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Abstract. We study nonlinear m-term approximation in a Banach space with regard to a basis. It is known that in the case of greedy basis (like the Haar basis $\mathcal{H}$ in $L_p([0,1])$, $1 < p < \infty$) a greedy type algorithm realizes near best m-term approximation for any individual function (element). In this paper we generalize this known result in two directions. First, instead of greedy algorithm we consider weak greedy algorithm. Second, we study in detail unconditional non-greedy bases (like the multivariate Haar basis $\mathcal{H}^d = \mathcal{H} \times \cdots \times \mathcal{H}$ in $L_p([0,1]^d)$, $1 < p < \infty$, $p \neq 2$). We prove some convergence results and also some results on convergence rate of weak type greedy algorithms. Our results are expressed in terms of properties of the basis with respect to a given weakness sequence.

1. Introduction

This paper deals with nonlinear m-term approximation with respect to a basis. Let X be an infinite dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_n\}_{n=1}^\infty$ be a normalized basis for X ($\|\psi_n\| = 1$, $n \in \mathbb{N}$). All bases considered in this paper are assumed to be normalized. For a given $f \in X$ we define the best m-term approximation with regard to Ψ as follows

$$\sigma_m(f,\Psi) := \sigma_m(f,\Psi)_X := \inf_{b_k,\Lambda} \|f - \sum_{k\in\Lambda} b_k\psi_k\|_X,$$

where inf is taken over coefficients b_k and sets of indices Λ with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f,\Psi)\psi_k. \tag{1.1}$$

We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \dots$, of the positive integers decreasing and write $\rho \in D(f)$ if

$$|c_{k_1}(f,\Psi)| \geq |c_{k_2}(f,\Psi)| \geq \dots \quad .$$

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In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f,\Psi) := G_m^X(f,\Psi) := G_m(f,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(f,\Psi)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme (it is not computationally ready) for m-term approximation of an element f. This algorithm is known in the theory of nonlinear approximation under the name Greedy Algorithm (see for instance [T2], [T3], [W]) and under the more specific name Thresholding Greedy Algorithm (TGA) (see [T8], [DKKT]). We will use the latter name in this paper. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f,\Psi,\rho)\|_X = \sigma_m(f,\Psi)_X,$$

or a little weaker

$$\|f - G_m(f,\Psi,\rho)\|_X \le G\sigma_m(f,\Psi)_X \tag{1.2}$$

for all elements $f \in X$ with a constant $G = C(X,\Psi)$ independent of f and m. The following concept of greedy basis has been introduced in [KT].

Definition 1.1. *We call a basis Ψ greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f,\Psi,\rho)\|_X \le G\sigma_m(f,\Psi)_X \tag{1.3}$$

holds with a constant independent of f, m.

The first result in this direction (see [T2]) established that the univariate Haar basis is a greedy basis. We remind the definition of the Haar basis. Denote $\mathcal{H} := \{H_k\}_{k=1}^{\infty}$ the Haar basis on $[0,1)$ normalized in $L_2(0,1)$: $H_1 = 1$ on $[0,1)$ and for $k = 2^n + l$, $n = 0,1,\dots$, $l = 1,2,\dots,2^n$

$$H_k(x) = \begin{cases} 2^{n/2}, & x \in [(2l-2)2^{-n-1},(2l-1)2^{-n-1}) \\ -2^{n/2}, & x \in [(2l-1)2^{-n-1},2l2^{-n-1}) \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{H}_p := \{H_{k,p}\}_{k=1}^{\infty}$ the Haar basis $\mathcal{H}$ renormalized in $L_p(0,1)$.

The following weak type greedy algorithm was considered in [T2]. Let $t \in (0,1]$ be a fixed parameter. For a given basis Ψ and a given $f \in X$ denote $\Lambda_m(t)$ any set of m indices such that

$$\min_{k\in\Lambda_m(t)} |c_k(f,\Psi)| \ge t \max_{k\notin\Lambda_m(t)} |c_k(f,\Psi)| \tag{1.4}$$

and define

$$G_m^{X,t}(f,\Psi) := \sum_{k\in\Lambda_m(t)} c_k(f,\Psi)\psi_k.$$

It was proved in [T2] that in the case of $X = L_p$, $1 < p < \infty$, and Ψ is the Haar system $\mathcal{H}$ we have for any $f \in L_p$

$$\|f - G_m^{L_p,t}(f,\mathcal{H})\|_{L_p} \le C(p,t)\sigma_m(f,\mathcal{H})_{L_p}. \tag{1.5}$$

We note here that the proof of (1.5) from [T2] works for any greedy basis instead of the Haar system $\mathcal{H}$. Thus for any greedy basis Ψ of a Banach space X and any $t \in (0,1]$ we have for each $f \in X$

$$\|f - G_m^{X,t}(f,\Psi)\|_X \le C(p,t)\sigma_m(f,\Psi)_X. \tag{1.6}$$

This means that for greedy bases we have more flexibility in constructing near best m-term approximants.

Recently, in the theory of greedy algorithms with regard to redundant systems the Weak Greedy Algorithm with an arbitrary weakness sequence $\tau := \{t_k\}_{k=1}^\infty$ has been studied (see [T7], [LTe], [T9]). In this paper we study a modification of the above weak type greedy algorithm in a way of further weakening the restriction (1.4). We call this modification the Weak Thresholding Greedy Algorithm (WTGA). Let a weakness sequence $\tau := \{t_k\}_{k=1}^\infty$, $t_k \in [0,1]$, $k = 1,\dots$ be given. We define the WTGA by induction. We take an element $f \in X$ and at the first step we let

$$\Lambda_1(\tau) := \{n_1\}; \qquad G_1^\tau(f,\Psi) := c_{n_1}\psi_{n_1}$$

with n_1 any satisfying

$$|c_{n_1}| \ge t_1 \max_n |c_n|$$

where we denote for brevity $c_n := c_n(f,\Psi)$. Assume we have already defined

$$G_{m-1}^\tau(f,\Psi) := G_{m-1}^{X,\tau}(f,\Psi) := \sum_{n\in\Lambda_{m-1}(\tau)} c_n\psi_n.$$

Then at the mth step we define

$$\Lambda_m(\tau) := \Lambda_{m-1}(\tau) \cup \{n_m\}; \qquad G_m^\tau(f,\Psi) := G_m^{X,\tau}(f,\Psi) := \sum_{n\in\Lambda_m(\tau)} c_n\psi_n$$

with $n_m \notin \Lambda_{m-1}(\tau)$ any satisfying

$$|c_{n_m}| \ge t_m \max_{n\notin\Lambda_{m-1}(\tau)} |c_n|.$$

Thus for an $f \in X$ the WTGA builds a rearrangement of a subsequence of the expansion (1.1). If Ψ is an unconditional basis then always $G_m^\tau(f,\Psi) \to f^*$. It is clear that in this case $f^* = f$ if and only if the sequence $\{n_k\}_{k=1}^\infty$ contains indices of all nonzero $c_n(f,\Psi)$. We say that the WTGA corresponding to Ψ and τ is convergent (converges) if for any realization $G_m^\tau(f,\Psi)$ we have

$$\|f - G_m^\tau(f,\Psi)\| \to 0 \quad \text{as} \quad m \to \infty$$

for all $f \in X$.

In Section 2 we prove the following three theorems on convergence of the WTGA. The first one deals with an arbitrary Banach space X and any basis Ψ.

Theorem 1. *Let X be a Banach space with a normalized basis Ψ. Let $\tau = \{t_n, n \geq 1\}$ be a weakness sequence. The following condition (D) is a necessary condition for the WTGA corresponding to Ψ and τ to be convergent.*

(D) For each subsequence $\{n_k, k \geq 1\}$ of different indices, the series $\sum_{k=1}^{\infty} t_k \psi_{n_k}$ diverges in X.

If the basis Ψ is unconditional, then the above condition (D) is also a sufficient condition for the WTGA corresponding to Ψ and τ to be convergent.

In the case $X = L_p([0,1]^d)$ we can derive from Theorem 1 a more specific condition in terms of τ.

Theorem 2. *Let $2 \leq p < \infty$, $d \geq 1$ and let Ψ be a normalized unconditional basis in $L_p([0,1]^d)$. Let $\tau = \{t_n, n \geq 1\}$ be a weakness sequence. Then the WTGA corresponding to Ψ and τ converges if and only if $\tau \notin l_p$.*

We do not have that simple criterion in terms of τ in the case $X = L_p([0,1]^d)$, $1 < p < 2$ and arbitrary unconditional basis Ψ. In this case we have the following result for the multivariate Haar basis $\mathcal{H}_p^d$ defined as the tensor product of the univariare Haar bases: $\mathcal{H}_p^d := \mathcal{H}_p \times \cdots \times \mathcal{H}_p$. To formulate the result, introduce the following notation. For a sequence $\{t_k, k \geq 1\}$ of nonnegative numbers such that $\lim_{k\to\infty} t_k = 0$, $\{t_k^*, k \geq 1\}$ is a nonincreasing rearrangement of the subsequence $\{t_{n_k}, k \geq 1\}$ consisting of positive elements of $\{t_k, k \geq 1\}$.

