{m}_1$, $\widetilde{m}_2$ which satisfy equation (30). Any bounded solution of (30) can be represented in the form of the first rows of the matrices*

$$\mathcal{M}_\psi(\omega) = P(\omega)\Lambda_1(\omega)G^*(\omega),$$

$$\widetilde{\mathcal{M}}_\psi(\omega) = \widetilde{P}(\omega)\Lambda_2(\omega)G^{-1}(\omega),$$

where

$$P(\omega) = \begin{pmatrix} \overline{\left(\dfrac{e^{i\omega}\widetilde{m}_0(\omega+\pi)}{\widetilde{B}(\omega)}\right)} & \dfrac{m_0(\omega)}{B(\omega)} \\ -\overline{\left(\dfrac{e^{i\omega}\widetilde{m}_0(\omega)}{\widetilde{B}(\omega)}\right)} & \dfrac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix}, \quad \widetilde{P}(\omega) = \begin{pmatrix} \overline{\left(\dfrac{e^{i\omega}m_0(\omega+\pi)}{B(\omega)}\right)} & \dfrac{\widetilde{m}_0(\omega)}{\widetilde{B}(\omega)} \\ -\overline{\left(\dfrac{e^{i\omega}m_0(\omega)}{B(\omega)}\right)} & \dfrac{\widetilde{m}_0(\omega+\pi)}{\widetilde{B}(\omega)} \end{pmatrix},$$

$$\Lambda_1(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1(\omega) \end{pmatrix}, \qquad \Lambda_2(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(\omega) \end{pmatrix},$$

where $B(\omega)$, $\widetilde{B}(\omega)$, $\lambda_1(\omega)$, $\lambda_2(\omega)$ are any π-periodic bounded solutions to

$$B(\omega)\overline{\widetilde{B}(\omega)} = m_0(\omega)\overline{\widetilde{m}_0(\omega)} + m_0(\omega+\pi)\overline{\widetilde{m}_0(\omega+\pi)}$$

and

$$\lambda_1(\omega)\bar{\lambda}_2(\omega) = \lambda(\omega) := 1 - m_0(\omega)\overline{\widetilde{m}_0(\omega)} - m_0(\omega+\pi)\overline{\widetilde{m}_0(\omega+\pi)}$$

with real Fourier coefficients, $G(\omega)$ is an arbitrary non-singular (a.e.) matrix with π-periodic measurable components with real Fourier coefficients such that $\Lambda_1(\omega)G^*(\omega)$ and $\Lambda_2(\omega)G^{-1}(\omega)$ are essentially bounded.

Proof. The fact that the proposed expressions give us a solution to (30) can be verified by a computation.

Let us prove that these expressions give all possible solutions. Obviously equation (30) can be rewritten in the form

$$\mathcal{M}_\psi(\omega)\widetilde{\mathcal{M}}_\psi^*(\omega) = \begin{pmatrix} 1 - m_0(\omega)\overline{\widetilde{m}_0(\omega)} & -m_0(\omega)\overline{\widetilde{m}_0(\omega+\pi)} \\ -m_0(\omega+\pi)\overline{\widetilde{m}_0(\omega)} & 1 - m_0(\omega+\pi)\overline{\widetilde{m}_0(\omega+\pi)} \end{pmatrix} = P(\omega)\Lambda(\omega)\widetilde{P}^*(\omega),$$

where

$$\Lambda(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda(\omega) \end{pmatrix}.$$

Taking into account that $|\det P(\omega)| = |\det \widetilde{P}(\omega)| = 1$, for those ω for which $\lambda(\omega) \neq 0$ we have

$$\left(\Lambda_1^{-1}(\omega)P^{-1}(\omega)\mathcal{M}_\psi(\omega)\right)\left(\widetilde{\mathcal{M}}_\psi^*(\omega)(\widetilde{P}^*(\omega))^{-1}(\Lambda_2^*(\omega))^{-1}\right) = I.$$

Thus, if we denote $\left(\Lambda_1^{-1}(\omega)P^{-1}(\omega)\mathcal{M}_\psi(\omega)\right)^*$ by $G(\omega)$, then we obtain necessary representations for $\mathcal{M}_\psi(\omega)$ and $\widetilde{\mathcal{M}}_\psi(\omega)$.

