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MULTIGRID ALGORITHMS FOR C^0 INTERIOR PENALTY METHODS

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Abstract. Multigrid algorithms for C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains are studied in this paper. It is shown that V-cycle, F-cycle and W-cycle algorithms are contractions if the number of smoothing steps is sufficiently large. The contraction numbers of these algorithms are bounded by $Cm^{-\alpha}$, where m is the number of pre-smoothing (and post-smoothing) steps, α is the index of elliptic regularity, and the positive constant C is mesh-independent. These estimates are established for a smoothing scheme that uses a Poisson solve as a preconditioner, which can be easily implemented because the C^0 finite element spaces are standard spaces for second order problems. Furthermore the variable V-cycle algorithm is also shown to be an optimal preconditioner.

1. Introduction

C^0 interior penalty methods [29, 24] are nonconforming finite element methods for fourth order problems. Consider the following variational problem on a bounded polygonal domain in $\mathbb{R}^2$: Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = \int_\Omega fv\,dx \qquad \forall\, v \in H_0^2(\Omega) \tag{1.1}$$

where

$$a(w,v) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\,dx + \int_\Omega b(x)\nabla w \cdot \nabla v\,dx \tag{1.2}$$

and $f \in L_2(\Omega)$. The function $b(x)$ in (1.2) belongs to $C^1(\bar{\Omega})$ and is nonnegative on Ω. Since $\partial\Omega$ is not smooth, the solution u of (1.1) does not belong to $H^4(\Omega)$ even if $f \in C^\infty(\bar{\Omega})$ [31, 37]. In general the shift theorem [27, 4] only holds for f belonging to the Sobolev space $H^{-2+\alpha}(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$, i.e., $u \in H^{2+\alpha}(\Omega)$ whenever $f \in H^{-2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|f\|_{H^{-2+\alpha}(\Omega)}. \tag{1.3}$$

(We follow the standard notation of Sobolev spaces [1, 26, 23] in this paper.)

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When $b = 0$, the variational problem defined by (1.1) corresponds to the biharmonic problem. When $b > 0$, it is a scalar analog of the elliptic system that appears in strain-gradient elasticity theory [30, 41, 44]. Within the framework of finite element methods, it can be solved numerically by conforming C^1 finite elements [8, 2], nonconforming finite elements [36, 3, 40, 39] and mixed finite elements [25].

Let $\mathcal{T}_h$ be either a simplicial triangulation or a convex quadrilateral triangulation of Ω. In the C^0 interior penalty method approach, the discrete space V_h is either a P_ℓ ($\ell \geq 2$) triangular Lagrange finite element space [26, 23] or a Q_ℓ ($\ell \geq 2$) quadrilateral Lagrange tensor product finite element space [26, 23] associated with $\mathcal{T}_h$. The discrete problem for (1.1) is then given by: Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{1.4}$$

where

$$\begin{aligned}\mathcal{A}_h(w, v) = &\sum_{D\in\mathcal{T}_h} \int_D \Big(\sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} + b(x)\nabla w\cdot\nabla v\Big) dx \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \left(\left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] + \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right) ds \\ &+ \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds,\end{aligned} \tag{1.5}$$

$\mathcal{E}_h$ is the set of all the edges of $\mathcal{T}_h$, $|e|$ is the length of the edge e, and $\eta > 0$ is a penalty parameter. The averages $\{\!\{\cdot\}\!\}$ and jumps $[\![\cdot]\!]$ in (1.5) are defined as follows.

Let e be an interior edge of $\mathcal{T}_h$ and n_e be a unit vector normal to e. Then e is shared by two elements $D_\pm \in \mathcal{T}_h$, where n_e is pointing from D_- to D_+, and we define on e

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2}\right), \tag{1.6}$$

where $v_\pm = v\big|_{D_\pm}$. For an edge e on $\partial\Omega$, we take n_e to be the outer unit normal vector and define

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Note that the values of the averages and jumps are independent of the choice of n_e in (1.6).

The C^0 interior penalty method is consistent, and for η sufficiently large (which is assumed to be the case) it is also stable. Therefore the error $u - u_h$ is quasi-optimal with respect to appropriate norms [29, 24]. The C^0 interior penalty approach has certain advantages over other finite element methods: (i) It is much simpler than C^1 finite element methods. (ii) The lowest order C^0 interior penalty methods (i.e., those based on the P_2 or Q_2 elements) are as simple as the classical nonconforming finite element methods, but unlike such methods, the C^0 interior penalty methods come in a natural hierarchy of arbitrary orders. (iii) Unlike mixed finite element methods, it is straight-forward to construct C^0 interior penalty methods for more complicated elliptic systems. (iv) The fact that the finite element spaces in the C^0 interior penalty approach are just standard finite element spaces for second order problems can be exploited in the design of effective smoothers for multigrid algorithms.

In this paper we extend the multigrid theory for classical nonconforming finite elements (cf. [19, 22] and the references therein) to the C^0 interior penalty methods. We will prove the convergence of V-cycle, F-cycle and W-cycle algorithms when the number of smoothing steps is sufficiently large and also the optimality of the variable V-cycle algorithm as a preconditioner. In all these multigrid algorithms we use a preconditioned relaxation scheme that is much more effective than classical smoothers (such as the Richardson and the Gauss-Seidel iterations) and at the same time can be easily implemented because the finite element spaces of the C^0 interior penalty methods are the standard spaces for second order problems.

The rest of the paper is organized as follows. We set the notation and state the multigrid algorithms in Section 2, and then we introduce the mesh-dependent norms and establish some basic estimates in Section 3. The analysis of W-cycle and variable V-cycle algorithms is carried out in Section 4. The analysis of V-cycle and W-cycle algorithms relies on the additive theory developed in [20, 22], which is recalled in Section 5. The convergence results for V-cycle and F-cycle algorithms are then established in Section 6. In Section 7 we present the results of numerical experiments. Appendix A contains some properties of multigrid Poisson solves relevant for the convergence analysis.

For future reference we state here two elementary inequalities:

$$2ab \le \theta^2 a^2 + \theta^{-2} b^2 \qquad \text{for} \quad a, b \in \mathbb{R} \quad \text{and} \quad \theta \in (0,1), \tag{1.7}$$

$$(a+b)^2 \le (1+\theta^2) a^2 + (1+\theta^{-2}) b^2 \qquad \text{for} \quad a, b \in \mathbb{R} \quad \text{and} \quad \theta \in (0,1). \tag{1.8}$$

2. Multigrid Algorithms

In this section we describe the multigrid algorithms. In view of their potential for 3D problems, we will focus on C^0 interior penalty methods that are based on quadrilateral elements. Similar results can of course be obtained for triangular elements.

Let $\mathcal{T}_0$ be a triangulation of Ω by convex quadrilaterals and the triangulations of $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be obtained from $\mathcal{T}_0$ through uniform subdivisions. The mesh sizes $h_k = \max_{D \in \mathcal{T}_k} \operatorname{diam} D$ thus satisfy the relation

$$h_k \approx 2^{-k} h_0. \tag{2.1}$$

Remark 2.1. In order to avoid the proliferation of constants, we will use the notation $A \lesssim B$ ($B \gtrsim A$) to represent the relation $A \le \text{constant} \times B$, where the positive constant is mesh-independent, i.e., it is independent of the mesh size h_k and the grid level k. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Let $V_k \subset H_0^1(\Omega)$ be the Q_ℓ ($\ell \ge 2$) finite element space associated with $\mathcal{T}_k$ and denote by $\mathcal{A}_k$ the symmetric bilinear form on V_k corresponding to the variational form (1.5) of the C^0 interior penalty method. The k-th level discrete problem for the C^0 interior penalty method is: Find $u_k \in V_k$ such that

$$\mathcal{A}_k(u_k, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_k. \tag{2.2}$$

For η sufficiently large, the bilinear form $\mathcal{A}_k(\cdot,\cdot)$ is positive definite on V_k and we can define the discrete energy norm $\|\cdot\|_{\mathcal{A}_k}$ by

$$\|v\|_{\mathcal{A}_k} = \sqrt{\mathcal{A}_k(v,v)} \qquad \forall\, v \in V_k. \tag{2.3}$$

Note that $\mathcal{A}_k(\zeta_1,\zeta_2)$ is well-defined for $\zeta_1,\zeta_2 \in H^{2+\alpha}(\Omega)\cap H^2_0(\Omega)$, and in fact, $\mathcal{A}_k(\zeta_1,\zeta_2) = a(\zeta_1,\zeta_2)$ because $[\![\partial\zeta_j/\partial n]\!] = 0$. In particular, in view of (1.2) and the Poincaré-Friedrichs inequality [38],

$$\mathcal{A}_k(\zeta,\zeta) = a(\zeta,\zeta) \approx |\zeta|^2_{H^2(\Omega)} \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega)\cap H^2_0(\Omega). \tag{2.4}$$

However, $\mathcal{A}_k(\cdot,\cdot)$ is not positive definite on the space $V_k + [H^{2+\alpha}(\Omega)\cap H^2_0(\Omega)]$. Therefore it is necessary to introduce the following norm $\|\cdot\|_k$ for functions in $V_k + [H^{2+\alpha}(\Omega)\cap H^2_0(\Omega)]$:

$$\begin{aligned}\|w\|_k^2 = \sum_{D\in\mathcal{T}_k} \Big(|w|^2_{H^2(D)} + |w|^2_{H^1(D)}\Big) \\ + \sum_{e\in\mathcal{E}_k} \Big(|e|\, \|\{\!\{\partial^2 w/\partial n^2\}\!\}\|^2_{L_2(e)} + |e|^{-1}\|[\![\partial w/\partial n]\!]\|^2_{L_2(e)}\Big).\end{aligned} \tag{2.5}$$

From (2.5) it is easy to see that

$$|\mathcal{A}_k(w_1,w_2)| \lesssim \|w_1\|_k\|w_2\|_k \qquad \forall\, w_1,w_2 \in V_k + [H^{2+\alpha}(\Omega)\cap H^2_0(\Omega)]. \tag{2.6}$$

Furthermore, on V_k itself, we have (cf. (4.18), (4.20) and (4.25) of [24])

$$|v|_{H^2(\Omega,\mathcal{T}_k)} \le \|v\|_k \approx \|v\|_{\mathcal{A}_k} \lesssim |v|_{H^2(\Omega,\mathcal{T}_k)} \qquad \forall\, v \in V_k, \tag{2.7}$$

where

$$|v|^2_{H^2(\Omega,\mathcal{T}_k)} = \sum_{D\in\mathcal{T}_k} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_k} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}. \tag{2.8}$$

We deduce immediately from (2.7) and (2.8) that

$$\|v\|_{\mathcal{A}_k} \lesssim \|v\|_{\mathcal{A}_{k-1}} \qquad \forall\, v \in V_{k-1} (\subset V_k). \tag{2.9}$$

Let the operator $A_k : V_k \longrightarrow V_k'$ be defined by

$$\langle A_k v_1, v_2\rangle = \mathcal{A}_k(v_1,v_2) \qquad \forall\, v_1,v_2 \in V_k, \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual. We can then rewrite the discrete problem (2.2) as

$$A_k u_k = \phi_k,$$

where $\phi_k \in V_k'$ is defined by

$$\langle \phi_k, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_k.$$

Multigrid algorithms [32, 35, 10, 17, 43] are iterative methods for the solution of equations of the form

$$A_k z = \psi, \tag{2.11}$$

where $\psi \in V_k'$ and $z \in V_k$. In the descriptions of the multigrid algorithms below we will denote the natural injection from V_{k-1} to V_k by I_{k-1}^k and its transpose from V_k' to V_{k-1}' by I_k^{k-1}, i.e.,

$$\langle \phi, I_{k-1}^k v\rangle = \langle I_k^{k-1}\phi, v\rangle \qquad \forall\, \phi \in V_k' \quad \text{and} \quad v \in V_{k-1}. \tag{2.12}$$

We also need an operator $B_k : V_k \longrightarrow V_k'$ in the preconditioned relaxation scheme used in the smoothing steps of the multigrid algorithms (cf. (2.18) and (2.20) below). Let $L_k : V_k \longrightarrow V_k'$ be the discrete Laplace operator, i.e.,

$$\langle L_k v_1, v_2\rangle = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \qquad \forall\, v_1, v_2 \in V_k. \tag{2.13}$$

Since V_k is a standard finite element space for second order problems, it is natural to consider L_k^{-1} as a preconditioner for the fourth order discrete differential operator A_k. In order to maintain the optimal complexity of multigrid algorithms, we use instead an approximation B_k of L_k with the following properties:

(i) B_k is symmetric positive definite, i.e.,

$$\langle B_k v_1, v_2\rangle = \langle B_k v_2, v_1\rangle \qquad \forall\, v_1, v_2 \in V_k, \tag{2.14}$$

$$\langle B_k v, v\rangle > 0 \qquad \forall\, v \in V_k \setminus \{0\}. \tag{2.15}$$

(ii) B_k is spectrally equivalent to the discrete Laplace operator in the sense that

$$\langle L_k v, v\rangle \le \langle B_k v, v\rangle \lesssim \langle L_k v, v\rangle = \|\nabla v\|_{L_2(\Omega)}^2 \qquad \forall\, v \in V_k. \tag{2.16}$$

(iii) B_k approximates L_k in the sense that, for some $\beta \in (0, 1/2)$,

$$|v - B_k^{-1} L_k v|_{H^1(\Omega)} \lesssim h_k^\beta \|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_k. \tag{2.17}$$

(iv) The cost for computing $B_k^{-1} v$ is of order $O(n_k)$, where n_k is the dimension of V_k.