Theorem 3. *Let $d \geq 1$ and $1 < p < 2$. The WTGA corresponding to $\mathcal{H}_p^d$ and a weakness sequence τ converges in $L_p([0,1]^d)$ if and only if one of the following conditions is satisfied:*

(i) *The sequence $\tau = \{t_k\}$ does not converge to* 0.

(ii) $\lim_{k\to\infty} t_k = 0$ *and*

$$\sum_{k=2}^{\infty} (t_k^*)^2 \left(k(\log k)^{(1-d)}\right)^{2/p-1} = \infty. \tag{1.7}$$

Along with convergence of the WTGA we study efficiency of approximation by $G_m^\tau(\cdot, \Psi)$. We compare accuracy of the WTGA with best m-term approximation. In the case of greedy basis and $\tau = \{t\}$, $t \in (0,1]$ the relation (1.6) shows that $G_m^\tau(\cdot, \Psi)$ realizes near best m-term approximation. There are two natural ways of adapting (1.6) to the case of nongreedy basis or to the case of general weakness sequence. In the first way (see [T5], [T3], [W], [Os]) we write (1.6) in the form

$$\|f - G_m^\tau(f, \Psi)\| \leq C(m, \tau, \Psi)\sigma_m(f, \Psi)$$

and look for the best (in the sense of order) constant $C(m, \tau, \Psi)$.

We now formulate the correspoding results. For a basis Ψ we define the fundamental function

$$\varphi(m) := \sup_{|A| \leq m} \|\sum_{k\in A} \psi_k\|.$$

We also need the following functions

$$\varphi^s(m) := \sup_{|A|=m} \|\sum_{k\in A} \psi_k\|, \qquad \varphi^i(m) := \inf_{|A|=m} \|\sum_{k\in A} \psi_k\|.$$

It is clear that

$$\varphi(m) = \sup_{n\le m} \varphi^s(n).$$

We now introduce some characteristics of a basis with respect to a weakness sequence τ. For a subset $V \subseteq [1,m]$ of integers we define

$$\phi(\tau,m,V) := \inf_{\{k_i\}} \|\sum_{i\in V} t_i\psi_{k_i}\|$$

where inf is taken over all sets $\{k_i\}$ of different indices. For two integers $1 \le n \le m$ we define

$$\phi(\tau,m,n) := \inf_{|V|=n} \phi(\tau,m,V),$$

and finally

$$\mu(\tau,m) := \sup_{n\le m} \frac{\varphi^s(n)}{\phi(\tau,m,n)}.$$

We have the following result.

Theorem 4. *Let Ψ be a normalized unconditional basis for X. Then we have*

$$\|f - G^\tau_m(f,\Psi)\| \le C(\Psi)\mu(\tau,m)\sigma_m(f,\Psi).$$

In the case $\tau = \{1\}$ Theorem 4 is known. The first result in this direction was obtained for the multivariate Haar basis $\mathcal{H}^d_p$ (see [T3]). Then it was generalized in [W] for other bases, in particular, for normalized unconditional bases. Moreover, it has been proved in [W] that $\mu_m(\{1\},m)$ is an optimal extra factor in the above inequality for $\tau = \{1\}$.

In Theorem 4 we compare efficiency of $G^\tau_m(\cdot,\Psi)$ with $\sigma_m(\cdot,\Psi)$. It is known in approximation theory that sometimes it is convenient to compare efficiency of an approximating operator which is characterized by m parameters with best possible approximation corresponding to smaller number of parameters $n \le m$. We use this idea in approximation by the WTGA. In this paper we study a setting when we write (1.6) in the form

$$\|f - G^\tau_{v_m}(f,\Psi)\| \le C(\Psi)\sigma_m(f,\Psi)$$

and look for the best (in the sense of order) sequence $\{v_m\}$ that is determined by the weakness sequence τ and the basis Ψ. We need some more notation. Define

$$\phi(\tau,N) := \phi(\tau,N,[1,N]) = \inf_{k_1,\dots,k_N} \|\sum_{j=1}^N t_j\psi_{k_j}\|.$$

Assume that $\phi(\tau,N) \to \infty$ as $N \to \infty$ and denote v_m the smallest N satisfying

$$\phi(\tau,N) \ge 2\varphi(m).$$

We have the following result in this case.

Theorem 5. *For any normalized unconditional basis Ψ we have*

$$\|f - G^{\tau}_{v_m}(f,\Psi)\| \le C(\Psi)\sigma_m(f,\Psi).$$

It is interesting to compare this result with some recent results from [DKKT]. It has been established in [DKKT] that the inequalities

$$\|f - G_{\lambda m}(f,\Psi)\| \le C(\Psi,\lambda)\sigma_m(f,\Psi) \tag{1.8}$$

with fixed $\lambda > 1$ are characteristic for a special class of bases. We describe this class now. Let us say that a basis Ψ is almost greedy if there is a constant C so that for any $f \in X$

$$\|f - G_m(f,\Psi)\| \le C \inf_{\Lambda, |\Lambda|=m} \|f - \sum_{k\in\Lambda} c_k(f,\Psi)\psi_k\|.$$

It is clear that each greedy basis is an almost greedy basis. It has been proved in [DKKT] that if (1.8) holds for some $\lambda > 1$ for all $f \in X$ then Ψ is almost greedy. It has also been proved in [DKKT] that (1.8) holds for any $\lambda > 1$ for all $f \in X$ provided Ψ is almost greedy.

In Section 4 we discuss the greedy properties of subsequencies of the Haar basis $\mathcal{H}^d_p := \mathcal{H}_p \times \cdots \times \mathcal{H}_p$ that is a tensor product of the univariate Haar bases $\mathcal{H}$. It is known (see [T2] and [T3]) that $\mathcal{H}_p$ is a greedy basis for $L_p([0,1])$, $1 < p < \infty$ and $\mathcal{H}^d_p$ is a greedy basis for $L_p([0,1]^d)$, $d \ge 2$ only for $p = 2$. Let $\mathcal{M}$ be a subset of the set of indices $\mathbf{n} \in \mathbb{Z}^d_+$. We denote

$$\mathcal{H}^d_p[\mathcal{M}] := \{H_{\mathbf{n},p}, \mathbf{n} \in \mathcal{M}\},$$

$$L_p[\mathcal{M}] := \{f \in L_p([0,1]^d) : \langle f, H_{\mathbf{n}}\rangle = 0, \mathbf{n} \notin \mathcal{M}\} = \overline{\operatorname{span}}\{\mathcal{H}^d_p[\mathcal{M}]\}$$

where closure is taken in $L_p([0,1]^d)$.

We introduce some more notation. Let us define the decomposition of $\mathcal{H}^d_p$ into dyadic blocks. First, define

$$U_0 := \{1,2\}, \quad \text{and} \quad U_s := \{n \in \mathbb{N} : 2^s + 1 \le n \le 2^{s+1}\} \quad \text{for} \quad s \ge 1. \tag{1.9}$$

For $\mathbf{s} = (s_1, \dots, s_d)$ we set

$$U_{\mathbf{s}} := \{\mathbf{n} = (n_1, \dots, n_d) : n_i \in U_{s_i} \quad \text{for} \quad i = 1, \dots, d\}. \tag{1.10}$$

We note that for each $\mathbf{s}$ the supports of the functions $\{H_{\mathbf{n},p}, \mathbf{n} \in U_{\mathbf{s}}\}$ have the same shape and measure $2^{-|\mathbf{s}|}$, where $|\mathbf{s}| = s_1 + \ldots + s_d$. Moreover, note that if $\mathbf{s} = (s_1, \dots, s_d)$ with $s_i \ne 0$ for all $1 \le i \le d$, then $\#U_{\mathbf{s}} = 2^{|\mathbf{s}|}$ and the supports of the functions $\{H_{\mathbf{n},p}, \mathbf{n} \in U_{\mathbf{s}}\}$ are disjoint. For general $\mathbf{s}$ we have $2^{|\mathbf{s}|} \le \#U_{\mathbf{s}} \le 2^{|\mathbf{s}|+d}$, and at most 2^d different functions from $\{H_{\mathbf{n},p}, \mathbf{n} \in U_{\mathbf{s}}\}$ have the same support.