Now we consider the case $\lambda(\omega_0) = 0$. Introducing the notation

$$A := P^{-1}(\omega_0)\mathcal{M}_\psi(\omega_0), \quad \widetilde{A} := \widetilde{P}^{-1}(\omega_0)\widetilde{\mathcal{M}}(\omega_0),$$

we have

$$A \cdot \widetilde{A}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It means that if we denote by $\vec{v}^1$,$\vec{v}^2$, $\vec{w}^1$,$\vec{w}^2$ rows of the matrices A and $\widetilde{A}$, then for the inner products of them relations $\langle \vec{v}^1, \vec{w}^1 \rangle = 1$ and $\langle \vec{v}^1, \vec{w}^2 \rangle = \langle \vec{v}^2, \vec{w}^1 \rangle = \langle \vec{v}^2, \vec{w}^2 \rangle = 0$ hold. This is possible if at least one of the vectors $\vec{v}^2$ and $\vec{w}^2$ is equal to $\vec{0}$. If both of them are equal to zero, we can take

$$\Lambda_1(\omega_0) = \Lambda_2(\omega_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G^*(\omega_0) = \begin{pmatrix} v_1^1 & v_2^1 \\ v_1' & v_2' \end{pmatrix},$$

where $\vec{v}'$ is an arbitrary non-zero vector which is orthogonal to $\vec{w}^1$.

We note that in this case the matrix $G^{-1}(\omega_0)$ has the same first row as the matrix $\tilde{A}$.

Let us assume that $\vec{v}^2 \neq 0$. In this case we have to take $G^*(\omega_0) = A$ and

$$\Lambda_1(\omega_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda_2(\omega_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we prove necessity of the boundedness of the matrix $\Lambda_1(\omega)G^*(\omega)$. Let us denote by $\lambda_{min}(\omega) > 0$ and $\lambda_{max}(\omega) > 0$ the least and the biggest singular numbers of the matrix $P(\omega)$. Since $|\det P(\omega)| = 1$, we have $\lambda_{min}(\omega) \cdot \lambda_{max}(\omega) = 1$. The boundedness of $\lambda_{max}(\omega)$ follows from the boundedness of elements of the matrix $P(\omega)$. It means that for some $\sigma > 0$ we have $\lambda_{min}(\omega) \geq \sigma > 0$. So if we assume that at least one element of $\Lambda_1(\omega)G^*(\omega)$ is unbounded, the matrix $P(\omega)\Lambda_1(\omega)G^*(\omega)$ is also contains unbounded elements.

The boundedness of the matrix $\Lambda_2(\omega)G^{-1}(\omega)$ follows from the similar reasons.

To prove π-periodicity of components of the matrix $G(\omega)$ we need just repeat the corresponding reasoning from the proof of Theorem 2. $\square$

Theorem 6. *Let trigonometric polynomials $m_0(\omega)$ and $\tilde{m}_0(\omega)$ satisfy (31) then there exist trigonometric polynomials $m_1(\omega)$, $\tilde{m}_1(\omega)$, $m_2(\omega)$, $\tilde{m}_2(\omega)$, satisfying (30).*

Proof. Here we keep the notation of Theorem 5. Additionally we suppose that $B(\omega)$, $\tilde{B}(\omega)$ are π-periodic trigonometric polynomials with real coefficients. We shall look for the matrix

$$G(\omega) = \begin{pmatrix} \dfrac{\overline{q_1}(\omega)}{\overline{B}(\omega)} & \dfrac{\overline{q_2}(\omega)}{\widetilde{B}(\omega)} \\ -\dfrac{q_3(\omega)}{\overline{B}(\omega)} & \dfrac{q_4(\omega)}{\widetilde{B}(\omega)} \end{pmatrix}, \qquad \widetilde{G}(\omega) = \begin{pmatrix} \dfrac{\overline{q_4}(\omega)}{\overline{\widetilde{B}}(\omega)} & \dfrac{\overline{q_3}(\omega)}{B(\omega)} \\ -\dfrac{q_2(\omega)}{\overline{\widetilde{B}}(\omega)} & \dfrac{q_1(\omega)}{B(\omega)} \end{pmatrix}$$

with π-periodic rational components such that the matrices $\mathcal{M}_\psi(\omega) = P(\omega)\Lambda(\omega)G^*(\omega)$ and $\widetilde{\mathcal{M}}_\psi(\omega) = \widetilde{P}(\omega)\widetilde{G}^*(\omega)$ consist of polynomials and $G(\omega)\widetilde{G}^*(\omega) = I$.