Remark 2.2. Let $B_k^{-1} : V_k' \longrightarrow V_k$ be the Poisson solve obtained by a symmetric V-cycle algorithm, a symmetric W-cycle algorithm or a symmetric variable V-cycle algorithm. Then B_k satisfies the properties (i), (ii) and (iv). If B_k^{-1} is the Poisson solve obtained by a symmetric W-cycle algorithm with a sufficiently large number of smoothing steps or a symmetric variable V-cycle algorithm, then the operator B_k also satisfies the property (iii). Details can be found in Appendix A.

Algorithm 2.3. (V-cycle Algorithm)

$MG_V(k, \psi, z_0, m)$ is the approximate solution of (2.11) with initial guess z_0 obtained as follows. If $k = 0$, we use a direct solve to obtain $A_0^{-1}\psi$ as the output of the V-cycle algorithm. If $k \ge 1$, we compute $MG_V(k, \psi, z_0, m)$ recursively in three steps.

Pre-smoothing For $1 \le j \le m$, compute z_j recursively by

$$z_j = z_{j-1} + \gamma_k B_k^{-1}(\psi - A_k z_{j-1}), \tag{2.18}$$

where γ_k^{-1} dominates the spectral radius of the operator $B_k^{-1}A_k : V_k \longrightarrow V_k$.

Coarse Grid Correction Compute

$$z_{m+1} = z_m + I_{k-1}^k MG_V(k-1, \varrho_{k-1}, 0, m), \tag{2.19}$$

where $\varrho_{k-1} = I_k^{k-1}(\psi - A_k z_m)$ is the transferred residual of z_m.

Post-smoothing For $m+2 \le j \le 2m+1$, compute z_j recursively by

$$z_j = z_{j-1} + \gamma_k B_k^{-1}(\psi - A_k z_{j-1}). \tag{2.20}$$

The final output of the V-cycle algorithm is

$$MG_V(k, \psi, z_0, m) = z_{2m+1}. \tag{2.21}$$

Algorithm 2.4. (W-cycle Algorithm)

If we replace the coarse grid correction step of Algorithm 2.3 by the following procedure, we have the W-cycle algorithm whose output will be denoted by $MG_W(k, \psi, z_0, m)$.

Coarse Grid Correction for the W-cycle Compute $e_1, e_2 \in V_{k-1}$ by

$$e_j = MG_W(k-1, \varrho_{k-1}, e_{j-1}, m) \qquad \text{for} \quad 1 \le j \le 2, \tag{2.22}$$

where $e_0 = 0$, and set

$$z_{m+1} = z_m + I_{k-1}^k e_2. \tag{2.23}$$

Algorithm 2.5. (F-cycle Algorithm)

If we replace the coarse grid correction step of Algorithm 2.3 by the following procedure, we have the F-cycle algorithm whose output will be denoted by $MG_F(k, \psi, z_0, m)$.

Coarse Grid Correction for the F-cycle Let $e_0 = 0 \in V_{k-1}$. Compute $e_1, e_2 \in V_{k-1}$ by

$$\begin{aligned} e_1 &= MG_F(k-1, \varrho_{k-1}, e_0, m), \\ e_2 &= MG_V(k-1, \varrho_{k-1}, e_1, m), \end{aligned}$$

and set z_{m+1} by (2.23).

Algorithm 2.6. (Variable V-cycle Algorithm)

If the numbers of smoothing steps in Algorithm 2.3 on different levels are allowed to be different, we have a variable V-cycle algorithm.

3. Mesh-Dependent Norms and Preliminary Estimates

In this section we introduce mesh-dependent norms and derive some preliminary estimates. First of all, because of (2.14) and (2.15), we can introduce a discrete inner product [5] related to the preconditioner in the smoothing steps:

$$(v_1, v_2)_k = \langle B_k v_1, v_2 \rangle \qquad \forall\, v_1, v_2 \in V_k. \tag{3.1}$$

It follows from (2.10) and (3.1) that the operator $\mathbb{A}_k = B_k^{-1} A_k : V_k \longrightarrow V_k$ satisfies

$$(\mathbb{A}_k v_1, v_2)_k = \mathcal{A}_k(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_k. \tag{3.2}$$

It is clear from (1.2) and (3.2) that $\mathbb{A}_k$ is symmetric positive definite with respect to the inner product $(\cdot, \cdot)_k$. Furthermore, it follows from (2.3), (2.8), (2.7), (2.16) and standard inverse estimates [26, 23] that the spectral radius $\rho(\mathbb{A}_k)$ of $\mathbb{A}_k$ satisfies

$$\rho(\mathbb{A}_k) \lesssim h_k^{-2}. \tag{3.3}$$

Therefore we can take the parameter γ_k in (2.18) and (2.20) to be Ch_k^2 $(\geq \rho(\mathbb{A}_k))$, where the positive constant C is mesh-independent.

Remark 3.1. In terms of the inner product $(\cdot, \cdot)_k$ the smoothing steps in (2.18) and (2.20) are just Richardson relaxation steps.

Remark 3.2. Using (3.3) it is not difficult to show that the condition number of $\mathbb{A}_k$ (in the energy norm) is of order $O(h_k^{-2})$. On the other hand the condition number of the fourth order discrete differential operator A_k (with respect to the natural nodal basis) is of order $O(h_k^{-4})$. The reduction in the order of the condition number of $\mathbb{A}_k$ greatly improves the performance of the multigrid algorithms (cf. Remark 4.2, and Table 1 and Table 4 in Section 7).

For $s \in \mathbb{R}$, we define the mesh-dependent norm $|\!|\!| \cdot |\!|\!|_{s,k}$ by

$$|\!|\!| v |\!|\!|_{s,k} = (\mathbb{A}_k^s v, v)_k^{1/2} \qquad \forall\, v \in V_k. \tag{3.4}$$

It is clear from (2.3), (2.16), (3.1), (3.2) and (3.4) that

$$|\!|\!| v |\!|\!|_{0,k} = \sqrt{(v, v)_k} = \sqrt{\langle B_k v, v \rangle} \approx |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k, \tag{3.5}$$

$$|\!|\!| v |\!|\!|_{1,k} = \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_k. \tag{3.6}$$

The following well-known properties [6] of mesh-dependent norms follow immediately from (3.2)–(3.4) and the Cauchy-Schwarz inequality:

$$|\!|\!| v |\!|\!|_{s,k} \lesssim h_k^{t-s} |\!|\!| v |\!|\!|_{t,k} \qquad \forall\, v \in V_k \quad \text{and} \quad 0 \leq t \leq s \leq 2, \tag{3.7}$$

$$|\!|\!| v |\!|\!|_{1+s,k} = \sup_{w \in V_k \setminus 0} \frac{\mathcal{A}_k(v, w)}{|\!|\!| w |\!|\!|_{1-s,k}} \qquad \forall\, v \in V_k \quad \text{and} \quad s \in \mathbb{R}. \tag{3.8}$$

Our convergence analysis in subsequent sections relies on the elliptic regularity estimate (1.3). Therefore a relation between the Sobolev norms and the mesh-dependent norms is crucial. For conforming methods such a relation is easy to derive. However, since $V_k \not\subset H_0^2(\Omega)$, additional work is required here.

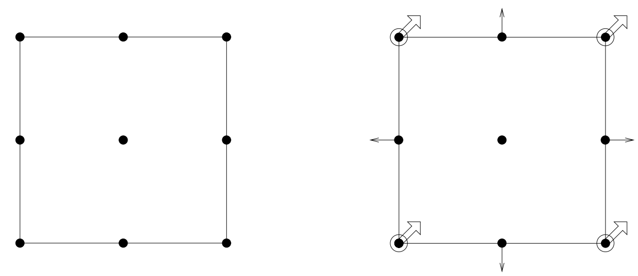

FIGURE 1. Q_2 Lagrange C^0 element and Q_4 Bogner-Fox-Schmit C^1 element

The key ingredient for building a link between Sobolev norms and mesh-dependent norms is the existence [24] of a C^1 finite element which is a *relative* [18, 19] of the Q_ℓ element in the sense that (i) the shape functions of the Q_ℓ element are also shape functions of the C^1 element and (ii) the nodal variables (degrees of freedom) of the Q_ℓ element are also nodal variables of the C^1 element.

Remark 3.3. For example a conforming relative (cf. Figure 1) for the rectangular Q_2 Lagrange element is the Q_4 element in the Bogner-Fox-Schmit family [8] whose nodal variables (degrees of freedom) are (i) the evaluations of the shape functions (denoted by the dot $\bullet$) at the nine nodes of the Q_2 Lagrange element, (ii) the evaluations of the normal derivatives of the shape functions (denoted by the arrow $\uparrow$) at the midpoints of the four edges, (iii) the evaluations of the gradients (denoted by the circle $\circ$) at the four corners, and (iv) the evaluations of the mixed derivatives (denoted by the tilted double arrow $\nearrow$) at the four corners.

Let $\tilde{V}_k \subset H_0^2(\Omega)$ be the finite element space defined by the C^1 element. Then we can construct a linear map $E_k : V_k \longrightarrow \tilde{V}_k$ by averaging such that the following properties hold:

$$\Pi_k E_k v = v \qquad \forall\, v \in V_k, \tag{3.9}$$

$$\|E_k v\|_{H^2(\Omega)} \lesssim \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_k, \tag{3.10}$$

$$\|E_k v\|_{H^{1+s}(\Omega)} \approx \|v\|_{H^{1+s}(\Omega)} \qquad \forall\, v \in V_k,\ 0 \le s < \frac{1}{2}, \tag{3.11}$$

where $\Pi_k : C^0(\bar{\Omega}) \longrightarrow V_k$ is the nodal interpolation operator.

Remark 3.4. The relation (3.9) and the estimate (3.10) can be found in [24] ((3.30) and Lemma 3). The estimate (3.11) can be proved by the arguments in Lemma 9 of [24], where the special case $s = 1 - \alpha$ is established.

Note also that the following estimates (cf. (3.16) and (3.18) of [24]) hold for Π_k:

$$\|\Pi_k \zeta\|_{\mathcal{A}_k} \lesssim \|\zeta\|_{H^2(\Omega)} \qquad \forall\, \zeta \in H_0^2(\Omega), \tag{3.12}$$

$$|\zeta - \Pi_k \zeta|_{H^1(\Omega)} \lesssim h_k \|\zeta\|_{H^2(\Omega)} \qquad \forall\, v \in H_0^2(\Omega), \tag{3.13}$$

$$\|\zeta - \Pi_k \zeta\|_k \lesssim h_k^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega), \tag{3.14}$$

where $\|\cdot\|_k$ is the norm defined in (2.5). Furthermore, because the finite elements are relatives, the following discrete estimate is a consequence of the equivalence of norms on finite dimensional vector spaces:

$$|\Pi_k\tilde{v}|_{H^1(\Omega)} \lesssim |\tilde{v}|_{H^1(\Omega)} \qquad \forall\, v \in \tilde{V}_k. \tag{3.15}$$

The following lemma gives useful two-level estimates for the nodal interpolation operator.

Lemma 3.5. *The following estimates hold for the nodal interpolation operator*:

$$\|\Pi_k\zeta - \Pi_{k-1}\zeta\|_{\mathcal{A}_k} \lesssim h_k^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega), \tag{3.16}$$

$$|\zeta - \Pi_{k-1}\Pi_k\zeta|_{H^2(\Omega,\mathcal{T}_{k-1})} \lesssim h_k^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)} \qquad \forall\, \zeta \in H^{2+\alpha}(\Omega), \tag{3.17}$$

$$\|\Pi_{k-1}v\|_{\mathcal{A}_{k-1}} \lesssim \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_k, \tag{3.18}$$

$$|v - \Pi_{k-1}v|_{H^1(\Omega)} \lesssim h_k \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_k, \tag{3.19}$$

$$\|v - \Pi_{k-1}v\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^\alpha \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_k. \tag{3.20}$$

Proof. We obtain from (2.7), (2.8) and scaling

$$\|\Pi_k\zeta - \Pi_{k-1}\zeta\|^2_{\mathcal{A}_k} \lesssim \sum_{D\in\mathcal{T}_k} \left(|\Pi_k\zeta - \Pi_{k-1}\zeta|^2_{H^2(D)} + (\operatorname{diam} D)^{-2}|\Pi_k\zeta - \Pi_{k-1}\zeta|^2_{H^1(D)}\right). \tag{3.21}$$

Since $(\Pi_k - \Pi_{k-1})p = 0$ on D for any quadratic polynomial p defined on the subdomain of $\mathcal{T}_{k-1}$ that contains $D \in \mathcal{T}_k$, the estimate (3.16) follows from (3.21), the Bramble-Hilbert lemma [11, 28] and the standard estimate [26, 23]

$$|\Pi_k\zeta|_{H^2(D)} \lesssim |\zeta|_{H^2(D)} \qquad \forall\, \zeta \in H^2_0(\Omega),\ D \in \mathcal{T}_k.$$

The proof of (3.17) is similar.