For a positive constant K we define two classes of subsequences $\mathcal{M}$:

$$\mathbf{R}(K) := \{\mathcal{M} : \forall n \quad \#\{\mathbf{s} : \mathcal{M} \cap U_{\mathbf{s}} \ne \emptyset, \quad |\mathbf{s}| = n\} \le K\};$$

$$\mathbf{J}(K) := \{\mathcal{M} : \forall \mathbf{s} \quad \#(\mathcal{M} \cap U_{\mathbf{s}}) \le K\}.$$

Denote by $\mathbf{G}(d)$ the set of all subsequences $\mathcal{M}$ representable in the form $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{M}_1 \in \mathbf{R}(K_1)$, $\mathcal{M}_2 \in \mathbf{J}(K_2)$ with some constants K_1, K_2.

Theorem 6. *Let $\mathcal{M} \in \mathbf{G}(d)$. Then $\mathcal{H}_p^d[\mathcal{M}]$ is a greedy basis for $L_p[\mathcal{M}]$, $1 < p < \infty$.*

It is clear that the condition $\mathcal{M} \in \mathbf{G}(d)$ is not a necessary condition for $\mathcal{H}_p^d[\mathcal{M}]$ to be a greedy basis for $L_p[\mathcal{M}]$. Indeed, we can find a sequence $\mathcal{M} \notin \mathbf{G}(d)$ with disjoint supports of $H_{\mathbf{n}}$, $\mathbf{n} \in \mathcal{M}$. However, we will show in Section 4 that Theorem 6 is sharp in a certain sense.

In Section 5 we present results on relations between $\{\sigma_m(f, \mathcal{H}^d)_p\}$ and $\{c_{\mathbf{n}}(f, \mathcal{H}_p^d)\}$. We give some embedding theorems in terms of the Lorentz spaces and their slight modifications.

Let us agree to denote by C various positive absolute constants and by C with arguments or indices $\big(C(q,p), C_r$ and so on$\big)$ positive numbers which depend on the arguments indicated. For two nonnegative sequences $a = \{a_n\}_{n=1}^{\infty}$ and $b = \{b_n\}_{n=1}^{\infty}$ the relation (order inequality) $a_n \ll b_n$ means that there is a number $C(a,b)$ such that for all n we have $a_n \le C(a,b)\, b_n$; and the relation $a_n \asymp b_n$ means that $a_n \ll b_n$ and $b_n \ll a_n$.

2. The convergence results

We will prove and discuss Theorems 1–3 in this section.

Proof of Theorem 1. We begin with the necessity part. Our proof is by contradiction. Suppose that $\sum_{k=1}^{\infty} t_k \psi_{n_k}$ converges in X for some sequence of different indices $\{n_k, k \ge 1\}$. First, we consider a special case. Let $\{n_k, k \ge 1\}$ be a sequence of different indices such that $\sum_{k=1}^{\infty} t_k \psi_{n_k}$ converges in X and there is a $\nu \in \mathbb{N}$ such that $n_k \ne \nu$ for all $k \in \mathbb{N}$. Take

$$f = \psi_\nu + \sum_{k=1}^{\infty} t_k \psi_{n_k}.$$

Then we can take the following realization of the WTGA

$$G_m^\tau(f, \Psi) = \sum_{k=1}^{m} t_k \psi_{n_k}.$$

Thus

$$f - G_m^\tau(f, \Psi) = \psi_\nu + \sum_{k=m+1}^{\infty} t_k \psi_{n_k}$$

and $\|f - G_m^\tau(f, \Psi)\| \not\to 0$. Consequently, the WTGA corresponding to Ψ and τ is not convergent.

We now reduce the general case to the considered above special case. Let $\{n_k, k \ge 1\}$ be a sequence of different indices such that $\sum_{k=1}^{\infty} t_k \psi_{n_k}$ converges in X. This implies that $\lim_{k\to\infty} t_k = 0$, so there is a subsequence $\{k_l, l \ge 1\}$ with $k_1 = 1$ such that

$$\sum_{l=1}^{\infty} t_{k_l} < \infty.$$

Clearly, then both

$$\sum_{l=1}^{\infty} t_{k_l}\psi_{n_{k_l}} \quad \text{and} \quad \sum_{l=1}^{\infty} t_{k_l}\psi_{n_{k_{l+1}}}$$

converge in X, and

$$\sum_{k=1}^{\infty} t_k\psi_{n_k} - \sum_{l=1}^{\infty} t_{k_l}\psi_{n_{k_l}} + \sum_{l=1}^{\infty} t_{k_l}\psi_{n_{k_{l+1}}} = \sum_{k=1}^{\infty} t_k\psi_{s_k},$$

where

$$s_k = \begin{cases} n_k & \text{if} \quad k \neq k_l \quad \text{for all} \quad l \geq 1, \\ n_{k_{l+1}} & \text{if} \quad k = k_l \quad \text{for some} \quad l \geq 1. \end{cases}$$

Note that $\{s_k, k \geq 1\}$ is a sequence of different indices such that $s_k \neq n_1$ for all $k \geq 1$. Therefore we are in the special case considered above. This completes the proof of the necessity part.

We now proceed to the sufficiency part. Our proof is again by contradiction. Assume that Ψ is an unconditional basis. Suppose that $f \in X$ is such that

$$G_m^{\tau}(f, \Psi) \not\to f$$

in X. By definition,

$$G_m^{\tau}(f, \Psi) = \sum_{k=1}^{m} c_{n_k}\psi_{n_k},$$

where

$$|c_{n_1}| \geq t_1 \sup_{n\in\mathbb{N}} |c_n|, \qquad \text{and} \quad |c_{n_k}| \geq t_k \sup_{n\neq n_1,\dots,n_{k-1}} |c_n| \quad \text{for} \quad k \geq 2. \tag{2.1}$$

As $G_m^{\tau}(f, \Psi) \not\to f$ and the basis Ψ is unconditional, there is $\mu \in \mathbb{N}$ with $c_\mu \neq 0$ such that $n_k \neq \mu$ for all $k \in \mathbb{N}$. This and (2.1) implies that $t_k \leq \frac{|c_{n_k}|}{|c_\mu|}$. Since the basis Ψ is unconditional, it follows that the series $\sum_{k=1}^{\infty} t_k\psi_{n_k}$ converges in X. Theorem 1 is now proved.

Remark 2.1 It is clear that in the case of TGA ($\tau = \{1\}$) the condition (D) of Theorem 1 is always satisfied. However, the TGA may not converge for some bases. For instant, it was proved in [T5] (see also [CF] for $1 \leq p < 2$) that the TGA may diverge in L_p, $p \neq 2$ for the trigonometric system. A basis for which the TGA converges is called quasi-greedy basis ([KT],[W]). It is clear that any unconditional basis is a quasi-greedy basis. It is known (see [KT]) that there is a quasi-greedy basis that is not an unconditional basis. For other examples of such bases see [DM].

We will prove one technical result that we will need later on. Let $\mathcal{M} = \{m_k, k \geq 1\}$ be a sequence of different indices, and let $\tau = \{t_k, k \geq 1\}$ be a weakness sequence. Consider a new weakness sequence $\tau(\mathcal{M}) = \{\eta_n, n \geq 1\}$, where

$$\eta_n = \begin{cases} t_k & \text{when} \quad n = m_k \quad \text{for} \quad k \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

We have the following corollary of Theorem 1.

Proposition 2.1. *Let Ψ be a normalized unconditional basis in a Banach space X. Then the WTGA corresponding to Ψ and $\tau(\mathcal{M})$ is convergent if and only if WTGA corresponding to Ψ and τ is convergent.*

Proof. It is clear that if $\tau(\mathcal{M})$ does not satisfy the necessary and sufficient condition (D) from Theorem 1, then τ also does not satisfy that condition. Thus if the WTGA diverges for $\tau(\mathcal{M})$ it diverges for τ. We now prove that if τ does not satisfy (D) then $\tau(\mathcal{M})$ also does not satisfy (D). Assume that $\sum_{k=1}^{\infty} t_k\psi_{n_k}$ converges. Then $t_k \to 0$ and we let $\mathcal{K} := \{k_j\}_{j=1}^{\infty}$ to be such that all k_j are even numbers and

$$\sum_{j=1}^{\infty} t_{k_j} < \infty.$$

Then the set $\mathcal{L} := \mathbb{N} \setminus \mathcal{K}$ is infinite and the series $\sum_{k\in\mathcal{L}} t_k\psi_{n_k}$ also converges.