Let us use the change of variable $z = e^{i\omega}$. Introducing functions

$$h(e^{i\omega}) = m_0(\omega), \quad \tilde{h}(e^{i\omega}) = \tilde{m}_0(\omega), \quad b(e^{2i\omega}) = B(\omega), \quad \tilde{b}(e^{2i\omega}) = \widetilde{B}(\omega),$$

$$a(z^2) = 1 - h(z)\tilde{h}(1/z) - h(-z)\tilde{h}(-1/z), \quad L(z) = \Lambda(\omega),$$

$$g_1(e^{2i\omega}) = q_1(\omega), \quad g_2(e^{2i\omega}) = q_2(\omega), \quad g_3(e^{2i\omega}) = q_3(\omega), \quad g_4(e^{2i\omega}) = q_4(\omega),$$

we obtain

$$P(\omega) = H(z) = \begin{pmatrix} \dfrac{\frac{1}{z}\tilde{h}\left(-\frac{1}{z}\right)}{\tilde{b}(1/z^2)} & \dfrac{h(z)}{b(z^2)} \\ -\dfrac{\frac{1}{z}\tilde{h}\left(\frac{1}{z}\right)}{\tilde{b}(1/z^2)} & \dfrac{h(-z)}{b(z^2)} \end{pmatrix}, \quad \widetilde{P}(\omega) = \widetilde{H}(z) = \begin{pmatrix} \dfrac{\frac{1}{z}h\left(-\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{\tilde{h}(z)}{\tilde{b}(z^2)} \\ -\dfrac{\frac{1}{z}h\left(\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{\tilde{h}(-z)}{\tilde{b}(z^2)} \end{pmatrix},$$

$$G(\omega) = J(z) = \begin{pmatrix} \dfrac{g_1(1/z^2)}{b(1/z^2)} & \dfrac{g_2(1/z^2)}{\tilde{b}(z^2)} \\ -\dfrac{g_3(z^2)}{b(1/z^2)} & \dfrac{g_4(z^2)}{\tilde{b}(z^2)} \end{pmatrix}, \quad \widetilde{G}(\omega) = \widetilde{J}(z) = \begin{pmatrix} \dfrac{g_4(1/z^2)}{\tilde{b}(1/z^2)} & \dfrac{g_3(1/z^2)}{b(z^2)} \\ -\dfrac{g_2(z^2)}{\tilde{b}(1/z^2)} & \dfrac{g_1(z^2)}{b(z^2)} \end{pmatrix}.$$

We consider here only the case when $a(z) \not\equiv 0$, otherwise we have the well-known case of a unique framelet.

Let us introduce polyphase representation of Laurent polynomials. For a polynomilal $p(z) = \sum_{k=m}^n p_k z^k$ we denote by $p_e(z)$ and $p_o(z)$ its even and odd parts,

$$p_e(z) = \sum_{m \le 2k \le n} p_{2k} z^k, \qquad p_o(z) = \sum_{m \le 2k+1 \le n} p_{2k+1} z^k.$$

Thus, polyphase representation $p(z) = p_e(z^2) + z p_o(z^2)$ takes place.

We shall prove that the functions

$$g_1(z) = -\sqrt{2} z h_o(z), \quad g_2(z) = \sqrt{2}\tilde{h}_e(1/z), \quad g_3(z) = \sqrt{2} h_e(1/z), \quad g_4(z) = -\sqrt{2} z \tilde{h}_o(z)$$

provide a solution to our problem.