In view of (2.7) and (2.8), the estimate (3.19) is a consequence of

$$|v - \Pi_{k-1}v|^2_{H^1(D)} \lesssim (\operatorname{diam} D)^2\Bigg(\sum_{\substack{D'\in\mathcal{T}_k\\ D'\subset D}} |v|^2_{H^2(D')} + \sum_{\substack{e\subset\mathcal{E}_k\\ e\subset D}} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\Bigg) \tag{3.22}$$

for all $v \in V_k$ and $D \in \mathcal{T}_{k-1}$. Since the quadrilaterals in $\mathcal{T}_k$ for $k \geq 0$ are shape regular, we can establish (3.22) by proving the following estimate on the reference square $\hat{D}$:

$$|\hat{v} - \hat{v}^I|^2_{H^1(\hat{D})} \lesssim \sum_{j=1}^4 |\hat{v}|^2_{H^2(\hat{D}_j)} + \sum_{j=1}^4 \|[\![\partial\hat{v}/\partial n]\!]\|^2_{L_2(\hat{e}_j)} \qquad \forall\, v \in \hat{V}, \tag{3.23}$$

where $\hat{V} \subset H^1(\hat{D})$ is the (finite dimensional) space of continuous functions whose members belong to the polynomial space $Q_\ell(\hat{D}_j)$ for each of the four sub-squares $\hat{D}_j$ (cf. Figure 2), $\hat{v}^I \in Q_\ell(\hat{D})$ agrees with $\hat{v}$ at the nodes of the Q_ℓ element on $\hat{D}$, and $\hat{e}_j$ for $1 \leq j \leq 4$ are interfaces of the sub-squares. Now the estimate (3.23) follows from the observation that the square root of the right-hand side of (3.23) defines a norm on the quotient space $\hat{V}/P_1(\hat{D})$ while the square root of the left-hand side defines a semi-norm on $\hat{V}/P_1(\hat{D})$.

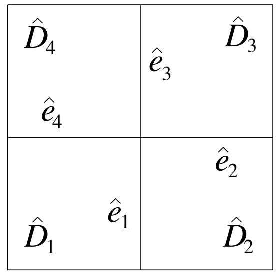

FIGURE 2. A subdivided referenced square $\hat{D}$

The estimate (3.20) follows from (3.19) and the inverse estimate [7]

$$|v|_{H^{1+s}(\Omega)} \lesssim h_k^{-s}|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k, \tag{3.24}$$

where $0 < s < 1/2$.

Finally we derive (3.18) using (2.7), (2.8), (3.22), a trace theorem (with scaling) and a standard inverse estimate:

$$\begin{aligned}
\|\Pi_{k-1}v\|^2_{\mathcal{A}_{k-1}} &\lesssim \sum_{D\in\mathcal{T}_{k-1}} |\Pi_{k-1}v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_k} |e|^{-1}\|[\![\partial(\Pi_{k-1}v)/\partial n]\!]\|^2_{L_2(e)} \\
&\lesssim \sum_{D\in\mathcal{T}_k} |v-\Pi_{k-1}v|^2_{H^2(D)} + \sum_{D\in\mathcal{T}_k} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_k} |e|^{-1}\|[\![\partial(v-\Pi_{k-1}v)/\partial n]\!]\|^2_{L_2(e)} \\
&\qquad + \sum_{e\in\mathcal{E}_k} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)} \\
&\lesssim \sum_{D\in\mathcal{T}_k} (\operatorname{diam} D)^{-2}|v-\Pi_{k-1}v|^2_{H^1(D)} + \|v\|^2_{\mathcal{A}_k} \lesssim \|v\|^2_{\mathcal{A}_k} \qquad \forall\, v\in V_k.
\end{aligned}$$

□

In the other direction we can also construct a map from the Sobolev spaces into V_k.

Lemma 3.6. *There exists a linear map $J_k : L_2(\Omega) \longrightarrow V_k$ with the following properties*:

$$J_kE_kv = v \qquad \forall\, v \in V_k, \tag{3.25}$$

$$\|J_kv\|_{\mathcal{A}_k} \lesssim \|v\|_{H^2(\Omega)} \qquad \forall\, v \in H^2_0(\Omega), \tag{3.26}$$

$$|J_kv|_{H^1(\Omega)} \lesssim |v|_{H^1(\Omega)} \qquad \forall\, v \in H^1_0(\Omega). \tag{3.27}$$

Proof. We define J_k by

$$J_kv = \Pi_kQ_kv \qquad \forall\, v \in L_2(\Omega), \tag{3.28}$$

where $Q_k : L_2(\Omega) \longrightarrow \tilde{V}_k$ is the L_2 orthogonal projection operator. The relation (3.25) is an obvious consequence of (3.9).

Regarding Q_k we have the estimates [16]

$$\|Q_kv\|_{H^2(\Omega)} \lesssim \|v\|_{H^2(\Omega)} \qquad \forall\, v \in H^2_0(\Omega), \tag{3.29}$$

$$\|Q_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)} \qquad \forall\, v \in H_0^1(\Omega). \tag{3.30}$$

The estimates (3.26) and (3.27) follow immediately from (3.12), (3.15) and (3.28)–(3.30). □

Lemma 3.7. *It holds that*

$$|\!|\!| v |\!|\!|_{s,k} \approx \|E_k v\|_{H^{1+s}(\Omega)} \qquad \forall\, v \in V_k \tag{3.31}$$

provided $0 \le s \le 1$ *and* $s \ne 1/2$.

Proof. From (3.5), (3.6), (3.10) and (3.11) we have

$$\|E_k v\|_{H^2(\Omega)} \lesssim |\!|\!| v |\!|\!|_{1,k} \qquad \forall\, v \in V_k,$$
$$\|E_k v\|_{H^1(\Omega)} \lesssim |\!|\!| v |\!|\!|_{0,k} \qquad \forall\, v \in V_k,$$

which implies, by operator interpolation theory for Hilbert scales [42, 33, 10],

$$\|E_k v\|_{H^{1+s}(\Omega)} \lesssim |\!|\!| v |\!|\!|_{s,k} \qquad \forall\, v \in V_k. \tag{3.32}$$

On the other hand, from (3.5), (3.6), (3.26), (3.27) and interpolation, we have

$$|\!|\!| J_k v |\!|\!|_{s,k} \lesssim \|v\|_{H^{1+s}(\Omega)} \qquad \forall\, v \in H_0^{1+s}(\Omega), \tag{3.33}$$

which together with (3.25) implies

$$|\!|\!| v |\!|\!|_{s,k} = |\!|\!| J_k E_k v |\!|\!|_{1+s,k} \lesssim \|E_k v\|_{H^{1+s}(\Omega)} \qquad \forall\, v \in V_k. \tag{3.34}$$

□

Remark 3.8. The norm equivalence (3.31) is also valid for $s = 1/2$ provided the norm on the right-hand side is replaced by the norm $\|\cdot\|_{H_{00}^{1+s}(\Omega)}$ (cf. [34, 42]).

From (3.11) and (3.31) we immediately obtain the following corollary which provides the link between mesh-dependent norms and Sobolev norms.

Corollary 3.9. *It holds that*

$$|\!|\!| v |\!|\!|_{s,k} \approx \|v\|_{H^{1+s}(\Omega)} \qquad \forall\, v \in V_k, \tag{3.35}$$

provided $0 \le s < 1/2$.

Let J_k^* be the adjoint of J_k with respect to the bilinear form $a(\cdot,\cdot)$ for the continuous problem and the bilinear form $\mathcal{A}_k(\cdot,\cdot)$ for the discrete problem, i.e., $J_k^* : V_k \longrightarrow H_0^2(\Omega)$ satisfies

$$a(J_k^* v, w) = \mathcal{A}_k(v, J_k w) \qquad \forall\, v \in V_k,\ w \in H_0^2(\Omega). \tag{3.36}$$

The following lemma on J_k^* will be useful in the convergence analysis of V-cycle and F-cycle algorithms.

Lemma 3.10. *Let* $\zeta_k \in V_k$ *and*

$$\phi(v) = \mathcal{A}_k(\zeta_k, J_k v) \qquad \forall\, v \in H_0^2(\Omega). \tag{3.37}$$

Then $\phi \in H^{-2+\alpha}(\Omega)$,

$$\|\phi\|_{H^{-2+\alpha}(\Omega)} \lesssim |\!|\!| \zeta_k |\!|\!|_{1+\alpha,k}, \tag{3.38}$$

and

$$\mathcal{A}_k(\zeta_k, v) = \phi(E_k v) \qquad \forall\, v \in V_k. \tag{3.39}$$

Furthermore, $\zeta = J_k^* \zeta_k \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$,

$$a(\zeta, w) = \phi(w) \qquad \forall\, w \in H_0^2(\Omega), \tag{3.40}$$

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim |\!|\!|\zeta_k|\!|\!|_{1+\alpha,k}, \tag{3.41}$$

and the following estimates hold:

$$\|\zeta - \zeta_k\|_k \lesssim h_k^\alpha |\!|\!|\zeta_k|\!|\!|_{1+\alpha,k}, \tag{3.42}$$

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\!|\!|\zeta|\!|\!|_{1+\alpha,k}. \tag{3.43}$$

Proof. From (3.8), (3.33) and (3.37) we have

$$\phi(v) \le |\!|\!|\zeta_k|\!|\!|_{1+\alpha,k} |\!|\!|J_k v|\!|\!|_{1-\alpha,k} \lesssim |\!|\!|\zeta_k|\!|\!|_{1+\alpha,k} \|v\|_{H^{2-\alpha}(\Omega)}, \tag{3.44}$$

which means that $\phi \in H^{-2+\alpha}(\Omega)$ and (3.38) is valid.

Equation (3.40) follows immediately from (3.36) and (3.37). Then $J_k^* \zeta_k \in H^{2+\alpha}(\Omega)$ by elliptic regularity and (3.41) follows from (1.3) and (3.38).

Finally (3.25) and (3.37) imply (3.39). Therefore ζ_k is the solution of a modified C^0 interior penalty method for (3.40) studied in [24] and the error estimates (3.42) and (3.43) follow from Theorem 4 and Theorem 6 of [24]. □

4. Results for W-Cycle and Variable V-Cycle Algorithms

In this section we establish the results for W-cycle and V-cycle algorithms. There are two ingredients in the analysis: the *smoothing property* and the *approximation property*.

The effect of one smoothing step in (2.18) and (2.20) is measured by the operator

$$R_k = Id_k - \gamma_k \mathbb{A}_k, \tag{4.1}$$

where Id_k is the identity operator on V_k. The proof of the following result which controls the effect of the smoothing steps can be found for example in [32, 23].

Lemma 4.1. *It holds that*

$$|\!|\!|R_k^m v|\!|\!|_{s,k} \lesssim h_k^{t-s} m^{(t-s)/2} |\!|\!|v|\!|\!|_{t,k} \qquad \forall\, v \in V_k \quad \textit{and} \quad 0 \le t \le s \le 2.$$

Remark 4.2. Without the preconditioner B_k^{-1}, the smoothing property becomes (for appropriately defined mesh-dependent norms)

$$|\!|\!|R_k^m v|\!|\!|_{s,k} \lesssim h_k^{t-s} m^{(t-s)/4} |\!|\!|v|\!|\!|_{t,k}.$$

In other words, the effect of m smoothing steps without preconditioning is equivalent to the smoothing effect of $\sqrt{m}$ many smoothing steps with preconditioning. Therefore the preconditioner greatly enhances the performance of the multigrid algorithms (cf. Table 1 and Table 4 in Section 7).

To control the effect of coarse grid correction, we first recall the following well-known relation between the exact solution of the coarse grid residual equation $A_{k-1}\hat{e}_{k-1} = \varrho_{k-1}$ and the error $z - z_m$:

$$\hat{e}_{k-1} = P_k^{k-1}(z - z_m),$$

where the operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ defined by

$$\mathcal{A}_{k-1}(P_k^{k-1}v, w) = \mathcal{A}_k(v, I_{k-1}^k w) \qquad \forall\, v \in V_{k-1},\ w \in V_k. \tag{4.2}$$

The approximation property in the following result then controls the effect of coarse grid correction.