We now assign for the sequence $\tau(\mathcal{M})$ to each $\eta_{m_k} = t_k$, $k \in \mathcal{L}$ a basic function ψ_{n_k}. We split the infinite set $\mathcal{K}$ into a union of two infinite sets $\mathcal{K}_1$ and $\mathcal{K}_2$. Then we set up a one-to-one correspondance $k \leftrightarrow k'$ between $\mathcal{K}$ and $\mathcal{K}_1$ and assign to each $\eta_{m_k} = t_k$, $k \in \mathcal{K}$ a basic function $\psi_{n_{k'}}$ and to different $\eta_l = 0$ we assign different basic functions ψ_s with $s \in \cup_{k\in\mathcal{K}_2}\{n_k\}$. Then the corresponding sum from the condition (D) for the $\tau(\mathcal{M})$ will be

$$\sum_{k\in\mathcal{L}} t_k\psi_{n_k} + \sum_{k\in\mathcal{K}} t_k\psi_{n_{k'}}.$$

This series converges and therefore $\tau(\mathcal{M})$ does not satisfy (D). By Theorem 1 we conclude that the WTGA corresponding to $\tau(\mathcal{M})$ diverges. This completes the proof of Proposition 2.1.

Proof of Theorem 2. Since Ψ is a normalized unconditional basis in $L_p([0,1]^d)$ with $p \geq 2$, we have for any set $\{n_k\}$ of different indices

$$\|\sum_{k=1}^{\infty} t_k\psi_{n_k}\|_p \geq C\Big(\int_{[0,1]^d}\big(\sum_{k=1}^{\infty}|t_k|^2|\psi_{n_k}(x)|^2\big)^{p/2}dx\Big)^{1/p} \geq C\Big(\sum_{k=1}^{\infty}|t_k|^p\Big)^{1/p}.$$

Therefore, by the sufficiency part of Theorem 1 the WTGA with a weakness sequence τ converges if $\tau \notin l_p$.

Let us assume that $\tau \in l_p$. Then it is known (see [KP]) that an unconditional basis $\Psi = \{\psi_n\}_{n=1}^{\infty}$ of $L_p([0,1]^d)$, $1 < p < \infty$ contains a subsequence $\{\psi_{n_k}\}_{k=1}^{\infty}$ such that each series $\sum_{k=1}^{\infty} a_k\psi_{n_k}$ converges provided $\{a_k\}_{k=1}^{\infty} \in l_p$. Specifying $a_k = t_k$ and applying Theorem 1 we obtain that the WTGA with the weakness sequence τ does not converge. This completes the proof of Theorem 2.

The case of $L_p([0,1]^d)$ with $1 < p < 2$ is different – the condition on the weakness sequence τ depends now on a particular unconditional basis Ψ. Let Ψ be a normalized unconditional basis in $L_p([0,1]^d)$ with $1 < p < 2$, then for any coefficients $\{a_n\}$ we have

$$C_1\Big(\sum_{n=1}^{\infty}|a_n|^2\Big)^{1/2} \leq \|\sum_{n=1}^{\infty} a_n\psi_n\|_p \leq C_2\Big(\sum_{n=1}^{\infty}|a_n|^p\Big)^{1/p}.$$

Thus by Theorem 1 if a weakness sequence $\tau \in l_p$, then the WTGA corresponding to Ψ and τ is not convergent in $L_p([0,1]^d)$. Also, if $\tau \notin l^2$ then the WTGA corresponding to Ψ and τ is convergent in $L_p([0,1]^d)$. In addition, as $L_p \oplus l_2$ is isomorphic to L_p, there is an unconditional basis Ψ in $L_p([0,1]^d)$ $(1<p<2)$ for which the condition $\tau \notin l_2$ is also a necessary condition for WTGA corresponding to Ψ and τ to be convergent.

Let us consider in detail the case of the Haar system $\mathcal{H}_p^d$.

Proof of Theorem 3. If $\tau = \{t_k, k \ge 1\}$ is a sequence of nonnegative numbers which does not converge to 0, then $\limsup_{k\to\infty} t_k > 0$, and convergence of the WTGA corresponding to $\mathcal{H}_p^d$ and τ is an immediate consequence of Theorem 1.

It remains to consider sequences τ such that $\lim_{k\to\infty} t_k = 0$. By Proposition 2.1, it is sufficient to consider a sequence τ with $t_k \neq 0$. Introduce the notation:

$$q_m := \#\{\mathbf{n} : |\operatorname{supp} H_{\mathbf{n}}| = 2^{-m}\}, \quad \nu_0 = 0, \quad \nu_m = \sum_{j=0}^{m-1} q_j, \quad \text{for} \quad m \ge 1.$$

Note that $q_m \asymp m^{d-1}2^m \asymp \nu_m$ and $\log q_m \asymp m \asymp \log \nu_m$ for $m \ge 1$. As the sequence $\{t_k^*, k \ge 0\}$ is nonincreasing, we have for $\nu_m \le k \le \nu_{m+1}$

$$C_1 (t^*_{\nu_{m+1}})^2 2^{m(2/p-1)} \le (t_k^*)^2 \big(k(\log k)^{(1-d)}\big)^{2/p-1} \le C_2 (t^*_{\nu_m})^2 2^{m(2/p-1)},$$

which implies that

$$\sum_{k=1}^{\infty} (t_k^*)^2 \big(k(\log k)^{(1-d)}\big)^{2/p-1} < \infty \quad \Longleftrightarrow \quad \sum_{m=1}^{\infty} (t^*_{\nu_{m+1}})^2 m^{d-1} 2^{2m/p} < \infty. \tag{2.2}$$

Let us recall (see Lemma 3.1 below for more detail) that for any N different indices $\mathbf{n}_1, \dots, \mathbf{n}_N$

$$\|\sum_{i=1}^{N} H_{\mathbf{n}_i,p}\|_p \ge C(p,d)(\log N)^{(d-1)(1/2-1/p)} N^{1/p}, \quad 1 < p \le 2. \tag{2.3}$$

For any sequence of different indices $\{\mathbf{n}_k, k \ge 1\}$

$$\|\sum_{k=1}^{\infty} t_k H_{\mathbf{n}_k,p}\|_p = \|\sum_{k=1}^{\infty} t_k^* H_{\mathbf{n}_k^*,p}\|_p \ge C(p,d)\Big(\sum_{m=0}^{\infty} \|\sum_{k=\nu_m+1}^{\nu_{m+1}} t_k^* H_{\mathbf{n}_k^*,p}\|_p^2\Big)^{1/2}$$

where $\mathbf{n}_k^*$ is such that $t_k^* = t_{k^*}$ and $\mathbf{n}_k^* = \mathbf{n}_{k^*}$. By (2.3) we continue

$$\ge C(p,d)\Big(\sum_{m=0}^{\infty} \big(t^*_{\nu_{m+1}} (\log q_m)^{(d-1)(1/2-1/p)} q_m^{1/p}\big)^2\Big)^{1/2} \ge$$

$$\ge C(p,d)\Big(\sum_{m=0}^{\infty} (t^*_{\nu_{m+1}})^2 (m+1)^{d-1} 2^{2m/p}\Big)^{1/2}.$$

Thus, it follows from (2.2) and Theorem 1 that if a weakness sequence τ satisfies the condition (1.7), then the WTGA corresponditng to $\mathcal{H}_p^d$ and τ converges.

Suppose now that

$$\sum_{k=1}^{\infty}(t_k^*)^2\big(k(\log k)^{(1-d)}\big)^{2/p-1}<\infty. \tag{2.4}$$

Take a sequence $\{\mathbf{n}_k, k\geq 1\}$ of different indices satisfying

$$|\operatorname{supp} H_{\mathbf{n}_k,p}|=2^{-m}\quad\text{for}\quad \nu_m+1\leq k\leq\nu_{m+1},$$

i.e. we order the functions $H_{\mathbf{n},p}$ according to the measure of their supports (more precisely, the sequence of measures $|\operatorname{supp} H_{\mathbf{n}_k,p}|$ is nonincreasing). Then, using unconditionality of $\mathcal{H}_p^d$, we obtain

$$\begin{aligned}\|\sum_{k=1}^{\infty}t_k^*H_{\mathbf{n}_k,p}\|_p^p &\leq C(p,d)\int_{[0,1]^d}\Big(\sum_{k=1}^{\infty}|t_k^*H_{\mathbf{n}_k,p}(x)|^2\Big)^{p/2}dx\\ &\leq C(p,d)\int_{[0,1]^d}\Big(\sum_{m=1}^{\infty}(t_{\nu_m+1}^*)^2\sum_{k=\nu_m+1}^{\nu_{m+1}}|H_{\mathbf{n}_k,p}(x)|^2\Big)^{p/2}dx\\ &\leq C(p,d)\Big(\sum_{m=0}^{\infty}(t_{\nu_m+1}^*)^2(m+1)^{d-1}2^{2m/p}\Big)^{p/2}.\end{aligned}$$

The above inequality combined with Theorem 1 and (2.2) implies that for τ satisfying (2.4) the corresponding WTGA is not convergent.

3. Proof of Theorems 4 and 5

This proof uses an idea from [T2]. The following proposition is a well known fact about unconditional bases (see [LT],v.I, p.19).