First, we check that $J(z)\widetilde{J}^*(z) = I$. It follows directly from the fact that the equality

$$h(z)\tilde{h}(1/z) + h(-z)\tilde{h}(-1/z) = b(z^2)\tilde{b}(1/z^2)$$

can be re-written in the form

$$2\left(h_e(z)\tilde{h}_e(1/z) + h_o(z)\tilde{h}_o(1/z)\right) = b(z)\tilde{b}(1/z).$$

Since we want the matrix $H(z)L(z)J^*(z)$ to be polynomial, the polynomials g_i have to satisfy the equations

$$\frac{1}{z}\tilde{h}\left(-\frac{1}{z}\right) g_1(z^2) + a(z^2)h(z)g_2(z^2) = b(z^2)\tilde{b}(1/z^2)f_1(z); \tag{32}$$

$$-\frac{1}{z}\tilde{h}\left(-\frac{1}{z}\right) g_3\left(\frac{1}{z^2}\right) + a(z^2)h(z)g_4\left(\frac{1}{z^2}\right) = b(z^2)\tilde{b}(1/z^2)f_2(z); \tag{33}$$

$$-\frac{1}{z}\tilde{h}\left(\frac{1}{z}\right) g_1(z^2) + a(z^2)h(-z)g_2(z^2) = b(z^2)\tilde{b}(1/z^2)f_3(z); \tag{34}$$

$$\frac{1}{z}\tilde{h}\left(\frac{1}{z}\right) g_3\left(\frac{1}{z^2}\right) + a(z^2)h(-z)g_4\left(\frac{1}{z^2}\right) = b(z^2)\tilde{b}(1/z^2)f_4(z); \tag{35}$$

where $f_1, f_2, f_3, f_4 \in \mathcal{L}$.

Let us check that g_1, g_2 satisfy (32). Using polyphase representation of the functions $h(z)$ and $\tilde{h}(z)$, (32) can be re-written in the equivalent form

$$\begin{cases} -\dfrac{1}{z}\tilde{h}_o\left(\dfrac{1}{z}\right) g_1(z) + a(z)h_e(z)g_2(z) = b(z)\tilde{b}\left(\dfrac{1}{z}\right) f_{1,e}(z^2), \\ \dfrac{1}{z}\tilde{h}_e\left(\dfrac{1}{z}\right) g_1(z) + a(z)h_o(z)g_2(z) = b(z)\tilde{b}\left(\dfrac{1}{z}\right) f_{1,o}(z^2), \end{cases}$$

After substitution of g_1 and g_2 in the last system we have

$$f_{1,e}(z) = \frac{1}{\sqrt{2}}\left(1 - h_e(z)\tilde{h}_e\left(\frac{1}{z}\right)\right), \qquad f_{1,o}(z) = -\frac{1}{\sqrt{2}}h_o(z)\tilde{h}_e\left(\frac{1}{z}\right).$$

Easily to see that (34) gives us

$$f_{3,e}(z) = f_{1,e}(z), \qquad f_{3,o}(z) = -f_{1,o}(z).$$

In analogous way from (33) and (35) we obtain

$$f_{2,e}(z) = \frac{1}{\sqrt{2}z}h_e(z)\tilde{h}_o\left(\frac{1}{z}\right), \qquad f_{2,o}(z) = -\frac{1}{\sqrt{2}z}\left(1 - h_o(z)\tilde{h}_o\left(\frac{1}{z}\right)\right).$$

$$f_{4,e}(z) = f_{2,e}(z), \qquad f_{4,o}(z) = -f_{2,o}(z).$$

Now we show that for polynomials g_1, g_2, g_3, g_4 the matrix $\widetilde{H}(z)\tilde{J}^*(z)$ is polynomial. To prove it we need to check that for the polynomials g_l the equalities

$$\frac{1}{z}h\left(-\frac{1}{z}\right) g_4(z^2) + \tilde{h}(z)g_3(z^2) = b(1/z^2)\tilde{b}(z^2)\tilde{f}_1(z); \tag{36}$$

$$-\frac{1}{z}h\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right)+\tilde{h}(z)g_1\left(\frac{1}{z^2}\right)=b(1/z^2)\tilde{b}(z^2)\tilde{f}_2(z); \tag{37}$$

$$-\frac{1}{z}h\left(\frac{1}{z}\right)g_4(z^2)+\tilde{h}(-z)g_3(z^2)=b(1/z^2)\tilde{b}(z^2)\tilde{f}_3(z); \tag{38}$$

$$\frac{1}{z}h\left(\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right)+h(-z)g_1\left(\frac{1}{z^2}\right)=b(1/z^2)\tilde{b}(z^2)\tilde{f}_4(z); \tag{39}$$