Lemma 4.3. *It holds that*

$$|\!|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!|\!|_{1-\alpha,k} \lesssim h_k^{2\alpha} |\!|\!|v|\!|\!|_{1+\alpha,k} \qquad \forall\, v \in V_k, \tag{4.3}$$

where α is the index of elliptic regularity in (1.3).

Proof. Let $v \in V_k$ be arbitrary. We will establish (4.3) by a duality argument. Using the norm equivalence in Corollary 3.9 (with $s = 1 - \alpha$) and duality, we find

$$\begin{aligned} |\!|\!|(Id_k - I_{k-1}^k P_k^{k-1})v|\!|\!|_{1-\alpha,k} &\approx \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{H^{2-\alpha}(\Omega)} \\ &= \sup_{\phi \in H^{-2+\alpha}(\Omega)\setminus\{0\}} \frac{\phi((Id_k - I_{k-1}^k P_k^{k-1})v)}{\|\phi\|_{H^{-2+\alpha}(\Omega)}}. \end{aligned} \tag{4.4}$$

Let $\phi \in H^{-2+\alpha}(\Omega)$ be arbitrary and define $\zeta \in H_0^2(\Omega)$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$a(\zeta, v) = \phi(v) \qquad \forall\, v \in H_0^2(\Omega), \tag{4.5}$$

$$\mathcal{A}_k(\zeta_k, v) = \phi(v) \qquad \forall\, v \in V_k, \tag{4.6}$$

$$\mathcal{A}_{k-1}(\zeta_{k-1}, v) = \phi(v) \qquad \forall\, v \in V_{k-1}. \tag{4.7}$$

In other words, ζ_k and ζ_{k-1} are the approximations of ζ obtained by the C^0 interior penalty method, and the following error estimates (cf. Theorem 5 of [24]) are valid:

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} \|\phi\|_{H^{-2+\alpha}(\Omega)}, \tag{4.8}$$

$$\|\zeta - \zeta_{k-1}\|_{H^{2-\alpha}(\Omega)} \lesssim h_{k-1}^{2\alpha} \|\phi\|_{H^{-2+\alpha}(\Omega)}. \tag{4.9}$$

From (4.6) and (4.7) we have

$$\mathcal{A}_{k-1}(\zeta_{k-1}, v) = \mathcal{A}_k(\zeta_k, I_{k-1}^k v) \qquad \forall\, v \in V_{k-1},$$

which implies (cf. (4.2))

$$\zeta_{k-1} = P_k^{k-1}\zeta_k. \tag{4.10}$$

We can now estimate the numerator in (4.4) by (2.1), (3.8), (4.2), (4.6) and (4.8)–(4.10) as follows:

$$\begin{aligned} \phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_k(\zeta_k, I_{k-1}^k P_k^{k-1}v) \\ &= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1}v) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= \mathcal{A}_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\leq |\!|\!| \zeta_k - \zeta_{k-1} |\!|\!|_{1-\alpha,k} |\!|\!| v |\!|\!|_{1+\alpha,k} \\
&\lesssim \|\zeta_k - \zeta_{k-1}\|_{H^{2-\alpha}(\Omega)} |\!|\!| v |\!|\!|_{1+\alpha,k} \\
&\leq \big(\|\zeta_k - \zeta\|_{H^{2-\alpha}(\Omega)} + \|\zeta - \zeta_{k-1}\|_{H^{2-\alpha}(\Omega)} \big) |\!|\!| v |\!|\!|_{1+\alpha,k} \\
&\lesssim h_k^{2\alpha} \|\phi\|_{H^{-2+\alpha}(\Omega)} |\!|\!| v |\!|\!|_{1+\alpha,k}.
\end{aligned} \tag{4.11}
$$

The estimate (4.3) follows from (4.4) and (4.11). □

The following corollary is immediate from (3.7) and (4.3).

Corollary 4.4. *It holds that*

$$
|\!|\!| (Id_k - I_{k-1}^k P_k^{k-1}) v |\!|\!|_{1-\alpha,k} \lesssim h_k^\alpha |\!|\!| v |\!|\!|_{1,k} \qquad \forall\, v \in V_k. \tag{4.12}
$$

We can now apply the theory developed in [19] (Theorem 4.3, Theorem 4.4, Lemma 4.7 and Theorem 4.8, where the results in [15] for the variable V-cycle is used) to derive the following results for W-cycle and V-cycle algorithms.

Theorem 4.5. *The output $MG_W(k, \psi, z_0, m)$ of the W-cycle algorithm (Algorithm 2.4) applied to (2.11) satisfies the following estimate:*

$$
\|z - MG_W(k, \psi, z_0, m)\|_{\mathcal{A}_k} \leq \frac{C}{m^\alpha} \|z - z_0\|_{\mathcal{A}_k},
$$

where the positive constant C is mesh-independent, provided that the number of smoothing steps m is greater than a positive integer m_* that is also mesh-independent.

Theorem 4.6. *The variable V-cycle algorithm (Algorithm 2.6) is an optimal preconditioner provided the following relation is satisfied by m_k (the number of smoothing steps on level k) :*

$$
\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k, \tag{4.13}
$$

where $1 < \beta_0 \leq \beta_1$.

Remark 4.7. Theorems 4.5 and 4.6 have been obtained for preconditioners that satisfy (2.14)–(2.16). Therefore they are valid for B_k^{-1} obtained by a symmetric V-cycle algorithm, a symmetric W-cycle algorithm or a variable V-cycle algorithm (cf. Remark 2.2 and Appendix A).

Finally we note that (3.7), Corollary 3.9 and Corollary 4.4 imply

$$
\begin{aligned}
|\!|\!| P_k^{k-1} v |\!|\!|_{1-\alpha,k} &\approx \|P_k^{k-1} v\|_{H^{2-\alpha}(\Omega)} \\
&\leq \|v\|_{H^{2-\alpha}(\Omega)} + \|v - P_k^{k-1} v\|_{H^{2-\alpha}(\Omega)} \\
&\lesssim |\!|\!| v |\!|\!|_{1-\alpha,k} + h_k^\alpha |\!|\!| v |\!|\!|_{1,k} \lesssim |\!|\!| v |\!|\!|_{1-\alpha,k} \qquad \forall\, v \in V_k.
\end{aligned} \tag{4.14}
$$

The estimate (4.14) will be used in the convergence analysis of V-cycle and F-cycle algorithms.

5. Additive Multigrid Theory

In this section we briefly review the additive multigrid theory [20, 22] which will be used in the convergence analysis of V-cycle and F-cycle algorithms in Section 6.

Let $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$ be the error propagation operator for the k-th level V-cycle algorithm, i.e., $z - MG_V(k, \psi, z_0, m) = \mathbb{E}_{k,m}(z - z_0)$, where $MG_V(k, \psi, z_0, m)$ is the approximate solution of (2.11) obtained by the V-cycle algorithm. The operators $\mathbb{E}_k$ satisfy the well-known recurrence relation [32, 35]

$$\mathbb{E}_{k,m} = R_k^m (Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k \mathbb{E}_{k-1,m} P_k^{k-1}) R_k^m \tag{5.1}$$

and the initial condition $\mathbb{E}_k = 0$. Iterating (5.1) leads to the following additive expression [20, 22] for $\mathbb{E}_k$:

$$\mathbb{E}_{k,m} = \sum_{j=2}^{k} T_{k,j,m} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m T_{j,k,m}, \tag{5.2}$$

where (for $j < k$) $T_{k,j,m}$ is the multilevel operator $R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1}$ from V_j into V_k, $T_{j,k,m} = P_{j+1}^j R_j^m \cdots P_k^{k-1} R_k^m$ is the adjoint operator of $T_{k,j,m}$ with respect to $\mathcal{A}_k(\cdot, \cdot)$, and $T_{k,k,m} = Id_k$.

A convergence theory for the V-cycle algorithm was developed in [20, 21] for second order problems. It yields the asymptotic behavior of the contraction numbers, which when combined with the results from the multiplicative theory [14, 45, 12, 13] provides a complete generalization of the classical result of Braess and Hackbusch [9] to the case of less than full elliptic regularity. This additive theory was extended to V-cycle and F-cycle algorithms for classical nonconforming finite elements in [22, 46, 47].

Note the operator $R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m$ that appears in (5.2) is already controlled by the smoothing property (Lemma 4.1) and the approximation property (Lemma 4.3). Therefore the key in the additive approach is to control the multilevel operators $T_{k,j,m}$ and $T_{j,k,m}$. This in turn requires a careful comparison of the mesh-dependent norms on consecutive levels. In this regard the following assumptions of the additive theory [20, 22] need to be verified:

$$\|I_{k-1}^k v\|_{1,k}^2 \le (1 + \theta^2) \|v\|_{1,k-1}^2 + C_1 \theta^{-2} h_k^{2\mu} \|v\|_{1+\mu,k-1}^2 \qquad \forall\, v \in V_{k-1}, \tag{5.3}$$

$$\|I_{k-1}^k v\|_{1-\tau,k}^2 \le (1 + \theta^2) \|v\|_{1-\tau,k-1}^2 + C_2 \theta^{-2} h_k^{2\tau} \|v\|_{1,k-1}^2 \qquad \forall\, v \in V_{k-1}, \tag{5.4}$$

$$\|P_k^{k-1} v\|_{1-\tau,k}^2 \le (1 + \theta^2) \|v\|_{1-\tau,k}^2 + C_3 \theta^{-2} h_k^{2\tau} \|v\|_{1,k}^2 \qquad \forall\, v \in V_k, \tag{5.5}$$

where $\theta \in (0, 1)$ is arbitrary, μ and τ are two parameters strictly between 0 and 1, and the positive constants C_1, C_2 and C_3 are independent of the meshes and θ.

Furthermore, we also need the following approximation property which is particular to nonconforming methods where the energy norm is not preserved by the coarse-to-fine intergrid transfer operator I_{k-1}^k:

$$\|(Id_{k-1} - P_k^{k-1} I_{k-1}^k) v\|_{1-\mu,k-1} \lesssim h_k^\mu \|v\|_{1,k-1} \qquad \forall\, v \in V_{k-1}. \tag{5.6}$$

Remark 5.1. The estimates (5.3) and (5.6) together imply that (cf. Lemma 4.2 of [22]), for $j \leq k$,

$$\|T_{k,j,m}v\|_{1,k} \lesssim \|v\|_{1,j} \qquad \forall\, v \in V_j, \tag{5.7}$$

provided that m is sufficiently large. We can then use (5.4), (5.5) and (5.7) to derive (cf. Lemmas 4.4–4.6 of [22]), for $j \leq k$, the following crucial estimate in the additive theory:

$$\|T_{j,k,m}T_{k,j,m}v\|_{1-\tau,j} \lesssim \|v\|_{1-\tau,j} \qquad \forall\, v \in V_j, \tag{5.8}$$

provided m is sufficiently large. The convergence of V-cycle algorithm for sufficiently large m follows from (5.8) and an argument based on a strengthened Cauchy-Schwarz inequality. The convergence of the F-cycle algorithm can then be established by a perturbation argument.

Therefore the heart of our convergence analysis of V-cycle and F-cycle algorithms is the derivation of the estimates (5.3)–(5.6), where there is a lot of freedom in choosing the parameters μ and τ.

We will prove the estimate (5.6) for $\mu = \alpha$ (the index of elliptic regularity in (1.3)) in this section and take up the estimates (5.3)–(5.5) in Section 6. The following lemma is a stronger version of (5.6).