Proposition 3.1. *Let Ψ be an unconditional basis for X. Then for every choice of bounded scalars $\{\lambda_k\}_{k=1}^{\infty}$, we have*

$$\|\sum_{k=1}^{\infty}\lambda_k a_k\psi_k\|\leq K\sup_k|\lambda_k|\|\sum_{k=1}^{\infty}a_k\psi_k\|.$$

Take any $\epsilon>0$ and find

$$p_m(f):=\sum_{k\in P}b_k\psi_k$$

such that $|P|=m$ and

$$\|f-p_m(f)\|\leq\sigma_m(f,\Psi)+\epsilon. \tag{3.1}$$

For any finite set of indices Λ we denote S_Λ the projector

$$S_\Lambda(f) := \sum_{k\in\Lambda} c_k(f,\Psi)\psi_k.$$

Proposition 3.1 implies that

$$\|f - S_P(f)\| \le K(\sigma_m(f,\Psi)+\epsilon). \tag{3.2}$$

Let

$$G_N^\tau(f,\Psi) = \sum_{k\in Q} c_k(f,\Psi)\psi_k = S_Q(f).$$

Then

$$\|f - G_N^\tau(f,\Psi)\| \le \|f - S_P(f)\| + \|S_P(f) - S_Q(f)\|. \tag{3.3}$$

The first term in the right side of (3.3) has been estimated in (3.2). We estimate now the second term. We have

$$S_P(f) - S_Q(f) = S_{P\setminus Q}(f) - S_{Q\setminus P}(f). \tag{3.4}$$

Similarly to (3.2) we have

$$\|S_{Q\setminus P}(f)\| \le K(\sigma_m(f,\Psi)+\epsilon). \tag{3.5}$$

We now estimate $\|S_{P\setminus Q}(f)\|$. Let J be the set of indices i such that elements of $P\cap Q$ where chosen at steps $i\in J$. Denote

$$a := \max_{k\in P\setminus Q} |c_k(f,\Psi)|.$$

Then from the definition of the WTGA we obtain that

$$S_{Q\setminus P}(f) = \sum_{k\in Q\setminus P} c_k(f,\Psi)\psi_k$$

and $\{c_k(f,\Psi)\}_{k\in Q\setminus P}$ can be enumerated by indices $i\in V := [1,N]\setminus J$ in such a way that

$$|c_{k_i}(f,\Psi)| \ge t_i a, \qquad i\in V.$$

Then by Proposition 3.1 we have

$$\|S_{Q\setminus P}(f)\| \ge K^{-1} a\phi(\tau,N,V) \tag{3.6}$$

and

$$\|S_{P\setminus Q}(f)\| \le Ka\varphi^s(|P\setminus Q|). \tag{3.7}$$

Thus in the case of $N = m$ (Theorem 4) denoting $n := |P \setminus Q| = |Q \setminus P|$ we get

$$\|S_{P\setminus Q}(f)\| \le K^2 \frac{\varphi^s(n)}{\phi(\tau,m,n)} \|S_{Q\setminus P}(f)\| \le$$

$$K^2 \mu(\tau,m) \|S_{Q\setminus P}(f)\|.$$

In the case of $N = v_m$ (Theorem 5) we obtain

$$\phi(\tau,N,V) \ge \phi(\tau,N) - \|\sum_{i\in J} t_i \psi_{k_i}\| \ge \phi(\tau,N) - \varphi(m) \ge \varphi(m). \tag{3.8}$$

Combining (3.6)–(3.8) we get

$$\|S_{P\setminus Q}(f)\| \le K^2 \|S_{Q\setminus P}\|.$$

It remains to substitute this inequality and the inequality (3.5) into (3.4) and use (3.3).

Theorems 4 and 5 are proved.

Let us make some comments on Theorems 4 and 5. First we consider the case when Ψ is a greedy basis. Then by Definition 1.1 we have (1.3) satisfied. Let us see what Theorem 4 gives in this case. We remind one result from [KT].

Definition 3.1. *We say that a normalized basis $\Psi = \{\psi_k\}_{k=1}^\infty$ is a democratic basis for X if there exists a constant $D := D(X,\Psi)$ such that for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have*

$$\|\sum_{k\in P} \psi_k\| \le D \|\sum_{k\in Q} \psi_k\|. \tag{3.9}$$

The following theorem was proved in [KT].

Theorem 3.1. *A normalized basis is greedy if and only if it is unconditional and democratic.*

Thus by Theorem 3.1 we have (3.9) satisfied for a greedy basis. It is easy to see that (3.9) implies $\varphi^s(m) \le D\phi(m)$ and therefore for $\tau = \{1\}$ we get $\mu(\{1\},m) \le D$. This means that Theorem 4 states that for any greedy basis Ψ we have (1.3) for any $\rho \in D(f)$.

We now apply Theorems 4 and 5 in the case of $\Psi = \mathcal{H}_p^d$, $1 < p < \infty$ with the weakness sequence $\tau = \{1\}$. We will use the following known inequalities.

Lemma 3.1. *Let $1 < p < \infty$. Then for any Λ, $|\Lambda| = m$, we have for $2 \le p < \infty$*

$$C^1_{p,d} m^{1/p} \min_{\mathbf{n}\in\Lambda} \|c_{\mathbf{n}} H_{\mathbf{n}}\|_p \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} H_{\mathbf{n}}\|_p \le C^2_{p,d} m^{1/p} (\log m)^{h(p,d)} \max_{\mathbf{n}\in\Lambda} \|c_{\mathbf{n}} H_{\mathbf{n}}\|_p,$$

and for $1 < p \le 2$

$$C^3_{p,d} m^{1/p} (\log m)^{-h(p,d)} \min_{\mathbf{n}\in\Lambda} \|c_{\mathbf{n}} H_{\mathbf{n}}\|_p \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} H_{\mathbf{n}}\|_p \le C^4_{p,d} m^{1/p} \max_{\mathbf{n}\in\Lambda} \|c_{\mathbf{n}} H_{\mathbf{n}}\|_p$$

where $h(p,d) := (d-1)|1/2-1/p|$.

Lemma 3.1 in the case $d=2$, $4/3 \le p \le 4$ has been proved in [T3] and in the general case in [W]. Lemma 3.1 implies that for $1<p<\infty$

$$\mu(\{1\},m) \asymp C(p,d)(\log m)^{(d-1)|1/2-1/p|}$$

and

$$v_m \asymp C(p,d)m(\log m)^{(d-1)|p/2-1|}.$$

Therefore Theorem 4 gives the known result (see [T3], [W])

$$\|f - G_m(f,\mathcal{H}_p^d)\|_p \le C(p,d)(\log m)^{(d-1)|1/2-1/p|}\sigma_m(f,\mathcal{H}_p^d)_p, \quad 1<p<\infty. \tag{3.10}$$

Theorem 5 gives a new result. We note that for functions f with slow decay of $\sigma_m(f,\mathcal{H}_p^d)_p$ Theorem 5 gives a better estimate than (3.10). Consider for example $\sigma_m(f,\mathcal{H}_p^d)_p \asymp m^{-\alpha}$. Then (3.10) gives

$$\|f - G_m(f,\mathcal{H}_p^d)\|_p \ll (\log m)^{(d-1)|1/2-1/p|}m^{-\alpha}, \quad 1<p<\infty \tag{3.11}$$

while Theorem 5 gives

$$\|f - G_m(f,\mathcal{H}_p^d)\|_p \ll \left(m(\log m)^{-(d-1)|p/2-1|}\right)^{-\alpha}, \quad 1<p<\infty. \tag{3.12}$$

For $\alpha < 1/p$ the estimate (3.12) is better than (3.11).

Theorem 3.2. *Let X be a Banach space with a normalized unconditional basis Ψ. Let $\tau = \{t_n, n \ge 1\}$ be a weakness sequence such that the WTGA with respect to Ψ and τ is convergent. Let $\{v_m, m \in \mathbb{N}\}$, be a sequence of natural numbers, $v_m \ge m$. Then the following two conditions are equivalent.*

(i) There is a constant C such that for each pair of natural numbers $n \le m$ and any set $V \subseteq [1, v_m]$, $|V| = v_m - m + n$ we have the following inequality

$$\|\sum_{j\in A}\psi_j\| \le C\|\sum_{i\in V} t_i\psi_{k_i}\|$$

for any two sets of indices A and $B := \{k_i, i \in V\}$ (all k_i, $i \in V$ are different) satisfying the conditions: $A \cap B = \emptyset$ and $|A| = n$.