where $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4 \in \mathcal{L}$, take place. Equalities (36) — (39) can be verified by computation. Moreover, it is easy to see that $\tilde{f}_{1,e}(z)=\tilde{f}_{3,e}(z)\equiv\frac{1}{\sqrt{2}}$, $\tilde{f}_{2,o}(z)=-\tilde{f}_{4,o}(z)=-\frac{1}{\sqrt{2}z}$, $\tilde{f}_{1,o}(z)=\tilde{f}_{2,e}(z)=\tilde{f}_{3,o}(z)=\tilde{f}_{4,e}(z)\equiv 0$. □

Remark. If the scaling functions $\varphi(x)$ and $\tilde{\varphi}(x)$ are even, then the choice of functions g_1, g_2, g_3, g_4 in the proof of Theorem 6 provides us with even (up to a shift) framelets.

In spite of the fact that the construction of framelets in the proof of Theorem 6 is degenerated for the dual multiresolution, it can be easily done more complex. Indeed, let us take any polynomial matrix $\mathcal{G}(\omega)$ with π-periodic polynomial elements and such that the inverse matrix $\mathcal{G}^{-1}(\omega)$ is also polynomial. Since we already have polynomial matrices $\mathcal{M}_\psi(\omega)$ and $\widetilde{\mathcal{M}}_\psi(\omega)$, the matrices $\mathcal{M}_\psi(\omega)\mathcal{G}^*(\omega)$ and $\widetilde{\mathcal{M}}_\psi(\omega)\mathcal{G}^{-1}(\omega)$ are also polynomial. The last procedure can be also implemented as a sequence of lifting steps (cf. [4]), applied to polyphase representation of the matrices $\mathcal{M}_\psi(\omega)$ and $\widetilde{\mathcal{M}}^*_\psi(\omega)$. However, it should be emphasized that such transforms do not lead to all possible polynomial solutions to (30). The point is that obviously $\det|\mathcal{G}(\omega)|$ is a monomial, $\det|G(\omega)|=\det|\widetilde{G}(\omega)|=1$. Thus, if instead of the pair of matrices $H(z)L(z)J^*(z)$ and $\widetilde{H}(z)\widetilde{J}^*(z)$ we consider the pair $H(z)J^*(z)$ and $\widetilde{H}(z)L(z)\widetilde{J}^*(z)$, they cannot be reduced one to another with the polynomial transform $\mathcal{G}(\omega)$ unless $\det|L(z)|$ is a monomial.

References

[1] C. de Boor, R.A. DeVore and A. Ron, On the construction of multivariate (pre)wavelets, Constr. Approx. **9** (1993), 123 – 166.

[2] M.Bownik, Tight frames of multidimensional wavelets, J. Fourier Anal. Appl. **3** (1997), 525 – 542.

[3] I. Daubechies, Ten lectures on wavelets, CBMF conference series in Applied Mathematics, Vol **61**, SIAM, Philadelphia, 1992.

[4] I. Daubechies, W. Sweldens, Factoring wavelet transforms into lifting steps, J. Fourier Anal. Appl. **4** (1998), # 3, 247 – 269.

[5] B.Han, On dual wavelet tight frames, Appl. and Comp. Harm. Anal. **4** (1997), 380 – 413.

[6] W.Lawton, Tight frames of compactly supported affine wavelets, J. Math. Phys. **31** (1990), 1898-1901.

[7] S. Mallat, Multiresolution approximation and wavelet orthonormal bases of $\mathbb{L}^2(\mathbb{R})$, Trans. Amer. Math. Soc. **315** (1989), 69 – 87.

[8] A. Ron and Z. Shen, Construction of compactly supported affine frames in $\mathbb{L}_2(R^d)$, *Springer-Verlag Electronic Production*, 23.12.1997

[9] A. Ron and Z. Shen, Affine systems in $\mathbb{L}_2(R^d)$ II: the analysis of the analysis operator, J. Func. Anal. **148** (1997), 408 – 447.

[10] A. Ron and Z. Shen, Affine systems in $\mathbb{L}_2(R^d)$ II: dual system, J. Fourier Anal. and Appl., **3**, 617 – 637.