Lemma 5.2. *It holds that*

$$\|(Id_{k-1} - P_k^{k-1}I_{k-1}^k)v\|_{1-\alpha,k-1} \lesssim h_k^{2\alpha}\|v\|_{1+\alpha,k-1} \qquad \forall\, v \in V_{k-1}. \tag{5.9}$$

Proof. Let $v \in V_{k-1}$ be arbitrary and define $\phi \in H_0^2(\Omega)$ by

$$\phi(w) = \mathcal{A}_{k-1}(v, J_{k-1}w) \qquad \forall\, w \in H_0^2(\Omega), \tag{5.10}$$

where $J_{k-1} : H_0^2(\Omega) \longrightarrow V_{k-1}$ is the map in Lemma 3.6. From Lemma 3.10 we have $\phi \in H^{-2+\alpha}(\Omega)$ and

$$\|\phi\|_{H^{-2+\alpha}(\Omega)} \lesssim \|v\|_{1+\alpha,k-1}. \tag{5.11}$$

Let $\zeta = J_{k-1}^* v$. Again, from Lemma 3.10 we have $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$, and

$$\|\zeta - v\|_{H^{2-\alpha}(\Omega)} \lesssim h_{k-1}^{2\alpha}\|v\|_{1+\alpha,k-1}. \tag{5.12}$$

Finally we define $\zeta_k \in V_k$ to be the solution of the following variational problem:

$$\mathcal{A}_k(\zeta_k, w) = \phi(w) \qquad \forall\, w \in V_k, \tag{5.13}$$

i.e., ζ_k is the solution of the C^0 interior penalty method for (4.5). Therefore we have the following error estimate (cf. Theorem 5 of [24]):

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha}\|\phi\|_{H^{-2+\alpha}(\Omega)}. \tag{5.14}$$

Moreover, from Corollary 3.9, (5.11) and (5.13) we have

$$\mathcal{A}_k(\zeta_k, w) \leq \|\phi\|_{H^{-2+\alpha}(\Omega)}\|w\|_{H^{2-\alpha}(\Omega)} \lesssim \|v\|_{1+\alpha,k-1}\|w\|_{1-\alpha,k} \qquad \forall\, w \in V_k,$$

which together with (3.8) implies that

$$\|\zeta_k\|_{1+\alpha,k} \lesssim \|v\|_{1+\alpha,k-1}. \tag{5.15}$$

We can now use (2.1), Corollary 3.9, (4.3), (4.14), (5.11), (5.12), (5.14) and (5.15) to complete the proof of the lemma as follows:

$$\begin{aligned}
⦀(Id_{k-1} - P_k^{k-1} I_{k-1}^k)v⦀_{1-\alpha,k-1} &\le ⦀v - P_k^{k-1}\zeta_k⦀_{1-\alpha,k-1} + ⦀P_k^{k-1}(\zeta_k - v)⦀_{1-\alpha,k-1} \\
&\lesssim \|v - P_k^{k-1}\zeta_k\|_{H^{2-\alpha}(\Omega)} + ⦀\zeta_k - v⦀_{1-\alpha,k} \\
&\lesssim \|v - \zeta_k\|_{H^{2-\alpha}(\Omega)} + \|\zeta_k - P_k^{k-1}\zeta_k\|_{H^{2-\alpha}(\Omega)} \\
&\lesssim \|v - \zeta\|_{H^{2-\alpha}(\Omega)} + \|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} + ⦀(Id_k - I_{k-1}^k P_k^{k-1})\zeta_k⦀_{1-\alpha,k} \\
&\lesssim h_k^{2\alpha} ⦀v⦀_{1+\alpha,k-1} + h_k^{2\alpha} ⦀\zeta_k⦀_{1+\alpha,k} \lesssim h_k^{2\alpha} ⦀v⦀_{1+\alpha,k-1}.
\end{aligned}$$

□

The following corollary is an immediate consequence of (3.7) and (5.9).

Corollary 5.3. *The estimate* (5.6) *holds for* $\mu = \alpha$.

Finally we prove a useful relation between the mesh-dependent norm $⦀\cdot⦀_{0,k}$ and the Sobolev norm $|\cdot|_{H^1(\Omega)}$ that will be used in the derivation of (5.3)–(5.5). We will use C in the proof of the following lemma (and others in Section 6) to denote a generic mesh-independent positive constant that can take different values at different occurrences.

Lemma 5.4. *It holds that*

$$⦀v⦀_{0,k}^2 \le (1+\theta^2)|v|_{H^1(\Omega)}^2 + C_4\theta^{-2}h_k^{2\beta}⦀v⦀_{\beta,k}^2 \qquad \forall\, v \in V_k,\ 0<\theta<1, \tag{5.16}$$

where β *is the number in* (2.17) *and the positive constant* C_4 *is mesh-independent.*

Proof. Let $\theta \in (0,1)$ and $v \in V_k$ be arbitrary. From (1.7), (2.13), (2.17), (3.5) and Corollary 3.9 we have

$$\begin{aligned}
⦀v⦀_{0,k}^2 &= \langle B_k v, v\rangle \\
&= \langle L_k v, v\rangle + \langle B_k(Id_k - B_k^{-1}L_k)v, v\rangle \\
&\le |v|_{H^1(\Omega)}^2 + ⦀(Id_k - B_k^{-1}L_k)v⦀_{0,k}⦀v⦀_{0,k} \\
&\le |v|_{H^1(\Omega)}^2 + \theta^2⦀v⦀_{0,k}^2 + C\theta^{-2}h_k^{2\beta}\|v\|_{H^{1+\beta}(\Omega)}^2 \\
&\le (1+C\theta^2)|v|_{H^1(\Omega)}^2 + C\theta^{-2}h_k^{2\beta}⦀v⦀_{\beta,k}^2,
\end{aligned}$$

which is equivalent to (5.16) because θ is arbitrary. □

6. Results for V-Cycle and F-Cycle Algorithms

In this section we will complete the convergence analysis of V-cycle and F-cycle algorithms by deriving the estimates (5.3)–(5.5). We shall take the parameter μ in (5.3) to be α and the parameter τ in (5.4)–(5.5) to be the number β that appears in (2.17).

Lemma 6.1. *The estimate* (5.3) *holds for* $\mu = \alpha$.

Proof. Let $v \in V_{k-1}$ and $\theta \in (0,1)$ be arbitrary, and $\zeta = J_{k-1}^* v$. It follows from (2.1) and Lemma 3.10 that $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$,

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim |||v|||_{1+\alpha,k-1}, \tag{6.1}$$

and

$$\|\zeta - v\|_{k-1} \lesssim h_k^\alpha |||v|||_{1+\alpha,k-1}. \tag{6.2}$$

Let $\Pi_k\zeta \in V_k$ and $\Pi_{k-1}\zeta \in V_{k-1}$ be the nodal interpolants of ζ. From (2.7), (3.14), (3.16), (6.1) and (6.2), we have

$$\|v - \Pi_{k-1}\zeta\|_{\mathcal{A}_{k-1}} \leq \|v - \zeta\|_{k-1} + \|\zeta - \Pi_{k-1}\zeta\|_{k-1} \lesssim h_k^\alpha |||v|||_{1+\alpha,k-1}, \tag{6.3}$$

$$\|\Pi_k\zeta - \Pi_{k-1}\zeta\|_{\mathcal{A}_k} \lesssim h_k^\alpha |||v|||_{1+\alpha,k-1}. \tag{6.4}$$

Using (1.8), (2.9), (3.6), (6.3) and (6.4) we find

$$\begin{aligned}
|||I_{k-1}^k v|||_{1,k}^2 &= \|v\|_{\mathcal{A}_k}^2 \\
&\leq \big(\|\Pi_{k-1}\zeta\|_{\mathcal{A}_k} + \|v - \Pi_{k-1}\zeta\|_{\mathcal{A}_k}\big)^2 \\
&\leq (1+\theta^2)\|\Pi_{k-1}\zeta\|_{\mathcal{A}_k}^2 + C\theta^{-2}\|v - \Pi_{k-1}\zeta\|_{\mathcal{A}_{k-1}}^2 \\
&\leq (1+\theta^2)\big(\|\Pi_k\zeta\|_{\mathcal{A}_k} + \|\Pi_k\zeta - \Pi_{k-1}\zeta\|_{\mathcal{A}_k}\big)^2 + C\theta^{-2}h_k^{2\alpha}|||v|||_{1+\alpha,k-1}^2 \\
&\leq (1+\theta^2)^2\|\Pi_k\zeta\|_{\mathcal{A}_k}^2 + C\theta^{-2}h_k^{2\alpha}|||v|||_{1+\alpha,k-1}^2.
\end{aligned} \tag{6.5}$$

On the other hand, from (1.7), (2.3), (2.4), (2.6), (2.7), (3.6), (3.12), (3.14), (6.1), we have

$$\begin{aligned}
\|\Pi_k\zeta\|_{\mathcal{A}_k}^2 &= \mathcal{A}_k(\Pi_k\zeta, \Pi_k\zeta) \\
&= \mathcal{A}_k(\zeta,\zeta) - \mathcal{A}_k(\Pi_k\zeta - \zeta, \Pi_k\zeta - \zeta) + 2\mathcal{A}_k(\Pi_k\zeta - \zeta, \Pi_k\zeta) \\
&\leq \mathcal{A}_k(\zeta,\zeta) + C\|\Pi_k\zeta - \zeta\|_k^2 + C\|\Pi_k\zeta - \zeta\|_k\|\Pi_k\zeta\|_{\mathcal{A}_k} \\
&\leq (1+\theta^2)\mathcal{A}_k(\zeta,\zeta) + C\theta^{-2}\|\Pi_k\zeta - \zeta\|_k^2 \\
&\leq (1+\theta^2)\mathcal{A}_{k-1}(\zeta,\zeta) + C\theta^{-2}h_k^{2\alpha}|||v|||_{1+\alpha,k-1}^2,
\end{aligned} \tag{6.6}$$

and similarly,

$$\begin{aligned}
\mathcal{A}_{k-1}(\zeta,\zeta) &= \mathcal{A}_{k-1}(v,v) + \mathcal{A}_{k-1}(\zeta - v, \zeta - v) + 2\mathcal{A}_{k-1}(\zeta - v, v) \\
&\leq \mathcal{A}_{k-1}(v,v) + C\|\zeta - v\|_k^2 + C\|\zeta - v\|_k\|v\|_{\mathcal{A}_k} \\
&\leq (1+\theta^2)\mathcal{A}_{k-1}(v,v) + C\theta^{-2}\|\zeta - v\|_k^2 \\
&\leq (1+\theta^2)|||v|||_{1,k-1}^2 + C\theta^{-2}h_k^{2\alpha}|||v|||_{1+\alpha,k-1}^2.
\end{aligned} \tag{6.7}$$

Combining (6.5)–(6.7) we find

$$|||I_{k-1}^k v|||_{1,k}^2 \leq (1+\theta^2)^4\|v\|_{\mathcal{A}_{k-1}}^2 + C\theta^{-2}h_k^{2\alpha}|||v|||_{1+\alpha,k-1}^2,$$

which implies (5.3) because $\theta \in (0,1)$ is arbitrary. $\square$

Since $0 < \beta < 1/2 < \alpha$, the estimates (3.7) and (5.3) imply the following corollary.

Corollary 6.2. *It holds that*

$$\|I^k_{k-1}v\|^2_{1,k} \le (1+\theta^2)\|v\|^2_{1,k-1} + C'_1\theta^{-2}h_k^{2\beta}\|v\|^2_{1+\beta,k-1} \qquad \forall\, v \in V_{k-1}, \tag{6.8}$$

where the positive constant C'_1 is mesh-independent.

Lemma 6.3. *The estimate* (5.4) *holds for $\tau = \beta$.*

Proof. Let $\theta \in (0,1)$ be arbitrary. From (2.16) and (5.16) we have

$$\|I^k_{k-1}v\|^2_{0,k} \le (1+\theta^2)\|v\|^2_{0,k-1} + C_4\theta^{-2}h_k^{2\beta}\|v\|^2_{\beta,k-1} \qquad \forall\, v \in V_{k-1}. \tag{6.9}$$

Let the inner product $(\!(\cdot,\cdot)\!)_{k-1,\theta}$ on V_{k-1} be defined by

$$(\!(v_1,v_2)\!)_{k-1,\theta} = (1+\theta^2)(v_1,v_2)_{k-1} + C_2\theta^{-2}h_k^{2\beta}(\mathbb{A}^{\beta}_{k-1}v_1,v_2)_{k-1} \qquad \forall\, v_1,v_2 \in V_{k-1}, \tag{6.10}$$

where $C_2 = \max(C'_1, C_4)$ is the maximum of the mesh-independent constants in (6.8) and (6.9). Note that $\mathbb{A}_{k-1}$ is symmetric positive definite with respect to the inner product $(\!(\cdot,\cdot)\!)_{k-1,\theta}$.

In view of (3.4) and (6.10), the estimates (6.8) and (6.9) imply

$$\|I^k_{k-1}v\|^2_{0,k} \le (\!(\mathbb{A}^0_k v, v)\!)_{k-1,\theta} \qquad \forall\, v \in V_{k-1}, \tag{6.11}$$

$$\|I^k_{k-1}v\|^2_{1,k} \le (\!(\mathbb{A}^1_k v, v)\!)_{k-1,\theta} \qquad \forall\, v \in V_{k-1}. \tag{6.12}$$

It follows from (3.4), (6.10)–(6.12) and interpolation between Hilbert scales that

$$\|I^k_{k-1}v\|^2_{1-\beta,k} \le (\!(\mathbb{A}^{1-\beta}_k v, v)\!)^2_{k-1,\theta} \le (1+\theta^2)\|v\|^2_{1-\beta,k} + C_2\theta^{-2}h_k^{2\beta}\|v\|^2_{1,k-1} \qquad \forall\, v \in V_{k-1}.$$

□

We now turn to the estimate (5.5). First we have to establish certain two-level estimates for the nodal interpolation operator with respect to the mesh-dependent norms.