(ii) There is a $C > 0$ such that for all $f \in X$

$$\|f - G_{v_m}^\tau(f,\Psi)\| \le C\sigma_m(f,\Psi). \tag{3.13}$$

Proof. The implication $(i) \Rightarrow (ii)$ can be proved in the same way as Theorem 5. We will not dwell on it here. We only note that we use (i) with $A = P\setminus Q$ and $B = Q \setminus P$ to get from the following analogs of (3.6) and (3.7)

$$\|S_{Q\setminus P}(f)\| \ge K^{-1}a\|\sum_{i\in V} t_i\psi_{k_i}\|, \quad B = \{k_i, i\in V\} \tag{3.14}$$

$$\|S_{P\setminus Q}(f)\| \le Ka\|\sum_{n\in A}\psi_n\| \tag{3.15}$$

the inequality

$$\|S_{P\setminus Q}(f)\| \le CK^2\|S_{Q\setminus P}(f)\|.$$

We now prove that $(ii) \Rightarrow (i)$. Let a pair of $n \le m$ be given and let V, A, B be any sets satisfying the conditions of (i). Let Y be such that $|Y| = m-n$ and $A \cap Y = \emptyset$ and $B \cap Y = \emptyset$. Consider

$$f := \sum_{n\in A\cup Y}\psi_n + \sum_{i\in V} t_i\psi_{k_i}.$$

We take the following realization of the WTGA. For steps $i \in V$ we take $n_i = k_i$ and for steps $i \notin V$ we take different $n_i \in Y$. Then we get

$$G^\tau_{v_m}(f,\Psi) = \sum_{n\in Y}\psi_n + \sum_{i\in V} t_i\psi_{k_i}.$$

This implies by (ii) that

$$\|\sum_{n\in A}\psi_n\| = \|f - G^\tau_{v_m}(f,\Psi)\| \le C\sigma_m(f,\Psi) \le \|\sum_{i\in V} t_i\psi_{k_i}\|.$$

This completes the proof of Theorem 3.2.

Let us make some more comments on Theorems 4 and 5. From the definition of $\varphi^s(m)$ and $\phi(m)$ we get immediately that $\varphi^s(m) \le Cm\phi(m)$ and therefore $\mu(\{1\},m) \le Cm$. Thus by Theorem 4 for any normalized unconditional basis Ψ we have

$$\|f - G_m(f,\Psi)\| \le C(\Psi)m\sigma_m(f,\Psi).$$

We will now construct an example of a Banach space X and unconditional basis Ψ such that Theorem 5 does not hold even for the TGA ($\tau = \{1\}$).

Example 3.1 Let X be the space of sequences $a = (a_{m,n}; n,m \ge 1)$ with the norm

$$\|a\| := \sum_{n=1}^\infty \Big(\sum_{m=1}^\infty |a_{m,n}|^n\Big)^{1/n} < \infty.$$

Let $\Psi = \{\psi_{k,l}\}$ where $\psi_{k,l} := (\delta_{(m,n),(k,l)}; n,m \ge 1)$. Clearly, Ψ is an unconditional basis of X. Therefore the TGA corresponding to Ψ converges. However, observe that there is no sequence $\{v_m, m \ge 1\}$ for which condition (i) of Theorem 3.2 is satisfied. To see this, consider $A_m := \{(k,1) : k = 1,\dots,m\}$ and $B_{\mu,n} = \{(l,n) : l = 1,\dots,\mu\}$. Then for each m

$$\|\sum_{(k,l)\in A_m}\psi_{k,l}\| = \|\sum_{k=1}^m \psi_{k,1}\| = m,$$

while for each μ

$$\inf_n \|\sum_{(k,l)\in B_{\mu,n}}\psi_{k,l}\| = \lim_{n\to\infty}\|\sum_{k=1}^{\mu}\psi_{k,n}\| = \lim_{n\to\infty}\mu^{1/n} = 1.$$

By Theorem 3.2 there is no sequence $\{v_m, m \ge 1\}$ such that the inequality (3.13) holds with a constant C independent of both $a \in X$ and m.

4. Greedy subsequences of the multivariate Haar basis

It is well known that, for $p \neq 2$ and $d \geq 2$ the d-variate tensor product Haar system $\mathcal{H}_p^d$ is not a greedy basis in the corresponding $L_p([0,1]^d)$ space. However, for some functions the Thresholding Greedy Algorithm may give an order of approximation comparable with the order of best approximation. In this section we address the question: For what functions the TGA realizes near best m-term approximation? Let us recall that for $\mathbf{s} = (s_1, \dots, s_d)$, the dyadic block $U_\mathbf{s}$ is defined by (1.10), and the Haar functions $H_{\mathbf{n},p}$ with $\mathbf{n} \in U_\mathbf{s}$ have the same shape of supports.

We are interested in the influence of some "structural constrains" imposed on a function on the efficiency of TGA with respect to the Haar system $\mathcal{H}_p^d$. By "structural constrains" we mean constrains imposed on the number of nonzero coefficients in dyadic blocks or on the number of dyadic blocks with nonzero coefficients. These constrains are expressed in terms of classes of sequences $\mathbf{R}(K)$, $\mathbf{J}(K)$, and $\mathbf{G}(d)$ (see the Introduction). We begin with proving Theorem 6.

Proof of Theorem 6. For any sequence $\mathcal{M}$ the system $\mathcal{H}_p^d[\mathcal{M}]$ is an unconditional basis for $L_p[\mathcal{M}]$, $1 < p < \infty$. Thus by Theorem 3.1 it is sufficient to establish that $\mathcal{H}_p^d[\mathcal{M}]$ is democratic provided $\mathcal{M} \in \mathbf{G}(d)$. This follows from Lemmas 4.1 and 4.2 below.

Lemma 4.1. *Let $1 < p < \infty$ and $\mathcal{M} \in \mathbf{R}(K)$. Then for any different $\mathbf{n}_1, \dots, \mathbf{n}_m \subset \mathcal{M}$ we have*

$$\|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \asymp m^{1/p}$$

with constants depending on K and p.

Lemma 4.2. *Let $1 < p < \infty$ and $\mathcal{M} \in \mathbf{J}(K)$. Then for any different $\mathbf{n}_1, \dots, \mathbf{n}_m \subset \mathcal{M}$ we have*

$$\|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \asymp m^{1/p}$$

with constants depending on K and p.

In the case $d = 1$ Lemma 4.1 with $\mathcal{M} = \mathbb{N}$ was proved in [T2]. That same proof works for $d \geq 2$ under assumption $\mathcal{M} \in \mathbf{R}(K)$. Let us prove Lemma 4.2.

Proof of Lemma 4.2. We recall (see Lemma 3.1) that by the Littlewood-Paley theory we have

$$\|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \leq C(p,d) m^{1/p} \quad \text{for} \quad 1 < p \leq 2$$

and

$$C(p,d) m^{1/p} \leq \|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \quad \text{for} \quad 2 \leq p < \infty$$

for any different $\mathbf{n}_1, \dots, \mathbf{n}_m$. To prove the upper estimate in case $2 < p < \infty$, we use the following inequality, which is a special case of Lemma 2.3 of [T1]: for $2 < p < \infty$ and $f = \sum_{\mathbf{s}} f_{\mathbf{s}}$ with $f_{\mathbf{s}} = \sum_{\mathbf{n}\in U_{\mathbf{s}}} c_{\mathbf{n}}(f) H_{\mathbf{n}}$

$$\|f\|_p \le C_{p,d}\Big(\sum_{\mathbf{s}} \big(2^{|\mathbf{s}|(1/2-1/p)}\|f_{\mathbf{s}}\|_2\big)^p\Big)^{1/p}. \tag{4.1}$$

For each $\mathbf{s}$, let $m_{\mathbf{s}}$ be the number of $\mathbf{n}_k$'s in $U_{\mathbf{s}}$. Note that

$$\|\sum_{k:\mathbf{n}_k\in U_{\mathbf{s}}} H_{\mathbf{n}_k,p}\|_2 = 2^{|\mathbf{s}|(1/p-1/2)} m_{\mathbf{s}}^{1/2},$$

and therefore by (4.1)

$$\|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \le C_{p,d}\Big(\sum_{\mathbf{s}} m_{\mathbf{s}}^{p/2}\Big)^{1/p}.$$

Taking into account that $m = \sum_{\mathbf{s}} m_{\mathbf{s}}$ and $m_{\mathbf{s}} \le K$ by assumption $\mathcal{M} \in \mathbf{J}(K)$ we get

$$\|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \le C_{p,d} m^{1/p} \quad \text{for} \quad 2 < p < \infty$$

with the constant depending only on p and K.

To complete the proof, recall that the lower estimate in the case $1 < p < 2$ follows from the upper estimates for all $2 < p < \infty$ by duality. Using the Hölder inequality we obtain

$$\begin{aligned} m = \sum_{k=1}^m \int_{[0,1]^d} H_{\mathbf{n}_k,p}(x)\cdot H_{\mathbf{n}_k,p'}(x)dx &= \int_{[0,1]^d} (\sum_{k=1}^m H_{\mathbf{n}_k,p}(x))\cdot(\sum_{k=1}^m H_{\mathbf{n}_k,p'}(x))dx \\ \le \|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p \cdot \|\sum_{k=1}^m H_{\mathbf{n}_k,p'}\|_{p'} &\le C m^{1/p'} \|\sum_{k=1}^m H_{\mathbf{n}_k,p}\|_p, \end{aligned}$$

which gives the lower estimate in case $1 < p < 2$ with a constant depending only on p and K.