Lemma 6.4. *The following estimate holds*:

$$\|\Pi_{k-1}v\|^2_{1,k-1} \le (1+\theta^2)\|v\|^2_{1,k} + C_\sharp h_k^{2\alpha}\|v\|^2_{1+\alpha,k} \qquad \forall\, v \in V_k,\ \theta \in (0,1), \tag{6.13}$$

where the constant $C_\sharp$ is mesh-independent.

Proof. Let $v \in V_k$ and $\theta \in (0,1)$ be arbitrary. Then $\zeta = J^*_k v \in H^{2+\alpha}(\Omega) \cap H^2_0(\Omega)$ by Lemma 3.10, and

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim \|v\|_{1+\alpha,k}, \tag{6.14}$$

$$\|\zeta - v\|_k \lesssim h_k^\alpha \|v\|_{1+\alpha,k}. \tag{6.15}$$

We see from (1.8), (2.7), (3.14), (3.18), (6.14) and (6.15) that

$$\begin{aligned}
\|\Pi_{k-1}v\|^2_{1,k-1} &\le \big(\|\Pi_{k-1}\Pi_k\zeta\|_{\mathcal{A}_{k-1}} + \|\Pi_{k-1}(v-\Pi_k\zeta)\|_{\mathcal{A}_{k-1}}\big)^2 \\
&\le (1+\theta^2)\|\Pi_{k-1}\Pi_k\zeta\|^2_{\mathcal{A}_{k-1}} + C\theta^{-2}\big(\|v-\zeta\|_k + \|\zeta-\Pi_k\zeta\|_k\big)^2 \\
&\le (1+\theta^2)\|\Pi_{k-1}\Pi_k\zeta\|^2_{\mathcal{A}_{k-1}} + C\theta^{-2}h_k^{2\alpha}\|v\|^2_{1+\alpha,k}.
\end{aligned} \tag{6.16}$$

We also have, from (1.7), (2.3), (2.4), (2.6), (2.7), (3.18) and (3.12),

$$
\begin{aligned}
\|\Pi_{k-1}\Pi_k\zeta\|^2_{\mathcal{A}_{k-1}} &= \mathcal{A}_{k-1}(\Pi_{k-1}\Pi_k\zeta, \Pi_{k-1}\Pi_k\zeta)\\
&= \mathcal{A}_{k-1}(\zeta,\zeta) - \mathcal{A}_{k-1}(\Pi_{k-1}\Pi_k\zeta-\zeta, \Pi_{k-1}\Pi_k\zeta-\zeta)\\
&\qquad + 2\mathcal{A}_{k-1}(\Pi_{k-1}\Pi_k\zeta-\zeta, \Pi_{k-1}\Pi_k\zeta)\\
&\le \mathcal{A}_{k-1}(\zeta,\zeta) + C\|\Pi_{k-1}\Pi_k\zeta-\zeta\|^2_{k-1}\\
&\qquad + C\|\Pi_{k-1}\Pi_k\zeta-\zeta\|_{k-1}\|\Pi_{k-1}\Pi_k\zeta\|_{\mathcal{A}_{k-1}}\\
&\le (1+\theta^2)\mathcal{A}_{k-1}(\zeta,\zeta) + C\theta^{-2}\|\Pi_{k-1}\Pi_k\zeta-\zeta\|^2_{k-1},
\end{aligned}
\tag{6.17}
$$

and from (2.7), (3.14) and (3.17),

$$
\begin{aligned}
\|\Pi_{k-1}\Pi_k\zeta-\zeta\|_{k-1} &\le \|\Pi_{k-1}\Pi_k\zeta-\Pi_{k-1}\zeta\|_{k-1} + \|\Pi_{k-1}\zeta-\zeta\|_{k-1}\\
&\lesssim |\Pi_{k-1}\Pi_k\zeta-\Pi_{k-1}\zeta|_{H^2(\Omega,\mathcal{T}_{k-1})} + \|\Pi_{k-1}\zeta-\zeta\|_{k-1}\\
&\lesssim |\Pi_{k-1}\Pi_k\zeta-\zeta|_{H^2(\Omega,\mathcal{T}_{k-1})} + \|\Pi_{k-1}\zeta-\zeta\|_{k-1} \lesssim h_k^\alpha |\!|\!| v |\!|\!|_{1+\alpha,k}.
\end{aligned}
\tag{6.18}
$$

Furthermore, the estimates (1.7), (2.6), (2.7), (6.15) and the relations (2.4) and (3.6) imply

$$
\begin{aligned}
\mathcal{A}_{k-1}(\zeta,\zeta) &= \mathcal{A}_k(\zeta,\zeta)\\
&= \mathcal{A}_k(v,v) + \mathcal{A}_k(\zeta-v,\zeta-v) + 2\mathcal{A}_k(\zeta-v,v)\\
&\le \mathcal{A}_k(v,v) + C\|\zeta-v\|_k^2 + C\|\zeta-v\|_k\|v\|_{\mathcal{A}_k}\\
&\le (1+\theta^2)\mathcal{A}_k(v,v) + C\theta^{-2}\|\zeta-v\|_k^2\\
&\le (1+\theta^2)|\!|\!| v |\!|\!|^2_{1,k} + C\theta^{-2}h_k^{2\alpha}|\!|\!| v |\!|\!|^2_{1+\alpha,k}.
\end{aligned}
\tag{6.19}
$$

Combining (6.16)–(6.19) we arrive at

$$
|\!|\!| \Pi_{k-1}v |\!|\!|^2_{1,k-1} \le (1+\theta^2)^3|\!|\!| v |\!|\!|^2_{1,k} + C\theta^{-2}h_k^{2\alpha}|\!|\!| v |\!|\!|^2_{1+\alpha,k},
$$

which is equivalent to (6.13) because $\theta\in(0,1)$ is arbitrary. □

Again, since $0<\beta<1/2<\alpha$, the estimates (3.7) and (6.13) imply the following corollary.

Corollary 6.5. *It holds that*

$$
|\!|\!| \Pi_{k-1}v |\!|\!|^2_{1,k-1} \le (1+\theta^2)|\!|\!| v |\!|\!|^2_{1,k} + C'_\sharp h_k^{2\beta}|\!|\!| v |\!|\!|^2_{1+\beta,k} \qquad \forall\, v\in V_k,\ \theta\in(0,1),
\tag{6.20}
$$

where the positive constant $C'_\sharp$ is mesh-independent.

Lemma 6.6. *The following estimate holds*:

$$
|\!|\!| \Pi_{k-1}v |\!|\!|^2_{0,k-1} \le (1+\theta^2)|\!|\!| v |\!|\!|^2_{0,k} + C_\flat h_k^{2\beta}|\!|\!| v |\!|\!|^2_{\beta,k} \qquad \forall\, v\in V_k,\ \theta\in(0,1),
\tag{6.21}
$$

where the positive constant $C_\flat$ is mesh-independent.

Proof. Let $\theta\in(0,1)$ and $v\in V_k$ be arbitrary. First we observe that, by (2.1), (3.6), (3.7), (3.19), (3.24) and Corollary 3.9,

$$
|\!|\!| \Pi_{k-1}v |\!|\!|_{\beta,k-1} \lesssim \|v\|_{H^{1+\beta}(\Omega)} + \|v-\Pi_{k-1}v\|_{H^{1+\beta}(\Omega)}
$$

$$
\begin{aligned}
&\lesssim |||v|||_{\beta,k} + h_k^{-\beta}|v-\Pi_{k-1}v|_{H^1(\Omega)} \qquad (6.22)\\
&\lesssim |||v|||_{\beta,k} + h_k^{1-\beta}|||v|||_{1,k} \lesssim |||v|||_{\beta,k}.
\end{aligned}
$$

The estimate (6.21) follows from (1.8), (2.1), (2.13), (2.16), (3.5), (3.7), (3.19) (5.16) and (6.22):

$$
\begin{aligned}
|||\Pi_{k-1}v|||^2_{0,k-1} &\le (1+\theta^2)|\Pi_{k-1}v|^2_{H^1(\Omega)} + C\theta^{-2}h_{k-1}^{2\beta}|||\Pi_{k-1}v|||^2_{\beta,k-1}\\
&\le (1+\theta^2)\big(|v|_{H^1(\Omega)} + |\Pi_{k-1}v - v|_{H^1(\Omega)}\big)^2 + C\theta^{-2}h_k^{2\beta}|||v|||^2_{\beta,k}\\
&\le (1+\theta^2)^2|v|^2_{H^1(\Omega)} + C\theta^{-2}h_k^2\|v\|^2_{\mathcal{A}_k} + C\theta^{-2}h_k^{2\beta}|||v|||^2_{\beta,k}\\
&\le (1+\theta^2)^2|||v|||^2_{0,k} + C\theta^{-2}h_k^2|||v|||^2_{1,k} + C\theta^{-2}h_k^{2\beta}|||v|||^2_{\beta,k}\\
&\le (1+\theta^2)^2|||v|||^2_{0,k} + C\theta^{-2}h_k^{2\beta}|||v|||^2_{\beta,k},
\end{aligned}
$$

which is equivalent to (6.21) because $\theta \in (0,1)$ is arbitrary. □

Corollary 6.7. *The following estimate holds*:

$$
|||\Pi_{k-1}v|||^2_{1-\beta,k-1} \le (1+\theta^2)|||v|||^2_{1-\beta,k} + C_\natural h_k^{2\beta}|||v|||^2_{1,k} \qquad \forall\, v\in V_k,\ \theta\in(0,1), \tag{6.23}
$$

where the constant $C_\natural$ is mesh-independent.

Proof. We use the technique in the proof of Lemma 6.3. For any $\theta \in (0,1)$, we define the inner product $(\!(\cdot,\cdot)\!)_{k,\theta}$ on V_k by

$$
(\!(v_1,v_2)\!)^2_{k,\theta} = (v_1,v_2)^2_k + C_\natural\theta^{-2}h_k^{2\beta}(\mathbb{A}_k^\beta v_1, v_2)_k \qquad \forall\, v_1,v_2\in V_k, \tag{6.24}
$$

where $C_\natural = \max(C'_\sharp, C_\flat)$. Then $\mathbb{A}_k$ is symmetric positive definite with respect to $(\!(\cdot,\cdot)\!)_{k,\theta}$.

In view of (3.4), (6.20), (6.24) and (6.21), we have

$$
|||\Pi_{k-1}v|||^2_{0,k-1} \le (\!(\mathbb{A}_k^0 v, v)\!)^2_{k,\theta} \qquad \forall\, v\in V_k, \tag{6.25}
$$

$$
|||\Pi_{k-1}v|||^2_{1,k-1} \le (\!(\mathbb{A}_k^1 v, v)\!)^2_{k,\theta} \qquad \forall\, v\in V_k. \tag{6.26}
$$

The estimate (6.23) follows from (6.25), (6.26) and interpolation between Hilbert scales. □

We are now ready to verify (5.5).

Lemma 6.8. *The estimate* (5.5) *holds for $\tau = \beta$.*

Proof. Let $v \in V_k$ and $\theta \in (0,1)$ be arbitrary. From (1.8), (3.7), (3.20), Corollary 3.9, (4.12) and (6.23) we have

$$
\begin{aligned}
|||P_k^{k-1}v|||^2_{1-\beta,k-1} &\le \big(|||\Pi_{k-1}v|||_{1-\beta,k-1} + |||P_k^{k-1}v - \Pi_{k-1}v|||_{1-\beta,k-1}\big)^2\\
&\le (1+\theta^2)|||\Pi_{k-1}v|||^2_{1-\beta,k-1} + C\theta^{-2}h_k^{2(\beta-\alpha)}|||P_k^{k-1}v - \Pi_{k-1}v|||^2_{1-\alpha,k-1}\\
&\le (1+\theta^2)|||\Pi_{k-1}v|||^2_{1-\beta,k-1} + C\theta^{-2}h_k^{2(\beta-\alpha)}\|P_k^{k-1}v - \Pi_{k-1}v\|^2_{H^{2-\alpha}(\Omega)}\\
&\le (1+\theta^2)^2|||v|||^2_{1-\beta,k} + C\theta^{-2}h_k^{2\beta}|||v|||^2_{1,k}\\
&\quad + C\theta^{-2}h_k^{2(\beta-\alpha)}\big(\|P_k^{k-1}v - v\|_{H^{2-\alpha}(\Omega)} + \|v-\Pi_{k-1}v\|_{H^{2-\alpha}(\Omega)}\big)^2
\end{aligned}
$$

$$\leq (1+\theta^2)^2 |\!|\!| v |\!|\!|_{1-\beta,k}^2 + C\theta^{-2} h_k^{2\beta} |\!|\!| v |\!|\!|_{1,k}^2,$$

which is equivalent to (5.5) because $\theta \in (0,1)$ is arbitrary. □

We have verified the assumptions (5.3)–(5.6) for the additive theory. Therefore we can apply the results in [22] to obtain the following convergence theorems for the V-cycle and F-cycle algorithms.

Theorem 6.9. *The output $MG_V(k,\psi,z_0,m)$ of the V-cycle algorithm* (*Algorithm 2.3*) *applied to* (2.11) *satisfies the following estimate*:

$$\|z - MG_V(k,\psi,z_0,m)\|_{\mathcal{A}_k} \leq \frac{C}{m^\alpha} \|z - z_0\|_{\mathcal{A}_k},$$

where the positive constant C is mesh-independent, provided that the number of smoothing steps m is greater than a positive integer m_* that is also mesh-independent.

Theorem 6.10. *The output $MG_F(k,\psi,z_0,m)$ of the F-cycle algorithm* (*Algorithm 2.5*) *applied to* (2.11) *satisfies the following estimate*:

$$\|z - MG_F(k,\psi,z_0,m)\|_{\mathcal{A}_k} \leq \frac{C}{m^\alpha} \|z - z_0\|_{\mathcal{A}_k},$$

where the positive constant C is mesh-independent, provided that the number of smoothing steps m is greater than a positive integer m_* that is also mesh-independent.

Remark 6.11. Theorems 6.9 and 6.10 have been obtained for preconditioners that satisfy (2.14)–(2.17). Therefore they are valid for a Poisson solve B_k^{-1} obtained by a symmetric W-cycle algorithm with a sufficiently large number of smoothing steps or a variable V-cycle algorithm (cf. Remark 2.2 and Appendix A). However, in practice these algorithms behave equally well when the preconditioner is a symmetric V-cycle algorithm with a few smoothing steps (cf. Section 7).

7. Numerical Experiments

In this section we report the results of some numerical experiments for the biharmonic problem. The finite element we use is the Q_2 rectangular element and the penalty parameter η is taken to be 5.

The first set of experiments involve the biharmonic problem on the unit square. We take $\mathcal{T}_0$ to be the triangulation with one element and we compute the contraction numbers of the V-cycle, F-cycle and W-cycle algorithms on the k-th level ($1 \leq k \leq 7$) with m pre-smoothing and m post-smoothing steps. We use the symmetric V-cycle algorithm for the Poisson problem with three pre-smoothing and three post-smoothing Richardson relaxation steps as the preconditioner in (2.18) and (2.20). The results are recorded in Tables 1–3. Convergence for the V-cycle, F-cycle and W-cycle algorithms is observed for $m = 4$, $m = 2$ and $m = 1$ respectively. We also observe that the performance of the F-cycle algorithm and the W-cycle algorithm are almost identical for $m \geq 6$.

Numerical experiments show that for moderate grid levels ($k \leq 7$) there is practically no difference in the performance of the multigrid algorithms whether we use a symmetric V-cycle or a symmetric W-cycle Poisson solve as the preconditioner in (2.18) and (2.20).

k \ m	4	5	6	7	8	9	10
1	0.08	0.04	0.02	0.011	0.006	0.0032	0.0017
2	0.27	0.22	0.18	0.15	0.13	0.11	0.09
3	0.43	0.32	0.29	0.26	0.23	0.21	0.19
4	0.56	0.35	0.34	0.31	0.28	0.25	0.23
5	0.64	0.42	0.37	0.34	0.31	0.29	0.27
6	0.70	0.43	0.39	0.35	0.33	0.30	0.27
7	0.75	0.44	0.39	0.36	0.34	0.31	0.29

TABLE 1. Contraction numbers for the V-cycle algorithm on the unit square

k \ m	2	3	4	5	6	7	8	9	10
1	0.28	0.15	0.08	0.04	0.02	0.01	0.0060	0.0032	0.0017
2	0.50	0.35	0.27	0.22	0.18	0.15	0.13	0.11	0.09
3	0.52	0.40	0.34	0.30	0.27	0.24	0.22	0.19	0.18
4	0.53	0.42	0.37	0.34	0.31	0.28	0.26	0.24	0.22
5	0.53	0.43	0.37	0.34	0.31	0.29	0.27	0.25	0.23
6	0.53	0.44	0.38	0.34	0.32	0.29	0.27	0.25	0.23
7	0.54	0.46	0.38	0.35	0.32	0.29	0.27	0.25	0.23

TABLE 2. Contraction numbers for the F-cycle algorithm on the unit square

k \ m	1	2	3	4	5	6	7	8	9	10
1	0.53	0.28	0.15	0.08	0.04	0.02	0.01	0.006	0.003	0.002
2	0.72	0.49	0.24	0.27	0.22	0.18	0.15	0.13	0.11	0.09
3	0.71	0.51	0.40	0.34	0.30	0.26	0.24	0.22	0.19	0.17
4	0.80	0.51	0.41	0.37	0.34	0.31	0.28	0.26	0.24	0.22
5	0.76	0.53	0.42	0.38	0.34	0.31	0.29	0.26	0.24	0.23
6	0.82	0.53	0.42	0.38	0.34	0.32	0.29	0.26	0.25	0.22
7	0.83	0.53	0.42	0.38	0.34	0.32	0.29	0.27	0.25	0.23

TABLE 3. Contraction numbers for the W-cycle algorithm on the unit square

k \ m	75	76	77	78	79	80	81	82	83
1	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05
2	0.47	0.47	0.46	0.46	0.46	0.46	0.46	0.45	0.45
3	0.64	0.47	0.42	0.64	0.42	0.40	0.41	0.36	0.63
4	0.60	0.60	0.58	0.57	0.54	0.52	0.50	0.51	0.49
5	0.71	0.69	0.66	0.64	0.63	0.61	0.57	0.56	0.52
6	0.76	0.74	0.72	0.70	0.68	0.65	0.62	0.60	0.56
7	0.80	0.78	0.76	0.73	0.71	0.68	0.65	0.61	0.56

TABLE 4. Contraction numbers for the V-cycle algorithm on the unit square without a preconditioner in the smoothing steps

For comparison we also report in Table 4 the contraction numbers of the V-cycle algorithm using the Richardson relaxation scheme without a preconditioner as the smoother. Convergence is observed only for $m \geq 75$.

k \ m	4	5	6	7	8	9	10
1	0.23	0.16	0.12	0.082	0.059	0.043	0.031
2	0.40	0.27	0.21	0.19	0.16	0.14	0.13
3	0.54	0.37	0.31	0.29	0.26	0.24	0.21
4	0.68	0.42	0.37	0.34	0.31	0.28	0.26
5	0.74	0.45	0.38	0.35	0.32	0.30	0.27
6	0.78	0.46	0.40	0.36	0.34	0.31	0.28
7	0.80	0.46	0.40	0.37	0.34	0.31	0.29

TABLE 5. Contraction numbers for the V-cycle algorithm on the L-shaped domain

k \ m	2	3	4	5	6	7	8	9	10
1	0.50	0.34	0.23	0.16	0.12	0.08	0.06	0.04	0.03
2	0.52	0.40	0.32	0.27	0.23	0.20	0.17	0.15	0.13
3	0.51	0.42	0.37	0.33	0.30	0.27	0.25	0.22	0.21
4	0.53	0.43	0.38	0.34	0.31	0.29	0.27	0.24	0.22
5	0.52	0.43	0.38	0.34	0.32	0.29	0.27	0.25	0.23
6	0.54	0.45	0.38	0.35	0.32	0.29	0.27	0.25	0.23
7	0.54	0.45	0.38	0.35	0.32	0.29	0.27	0.25	0.23

TABLE 6. Contraction numbers for the F-cycle algorithm on the L-shaped domain

In the second set of experiments we study the biharmonic problem on the L-shaped domain with vertices $(-1,-1)$, $(1,-1)$, $(1,0)$, $(0,0)$, $(0,1)$ and $(-1,1)$. We take $\mathcal{T}_0$ to be the triangulation with three elements. Again we use the symmetric V-cycle algorithm for the Poisson problem with three pre-smoothing and three post-smoothing steps as the preconditioner. The contraction numbers of the V-cycle, F-cycle and W-cycle algorithms on the k-th level with m pre-smoothing and m post-smoothing steps are reported in Tables 5–7. The relative performance of these algorithms is similar to the case of the unit square.

k \ m	1	2	3	4	5	6	7	8	9	10
1	0.75	0.50	0.34	0.23	0.16	0.12	0.08	0.06	0.04	0.03
2	0.71	0.51	0.39	0.32	0.27	0.23	0.20	0.17	0.15	0.13
3	0.78	0.51	0.42	0.37	0.33	0.30	0.27	0.25	0.22	0.20
4	0.73	0.53	0.42	0.38	0.34	0.31	0.29	0.26	0.24	0.22
5	0.82	0.52	0.42	0.38	0.34	0.32	0.29	0.27	0.25	0.23
6	0.74	0.52	0.42	0.38	0.34	0.32	0.29	0.27	0.25	0.23
7	0.81	0.52	0.42	0.38	0.34	0.32	0.29	0.27	0.25	0.23

Table 7. Contraction numbers for the W-cycle algorithm on the L-shaped domain

Appendix A. Some Properties of Multigrid Poisson Solves

In this appendix we consider multigrid Poisson solves as the preconditioner in (2.18) and (2.20). We will show that properties (i)–(iv) in Section 2 are satisfied by such preconditioners.

Consider the discrete Poisson problem: Find $z \in V_k$ such that

$$L_k z = \psi \qquad \forall\, v \in V_k, \tag{A.1}$$

where L_k is defined in (2.13) and $\psi \in V_k'$.

Let $S_k : V_k' \longrightarrow V_k$ be the solution operator for (A.1) generated by either a symmetric V-cycle algorithm, a symmetric W-cycle algorithm or a symmetric variable V-cycle algorithm (that satisfies (4.13)), with 0 as the initial guess and Richardson relaxation as the smoother. In terms of S_k the output $MG(k,\psi,z_0)$ of the multigrid method can be written as

$$MG(k,\psi,z_0) = z_0 + S_k(\psi - L_k z_0), \tag{A.2}$$

and $Id_k - S_k L_k$ is the error propagation operator.

The operator S_k is symmetric, or equivalently the operator $Id_k - S_kL_k$ is symmetric with respect to the bilinear form $\langle L_k\cdot,\cdot\rangle$ (cf. Lemma 7.1 of [17] where S_k is denoted by B_k). Furthermore (cf. Theorems 5.1, 7.1 and 7.2 of [17]) there exists a number $\delta \in (0,1)$ independent of k such that

$$0 \le \langle L_k(Id_k - S_kL_k)v, v\rangle \le \delta\langle L_kv, v\rangle \qquad \forall\, v \in V_k. \tag{A.3}$$

We see from (A.3) that S_k is positive definite. Therefore we can define $B_k = S_k^{-1}$ and the operator $B_k : V_k \longrightarrow V_k'$ is symmetric positive definite. Moreover (A.3) implies that the eigenvalues of the operator $Id_k - B_k^{-1}L_k : V_k \longrightarrow V_k$ lie between 0 and δ. Since $Id_k - B_k^{-1}L_k$ is also symmetric with respect to $\langle B_k\cdot,\cdot\rangle$, we deduce that

$$0 \le \langle B_k(Id_k - B_k^{-1}L_k)v, v\rangle \le \delta\langle B_kv, v\rangle \qquad \forall\, v \in V_k. \tag{A.4}$$

The estimate (2.16) follows from (A.4) immediately.

Hence the operator B_k satisfies properties (i) and (ii) in Section 2. Property (iv), which states that multigrid algorithms have optimal complexity, is also standard [32]. In particular Theorem 4.5 and Theorem 4.6 are valid for all three types of multigrid preconditioners.

On the other hand, the proofs of Theorem 6.9 and Theorem 6.10 require property (iii). Below we will demonstrate that (2.17) is satisfied by the B_k generated by W-cycle or variable V-cycle Poisson solves.

Let $B_k^{-1} : V_k' \longrightarrow V_k$ be the preconditioner obtained by a symmetric W-cycle algorithm. We have a well-known recurrence relation [32]:

$$Id_k - B_k^{-1}L_k = R_k^m(Id_k - I_{k-1}^kP_k^{k-1})R_k^m + R_k^mI_{k-1}^k(Id_{k-1} - B_{k-1}^{-1}L_{k-1})^2P_k^{k-1}R_k^m, \tag{A.5}$$

where $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ is the natural injection, $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the adjoint of I_k^{k-1} with respect to the bilinear form $\langle L_k\cdot,\cdot\rangle$ and $\langle L_{k-1}\cdot,\cdot\rangle$, and R_k is the error reduction operator of one Richardson relaxation step. Of course at the coarsest level we have $B_0^{-1} = S_0 = L_0^{-1}$ and hence

$$Id_0 - B_0^{-1}L_0 = 0. \tag{A.6}$$

Let β be any number in $(0, 1/2)$. The following estimates are valid [20]:

$$|R_k^m(Id_k - I_{k-1}^kP_k^{k-1})R_k^mv|_{H^1(\Omega)} \lesssim h_k^\beta m^{-\beta/2}\|v\|_{H^{1+\beta}(\Omega)} \qquad \forall v \in V_k \text{ and } k \ge 1, \tag{A.7}$$

$$\|P_k^{k-1}v\|_{H^{1+\beta}(\Omega)} \lesssim \|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_k \text{ and } k \ge 1, \tag{A.8}$$

$$|(Id_k - B_k^{-1}L_k)v|_{H^1(\Omega)} \lesssim m^{-\alpha_*}|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k \text{ and } k \ge 1, \tag{A.9}$$

where $\alpha_* \in (1/2, 1]$ is the index of elliptic regularity for the Poisson problem. Furthermore, we have

$$|R_k^mv|_{H^1(\Omega)} \le C|v|_{H^1(\Omega)} \qquad \forall\, v \in V_k\,,\ k \ge 1,\ \text{and } m \ge 1, \tag{A.10}$$

$$\|R_k^mv\|_{H^{1+\beta}(\Omega)} \le C\|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_k,\ k \ge 1,\ \text{and } m \ge 1, \tag{A.11}$$

where the positive constant C is independent of the meshes.