This completes the proof of Theorem 6.

We now proceed to a discussion of in what sense Theorem 6 is sharp. We need some more notation describing the structural constrains on functions.

Let $\Lambda = \{\lambda_{\mathbf{s}}, \mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}_+^d\}$ be a sequence of integers, satisfying

$$0 \le \lambda_{\mathbf{s}} \le \#U_{\mathbf{s}}. \tag{4.2}$$

Denote

$$\mathcal{V}(\Lambda) := \{\mathcal{M} : |\mathcal{M}\cap U_{\mathbf{s}}| \le \lambda_{\mathbf{s}}\}.$$

For $1 < p < \infty$ consider the following sets of functions

$$L_p(\Lambda) = \bigcup_{\mathcal{M} \in \mathcal{V}(\Lambda)} L_p[\mathcal{M}], \tag{4.3}$$

i.e. $L_p(\Lambda)$ consists of $f \in L_p([0,1]^d)$ with at most $\lambda_{\mathbf{s}}$ non-zero coefficients in blocks $U_{\mathbf{s}}$, $\mathbf{s} \in \mathbb{Z}_+^d$.

We describe a distribution of $\lambda_{\mathbf{s}}$'s for a given sequence Λ by defining for nonnegative integers μ, M

$$\alpha_{\mu,M}(\Lambda) := \#\{\mathbf{s} : |\mathbf{s}| = \mu \quad \text{and} \quad \lambda_{\mathbf{s}} \geq M\}. \tag{4.4}$$

Now, let $A := \{a_{\mu,M}\}$ be a sequence of nonnegative integers satysfying the conditions

$$a_{\mu,M_1} \leq a_{\mu,M_2} \quad \text{for} \quad M_1 \geq M_2 \tag{4.5}$$

$$a_{\mu,0} = \#\{\mathbf{s} = (s_1, \dots, s_d) : |\mathbf{s}| = \mu\}, \quad a_{\mu,M} = 0 \quad \text{for} \quad M > \max_{|\mathbf{s}|=\mu} \#U_{\mathbf{s}}. \tag{4.6}$$

Let us note that the sequence $\{\alpha_{\mu,M}(\Lambda)\}$ defined above satisfies these conditions for any Λ.

To formulate the main result of this section, we define a type of a sequence Λ and full range sequences.

Definition 4.1. *Let $A = \{a_{\mu,M}\}$ be a sequence satisfying (4.5) and (4.6), and let $\Lambda = \{\lambda_{\mathbf{s}}\}$ be a sequence of integers satisfying (4.2). Λ is called a type A sequence if $\alpha_{\mu,M}(\Lambda) = a_{\mu,M}$ for all $\mu, M \geq 0$ (where $\alpha_{\mu,M}(\Lambda)$ is given by formula (4.4)).*

Definition 4.2. *Let $A = \{a_{\mu,M}\}$ be a sequence satisfying (4.5) and (4.6). The sequence A is called a full range sequence if for each $M > 0$ we have $\limsup_{\mu \to \infty} a_{\mu,M} = \infty$.*

Let us take a sequence $\mathcal{M} \in \mathbf{G}(d)$ and define

$$a_{\mu,M}(\mathcal{M}) := \#\{\mathbf{s} : |\mathbf{s}| = \mu \quad \text{and} \quad \#(\mathcal{M} \cap U_{\mathbf{s}}) \geq M\}.$$

From the definition of $\mathbf{G}(d)$ we get that

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2, \quad \mathcal{M}_1 \in \mathbf{R}(K_1), \quad \mathcal{M}_2 \in \mathbf{J}(K_2).$$

Thus for $M > K_2$ we have $a_{\mu,M}(\mathcal{M}) \leq K_1$. Therefore any $\mathcal{M} \in \mathbf{G}(d)$ has a distribution that is not a full range sequence. It follows from the Definition 4.2 that the opposite is also true: if $\{a_{\mu,M}(\mathcal{M})\}$ is not a full range sequence then $\mathcal{M} \in \mathbf{G}(d)$. Theorem 4.1 below states that if constrains on the structure of a function are given in terms of the distribution sequence $\{a_{\mu,M}(\mathcal{M})\}$ then Theorem 6 is the best possible.

Theorem 4.1. *Let $A = \{a_{\mu,M}\}$ be a sequence satisfying (4.5) and (4.6). Let $d \geq 2$, $1 < p < \infty$, $p \neq 2$. Then the following conditions are equivalent:*

(i) A is not a full range sequence.

(ii) There is a constant $C = C(A, d, p)$ (depending only on A, d and p) such that for each $\Lambda = \{\lambda_{\mathbf{s}}, \mathbf{s} = (s_1, \dots, s_d)\}$ of type A we have for all $f \in L_p(\Lambda)$ and $m \in \mathbb{N}$

$$\|f - G_m(f, \mathcal{H}_p^d)\|_p \leq C\sigma_m(f, \mathcal{H}_p^d). \tag{4.7}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows from Theorem 6. We now prove that $(ii) \Rightarrow (i)$. For any given sequence A of a full range we will construct a Λ of type A such that (4.7) does not hold. We begin with a construction which will provide us with building blocks of the counterexample sequence $\mathcal{M}$. This construction is a modification of a construction from [6,Section 4].

For a given pair of natural numbers k and l such that $l < k$ we consider the following special polynomials. First, we define a set

$$I(k, l) := \{\mathbf{s} : |\mathbf{s}| = kd, \quad s_j \geq k - l, \quad j = 1, \dots, d\}.$$

Then

$$\#I(k, l) \asymp l^{d-1}.$$

Consider the cube $[0, 2^{l-k})^d$ and define

$$U_{\mathbf{s}}(k, l) := \{\mathbf{n} : \mathbf{n} \in U_{\mathbf{s}} \quad \text{and} \quad \operatorname{supp} H_{\mathbf{n}} \subseteq [0, 2^{l-k})^d\};$$

$$E(k, l) := \cup_{\mathbf{s} \in I(k,l)} U_{\mathbf{s}}(k, l).$$

Define a polynomial

$$g_{k,l} := \sum_{\mathbf{n} \in E(k,l)} H_{\mathbf{n},p}.$$

By the Littlewood-Paley theory we have

$$\|g_{k,l}\|_p \asymp \|\big(\sum_{\mathbf{n} \in E(k,l)} |H_{\mathbf{n},p}(x)|^2\big)^{1/2}\|_p, \quad 1 < p < \infty. \tag{4.8}$$

The supports of $\sum_{\mathbf{n} \in U_{\mathbf{s}}(k,l)} H_{\mathbf{n},p}$, $\mathbf{s} \in I(k, l)$ cover the cube $[0, 2^{l-k})^d$ and therefore we obtain from (4.8)

$$\|g_{k,l}\|_p \asymp 2^{ld/p} l^{(d-1)/2}. \tag{4.9}$$

The number $m := m(k, l)$ of terms of the polynomial $g_{k,l}$ satisfies the inequalities

$$C_1(d) l^{(d-1)} 2^{ld} \leq m \leq C_2(d) l^{(d-1)} 2^{ld}. \tag{4.10}$$

Let us take a companion to the $g_{k,l}$ polynomial

$$h_m := \sum_{i=1}^{m} H_{\mathbf{n}_i,p}$$

such that $\mathbf{n}_i \notin E(k,l)$, $i = 1,\dots,m$ and

$$\operatorname{supp} H_{\mathbf{n}_i} \cap \operatorname{supp} H_{\mathbf{n}_j} = \emptyset, \qquad i \neq j.$$

Then

$$\|h_m\|_p \asymp m^{1/p}. \tag{4.11}$$

Considering the function $f := g_{k,l} + 2h_m$ in the case $2 < p < \infty$ and the function $f := 2g_{k,l} + h_m$ in the case $1 < p < 2$ we will get for an $\mathcal{M}$ containing $E(k,l)$ and $\{\mathbf{n}_i\}_{i=1}^m$

$$\|f - G_m(f, \mathcal{H}_p^d[\mathcal{M}])\|_p / \sigma_m(f, \mathcal{H}_p^d[\mathcal{M}])_p \gg (\log m)^{(d-1)|1/p-1/2|}.$$

Let A be a full range sequence. Then there is an increasing sequence $\{\mu_l\}$ such that

$$a_{\mu_l, 2^{ld}} \geq 2C_2(d) l^{(d-1)} 2^{ld}. \tag{4.12}$$

We define

$$\mathcal{M} := \left(\cup_{l=1}^\infty E(\mu_l, l)\right) \cup \left(\{\mathbf{n}_j\}_{j=1}^\infty\right),$$

where $\{\mathbf{n}_j\}_{j=1}^\infty$ is such that

$$\operatorname{supp} H_{\mathbf{n}_j} \subset [1/2, 1)^d \quad \text{and} \quad \operatorname{supp} H_{\mathbf{n}_i} \cap \operatorname{supp} H_{\mathbf{n}_j} = \emptyset, \qquad i \neq j. \tag{4.13}$$

It is clear that $\{\mathbf{n}_j\}_{j=1}^\infty$ with the properties (4.13) can be chosen in a way that $\mathcal{M}$ will be of type A. This completes the proof of Theorem 4.1.