It follows from (2.1), (A.5), and (A.7)–(A.11) that

$$|v - B_k^{-1}L_kv|_{H^1(\Omega)} \le C_*h_k^\beta\big[m^{-\beta/2} + \sigma m^{-\alpha_*/2}\big]\|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_k, \tag{A.12}$$

where C_* is a mesh-independent positive constant, provided that

$$|v - B_{k-1}^{-1} L_{k-1} v|_{H^1(\Omega)} \le \sigma h_{k-1}^{\beta} \|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_{k-1}.$$

Hence, if m is sufficiently large, we obtain from (A.6), (A.12) and mathematical induction that

$$|v - B_k^{-1} L_k v|_{H^1(\Omega)} \le \sigma h_k^{\beta} \|v\|_{H^{1+\beta}(\Omega)} \qquad \forall\, v \in V_k,\ k \ge 0, \tag{A.13}$$

if σ is the number defined by

$$\sigma = \frac{C_* m^{-\beta/2}}{1 - C_* m^{-\alpha_*/2}}.$$

Therefore (2.17) is satisfied by the W-cycle preconditioner provided that m is sufficiently large.

Now we consider the preconditioner B_k^{-1} obtained from a variable V-cycle algorithm. Given a positive integer k, we assume that the number m_j of smoothing steps on level j satisfies

$$(1+\epsilon) m_{j+1} \le m_j \qquad \text{for} \quad 0 \le j \le k-1, \tag{A.14}$$

where ϵ is a positive number. We have an additive expression for the error propagation operator:

$$\begin{aligned} Id_k - B_k^{-1} L_k &= R_k^{m_k} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^{m_k} \\ &\quad + R_k^{m_k} I_{k-1}^k R_{k-1}^{m_{k-1}} (Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2}) R_{k-1}^{m_{k-1}} P_k^{k-1} R_k^{m_k} \\ &\quad + R_k^{m_k} I_{k-1}^k R_{k-1}^{m_{k-1}} I_{k-2}^{k-1} R_{k-2}^{m_{k-2}} (Id_{k-2} - I_{k-3}^{k-2} P_{k-2}^{k-3}) R_{k-2}^{m_{k-2}} P_{k-1}^{k-2} R_{k-1}^{m_{k-1}} P_k^{k-1} R_k^{m_k} \\ &\quad + \cdots . \end{aligned} \tag{A.15}$$

The following estimates are valid [20]:

$$\|R_k^{m_k} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^{m_k} v\|_{H^{1-\beta}(\Omega)} \lesssim h_k^{\beta} m_k^{-\alpha_* + (\beta/2)} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k,\ k \ge 1, \tag{A.16}$$

$$\|R_k^{m_k} I_{k-1}^k \cdots R_{j+1}^{m_{j+1}} I_j^{j+1} v\|_{H^{1-\beta}(\Omega)} \lesssim \|v\|_{H^{1-\beta}(\Omega)} \qquad \forall\, v \in V_j,\ j \le k, \tag{A.17}$$

$$|P_{j+1}^{j} R_{j+1}^{m_{j+1}} \cdots P_k^{k-1} R_k^{m_k} v|_{H^1(\Omega)} \lesssim |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k,\ j \le k. \tag{A.18}$$

Combining (2.1), (A.6) and (A.15)–(A.18), we find, for any $\beta \in (0, 1/2)$,

$$\begin{aligned} |v - B_k^{-1} L_k v|_{H^{1-\beta}(\Omega)} &\le C_\beta |v|_{H^1(\Omega)} \sum_{j=2}^{k} h_j^{\beta} m_j^{-\alpha_* + (\beta/2)} \\ &\le C_\beta |v|_{H^1(\Omega)} \sum_{j=2}^{k} (2^{(j-k)} h_k)^{\beta} ((1+\epsilon)^{(j-k)} m_k)^{-\alpha_* + (\beta/2)} \\ &\le C_\beta m_k^{-\alpha_* + (\beta/2)} h_k^{\beta} |v|_{H^1(\Omega)} \sum_{j=2}^{k} [2^{\beta} (1+\epsilon)^{(-\alpha_* + \beta/2)}]^{j-k}, \end{aligned} \tag{A.19}$$

where C_β depends on β but not the meshes. It follows from (A.19) that there exists a positive mesh-independent constant C such that

$$|v - B_k^{-1} L_k v|_{H^{1-\beta}(\Omega)} \leq C h_k^\beta |v|_{H^1(\Omega)} \qquad \forall\, v \in V_k \tag{A.20}$$

if we choose $\beta > 0$ so that

$$\beta < \max\Big(\frac{1}{2}, \frac{\alpha_* \ln(1+\epsilon)}{\ln(2\sqrt{1+\epsilon})}\Big).$$

The estimate (2.17) follows from (A.20) and duality. In other words, property (iii) is satisfied by the variable V-cycle preconditioner under condition (A.14).

References

[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces* (*Second Edition*). Academic Press, Amsterdam, 2003.

[2] J.H. Argyris, I. Fried, and D.W. Scharpf. The TUBA family of plate elements for the matrix displacement method. *Aero. J. Roy. Aero. Soc.*, 72:701–709, 1968.

[3] D.N. Arnold and F. Brezzi. Mixed and nonconforming finite element methods: implementation, post-processing and error estimates. *R.A.I.R.O Modél. Math. Anal. Numér.*, 19:7–32, 1985.

[4] C. Bacuta, J.H. Bramble, and J.E. Pasciak. Shift theorems for the biharmonic Dirichlet problem. In *Recent Progress in Computational and Applied PDEs*, pages 1–26. Kluwer/Plenum, New York, 2002.

[5] R.E. Bank and C.C. Douglas. Sharp estimates for multigrid rates of convergence with general smoothing and acceleration. *SIAM J. Numer. Anal.*, 22:617–633, 1985.

[6] R.E. Bank and T.F. Dupont. An optimal order process for solving finite element equations. *Math. Comp.*, 36:35–51, 1981.

[7] F. Ben Belgacem and S.C. Brenner. Some nonstandard finite element estimates with applications to 3D Poisson and Signorini problems. *Electronic Transactions on Numerical Analysis* (*ETNA*), 12:134–148, 2001.

[8] F.K. Bogner, R.L. Fox, and L.A. Schmit. The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas. In *Proceedings Conference on Matrix Methods in Structural Mechanics*, pages 397–444. Wright Patterson A.F.B., Dayton, OH, 1965.

[9] D. Braess and W. Hackbusch. A new convergence proof for the multigrid method including the V-cycle. *SIAM J. Numer. Anal.*, 20:967–975, 1983.

[10] J.H. Bramble. *Multigrid Methods*. Longman Scientific & Technical, Essex, 1993.

[11] J.H. Bramble and S.R. Hilbert. Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. *SIAM J. Numer. Anal.*, 7:113–124, 1970.

[12] J.H. Bramble and J.E. Pasciak. New estimates for multigrid algorithms including the V-cycle. *Math. Comp.*, 60:447–471, 1993.

[13] J.H. Bramble and J.E. Pasciak. Uniform convergence estimates for multigrid V-cycle algorithms with less than full elliptic regularity. In A. Quarteroni et. al., editor, *Domain Decomposition Methods in Science and Engineering*, pages 17–26. Amer. Math. Soc., Providence, 1994. Contemporary Mathematics 157.

[14] J.H. Bramble, J.E. Pasciak, J. Wang, and J. Xu. Convergence estimates for product iterative methods with applications to domain decomposition and multigrid. *Math. Comp.*, 57:1–21, 1991.

[15] J.H. Bramble, J.E. Pasciak, and J. Xu. The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms. *Math. Comp.*, 56:1–34, 1991.

[16] J.H. Bramble and J. Xu. Some estimates for a weighted L^2 projection. *Math. Comp.*, 56:463–476, 1991.

[17] J.H. Bramble and X. Zhang. The Analysis of Multigrid Methods. In P.G. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, VII*, pages 173–415. North-Holland, Amsterdam, 2000.

[18] S.C. Brenner. Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comp.*, 65:897–921, 1996.

[19] S.C. Brenner. Convergence of nonconforming multigrid methods without full elliptic regularity. *Math. Comp.*, 68:25–53, 1999.

[20] S.C. Brenner. Convergence of the multigrid V-cycle algorithm for second order boundary value problems without full elliptic regularity. *Math. Comp.*, 71:507–525, 2002.

[21] S.C. Brenner. Smoothers, mesh dependent norms, interpolation and multigrid. *Appl. Numer. Math.*, 43:45–56, 2002.

[22] S.C. Brenner. Convergence of nonconforming V-cycle and F-cycle multigrid algorithms for second order elliptic boundary value problem. *Math. Comp.*, 73:1041–1066, 2004.

[23] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods* (*Second Edition*). Springer-Verlag, New York-Berlin-Heidelberg, 2002.

[24] S.C. Brenner and L.-Y. Sung. C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. IMI Research Report 2003:10 (http://www.math.sc.edu/∼imip/03.html), Department of Mathematics, University of South Carolina, 2003 (to appear in *J. Sci. Comput.*).

[25] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York-Berlin-Heidelberg, 1991.

[26] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[27] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains,* Lecture Notes in Mathematics 1341. Springer-Verlag, Berlin-Heidelberg, 1988.

[28] T. Dupont and R. Scott. Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, 34:441–463, 1980.

[29] G. Engel, K. Garikipati, T.J.R. Hughes, M.G. Larson, L. Mazzei, and R.L. Taylor. Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Methods Appl. Mech. Engrg.*, 191:3669–3750, 2002.

[30] N.A. Fleck and J.W. Hutchinson. Strain gradient plasticity. *Adv. Appl. Mech.*, 33:295–361, 1997.

[31] P. Grisvard. *Elliptic Problems in Non Smooth Domains*. Pitman, Boston, 1985.

[32] W. Hackbusch. *Multi-grid Methods and Applications*. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[33] S.G. Kreĭn, Ju.I. Petunin, and E.M. Semenov. Interpolation of Linear Operators. In *Translations of Mathematical Monographs 54*. American Mathematical Society, Providence, 1982.

[34] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications I*. Springer-Verlag, New York, 1972.

[35] J. Mandel, S. McCormick, and R. Bank. Variational Multigrid Theory. In S. McCormick, editor, *Multigrid Methods, Frontiers In Applied Mathematics 3*, pages 131–177. SIAM, Philadelphia, 1987.

[36] L.S.D. Morley. The triangular equilibrium problem in the solution of plate bending problems. *Aero. Quart.*, 19:149–169, 1968.

[37] S.A. Nazarov and B.A. Plamenevsky. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. de Gruyter, Berlin-New York, 1994.

[38] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques*. Masson, Paris, 1967.

[39] T.K. Nilssen, X.-C. Tai, and R. Winther. A robust nonconforming H^2-element. *Math. Comp.*, 70:489–505, 2001.

[40] Z. Shi. On the convergence of the incomplete biquadratic nonconforming plate element. *Math. Numer. Sinica*, 8:53–62, 1986.

[41] J.Y. Shu, W.E. King, and N.A. Fleck. Finite elements for materials with strain gradient effects. *Internat. J. Numer. Methods Engrg.*, 44:373–391, 1999.

[42] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Amsterdam, 1978.

[43] U. Trottenberg, C. Oosterlee, and A. Schüller. *Multigrid*. Academic Press, San Diego, 2001.

[44] G.N. Wells, K. Garikipati, and L. Molari. A discontinuous Galerkin formulation for a strain gradient-dependent damage model. *Comput. Methods Appl. Mech. Engrg*, 2004 (to appear).
[45] X. Zhang. Multilevel Schwarz methods. *Numer. Math.*, 63:521–539, 1992.
[46] J. Zhao. Convergence of nonconforming V-cycle and F-cycle methods for the biharmonic problem using the Morley element. *ETNA*, 17:112–132, 2004.
[47] J. Zhao. *Multigird Methods for Fourth Order Problems.* PhD thesis, University of South Carolina, 2004.

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