We note that the above argument implies even more.

Proposition 4.1. *Let $A = \{a_{\mu,M}\}$ be a full range sequence and $d \geq 2$, $1 < p < \infty$. Let $\{C(m, A, d, p), m \in \mathbb{N}\}$ be a sequence of reals such that for each $\Lambda = \{\lambda_{\mathbf{s}}\}$ of type A, $f \in L_p(\Lambda)$ and $m \in \mathbb{N}$*

$$\|f - G_m(f, \mathcal{H}_p^d)\|_p \leq C(m, A, p) \sigma_m(f, \mathcal{H}_p^d)_p.$$

Then

$$C(m, A, d, p) \asymp (\log m)^{(d-1)|1/2-1/p|}.$$

5. Some direct and inverse theorems in m-term approximation with regard to $\mathcal{H}_p^d$

In the case $d = 1$ the Haar basis is a greedy basis for L_p, $1 < p < \infty$. The following characterization theorem has been established in [T3] (for the case $p = 2$ see [St], [DT]). We will use the notation

$$a_n(f, p) := |c_{k_n}(f, \mathcal{H}_p^d)|$$

for the decreasing rearrangement of the coefficients of f.

Theorem 5.1. *Let $1 < p < \infty$ and $0 < q < \infty$. Then, for any positive r we have the equivalence relation*

$$\sum_{m=1}^{\infty} \sigma_m(f)_p^q m^{rq-1} < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} a_n(f,p)^q n^{rq-1+q/p} < \infty.$$

Let us remind the definition of the Lorentz spaces of sequences and introduce new spaces which provide finer (logarithmic) scale. Let for a sequence $\{x_k\}_{k=1}^{\infty}$ a sequence $\{x_{\rho(k)}\}_{k=1}^{\infty}$ be a decreasing rearrangement

$$|x_{\rho(1)}| \ge |x_{\rho(2)}| \ge \dots \quad .$$

For $r > 0$, $0 < q < \infty$ denote

$$\ell_q^r := \{\{x_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_{\rho(k)}|^q k^{rq-1} < \infty\}$$

or, equivalently,

$$\ell_q^r := \{\{x_k\}_{k=1}^{\infty} : \sum_{s=0}^{\infty} |x_{\rho(2^s)}|^q 2^{rqs} < \infty\}.$$

For $r > 0$, $b \in \mathbb{R}$, $0 < q < \infty$ define

$$\ell_q^{r,b} := \{\{x_k\}_{k=1}^{\infty} : \sum_{s=1}^{\infty} \big(|x_{\rho(2^s)}| 2^{rs} s^b\big)^q < \infty\}.$$

It is clear that $\ell_q^{r,0} = \ell_q^r$.

The proof of Theorem 5.1 was based on the following two lemmas.

Lemma 5.1. *For any two positive integers $N < M$ we have*

$$a_M(f,p) \le C(p)\sigma_N(f,\mathcal{H})_p (M-N)^{-1/p}.$$

Lemma 5.2. *For any sequence $m_0 < m_1 < \dots$ of nonnegative integers we have*

$$\sigma_{m_s}(f,\mathcal{H})_p \le C(p) \sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}.$$

We will prove in this section the following multivariate analogs of the above lemmas.

Lemma 5.3. *For any two positive integers $N < M$ we have*

$$a_M(f,p) \le C(p,d)\sigma_N(f,\mathcal{H}^d)_p (M-N)^{-1/p}, \quad 2 \le p < \infty;$$

$$a_M(f,p) \le C(p,d)\sigma_N(f,\mathcal{H}^d)_p (M-N)^{-1/p} (\log M)^{h(p,d)}, \quad 1 < p \le 2$$

with $h(p,d) := (d-1)|1/2 - 1/p|$.

Lemma 5.4. *For any sequence $m_0 < m_1 < \dots$ of non-negative integers we have*

$$\sigma_{m_s}(f,\mathcal{H}^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}(\log m_{i+1})^{h(p,d)}, \quad 2 \le p < \infty;$$

$$\sigma_{m_s}(f,\mathcal{H}^d)_p \le C(p,d)\sum_{i=s}^{\infty} a_{m_i}(f,p)(m_{i+1}-m_i)^{1/p}, \quad 1 < p \le 2.$$

Proof of Lemmas 5.3 and 5.4. These two lemmas follow from the well-known inequalities

$$C_1(p,d)\big(\sum_{\mathbf{n}}\|c_{\mathbf{n}}H_{\mathbf{n}}\|_p^{p_l}\big)^{1/p_l} \le \|\sum_{\mathbf{n}} c_{\mathbf{n}}H_{\mathbf{n}}\|_p \le C_2(p,d)\big(\sum_{\mathbf{n}}\|c_{\mathbf{n}}H_{\mathbf{n}}\|_p^{p_u}\big)^{1/p_u}$$

where $1 < p < \infty$ and $p_l := \max(2,p)$; $p_u := \min(2,p)$, and from Lemma 3.1.

Using Lemmas 5.3 and 5.4 one can establish the following embedding theorem in the same way as Theorem 5.1 was deduced from Lemmas 5.1 and 5.2 in [T3].

Theorem 5.2. *Let $1 < p < \infty$. Denote*

$$\sigma(f)_p := \{\sigma_m(f,\mathcal{H}^d)_p\}_{m=1}^{\infty} \quad \textit{and} \quad a(f,p) := \{a_n(f,p)\}_{n=1}^{\infty}.$$

Then we have the implications:

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b}, \quad 2 \le p < \infty; \tag{5.1}$$

$$\sigma(f)_p \in \ell_q^{r,b} \quad \Rightarrow \quad a(f,p) \in \ell_q^{r+1/p,b-h(p,d)}, \quad 1 < p \le 2; \tag{5.2}$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b-h(p,d)}, \quad 2 \le p < \infty; \tag{5.3}$$

$$a(f,p) \in \ell_q^{r+1/p,b} \quad \Rightarrow \quad \sigma(f)_p \in \ell_q^{r,b}, \quad 1 < p \le 2. \tag{5.4}$$

Let us discuss in more detail the implication (5.1). We want to understand what smoothness classes are natural for m-term approximation with regard to the basis $\mathcal{H}^d$ which is a tensor product of the univariate Haar basis $\mathcal{H}$. We consider the relation $a(f,p) \in \ell_q^{r+1/p,b}$ for a special choice of $b=0$ and $q = \xi := (r+1/p)^{-1}$. Then $a(f,p) \in \ell_q^{r+1/p}$ is equivalent to $\sum_n a_n(f,p)^{\xi} < \infty$ or

$$\sum_{\mathbf{n}}\|c_{\mathbf{n}}(f)H_{\mathbf{n}}\|_p^{\xi} < \infty \quad \text{where} \quad f = \sum_{\mathbf{n}} c_{\mathbf{n}}(f)H_{\mathbf{n}}. \tag{5.5}$$

Next, we have for $\mathbf{n} \in U_{\mathbf{s}}$

$$\|c_{\mathbf{n}}(f)H_{\mathbf{n}}\|_p = \|c_{\mathbf{n}}(f)H_{\mathbf{n}}\|_{\xi}2^{-|\mathbf{s}|(1/p-1/\xi)} = \|c_{\mathbf{n}}(f)H_{\mathbf{n}}\|_{\xi}2^{-r|\mathbf{s}|}.$$

Thus (5.5) is equivalent to

$$\sum_{\mathbf{s}}\big(2^{-r|\mathbf{s}|}\sum_{\mathbf{n}\in U_{\mathbf{s}}}\|c_{\mathbf{n}}(f)H_{\mathbf{n}}\|_{\xi}\big)^{\xi}<\infty. \tag{5.6}$$

The above relation is the same as to say that f belongs to the mixed smoothness Besov class $MB^r_\xi(L_\xi)$. Thus we conclude that the multivariate classes with mixed smoothness are natural for studying nonlinear m-term approxmation with regard to a basis which is a tensor product of univariate bases. There is an extensive literature in approximation theory on function classes with mixed smoothness. For the linear theory see [Te1], [Te2] and for some results in nonlinear m-term approximation see [T4] and [T6].

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