IMI

Preprint Series

Industrial Mathematics Institute

2004:15

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TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONERS FOR C^0 INTERIOR PENALTY METHODS

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ABSTRACT. We study two-level additive Schwarz preconditioners that can be used in the iterative solution of the discrete problems resulting from C^0 interior penalty methods for fourth order elliptic boundary value problems. We show that the condition number of the preconditioned system is bounded by $C(1+(H^3/\delta^3))$, where H is the typical diameter of a subdomain, δ measures the overlap among the subdomains and the positive constant C is independent of the mesh sizes and the number of subdomains.

1. INTRODUCTION

C^0 interior penalty methods for fourth order elliptic boundary value problems have recently been analyzed in [19, 12]. The idea behind this approach can be explained in terms of the following model problem:
Find $u \in H_0^2(\Omega)$ such that

$$\sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^2(\Omega), \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ and $f \in L_2(\Omega)$.

Let $\mathcal{T}_h$ be a (simplicial or convex quadrilateral) triangulation of Ω and $V_h \subset H_0^1(\Omega)$ be a Lagrange (triangular or tensor product) finite element space associated with $\mathcal{T}_h$. By a careful integration by parts argument [12], it can be shown that the solution u of (1.1), which by elliptic regularity [24, 15, 26, 3] belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha > 1/2$, satisfies

$$\mathcal{A}_h(u,v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{1.2}$$

where

$$\mathcal{A}_h(w,v) = a_h(w,v) + b_h(w,v) + \eta\, c_h(w,v), \tag{1.3}$$

$$a_h(w,v) = \sum_{D \in \mathcal{T}_h} \sum_{i,j=1}^{2} \int_D \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx, \tag{1.4}$$

Date: September 24, 2004.
1991 *Mathematics Subject Classification.* 65N55, 65N30.
Key words and phrases. domain decomposition, two-level additive Schwarz preconditioners, C^0 interior penalty methods, fourth order problems.
This work was supported in part by the National Science Foundation under Grant No. DMS-03-11790.

$$b_h(w,v) = \sum_{e\in\mathcal{E}_h} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2 w}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] \right) ds, \tag{1.5}$$

$$c_h(w,v) = \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] ds. \tag{1.6}$$

$\mathcal{E}_h$ in (1.5) and (1.6) is the set of all the edges of $\mathcal{T}_h$, and η in (1.3) is a penalty parameter. The jumps $[\![\cdot]\!]$ and averages $\{\!\{\cdot\}\!\}$ in (1.5) and (1.6) are defined as follows.

Let e be an interior edge of $\mathcal{T}_h$ and n_e be a unit normal vector of e. Then e is shared by two elements D_+ and D_- in $\mathcal{T}_h$ where the normal vector n_e points from D_- to D_+. We define on e, for any function v that is piecewise H^s with respect to the triangulation $\mathcal{T}_h$ for some $s > \frac{5}{2}$,

$$\left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} = \frac{1}{2}\Big[\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2} \Big], \tag{1.7}$$

where $v_\pm = v\big|_{D_\pm}$. Note that $[\![\partial v/\partial n]\!]$ and $\{\!\{\partial^2 v/\partial n^2\}\!\}$ are independent of the choice of n_e. For an edge e that is a subset of $\partial\Omega$, we take n_e to be the outward pointing unit normal vector and define

$$\left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}. \tag{1.8}$$

In the C^0 interior penalty approach, the discrete problem for (1.1) is:
Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h. \tag{1.9}$$

In view of (1.2), the C^0 interior penalty method defined by (1.9) is consistent and, for a sufficiently large η, it is also stable. Therefore the discretization error $u - u_h$ is quasi-optimal with respect to appropriate norms [19, 12].

The C^0 interior approach has certain advantages: (i) The simplest C^0 interior penalty methods for (1.1), i.e., those based on the P_2 Lagrange triangular element or the Q_2 Lagrange tensor product element, are as simple as classical nonconforming finite elements. (ii) Unlike nonconforming finite elements, the C^0 interior penalty methods come in arbitrary orders. For smooth solutions, the higher order C^0 interior penalty methods have the same convergence rate as higher order C^1 finite elements for smooth solutions and at the same time are much simpler. (iii) Because the finite element spaces in the C^0 interior penalty approach are the standard spaces for second order problems, Poisson solves can be used naturally as preconditioners [13] in iterative methods for (1.9). (iv) Unlike mixed methods, this approach can be extended in a straight-forward way to more complicated fourth order problems, such as the fourth order elliptic systems that appear in strain-gradient elasticity and plasticity theory [22, 29].

Multigrid methods for (1.9) have been analyzed in [13]. In this paper we construct two-level additive Schwarz preconditioners [16, 17, 27] for the discrete problem (1.9). We show that the classical results for this overlapping domain decomposition method can be extended to (1.9). More precisely, we prove that the condition number of the preconditioned system is bounded by $C(1+(H^3/\delta^3))$, where H is the typical diameter of a subdomain, δ measures the overlap among the subdomains and the positive constant C is independent of the mesh sizes and the number of subdomains.

The rest of the paper is organized as follows. We recall two-level additive Schwarz preconditioners in Section 2. Then we define the coarse spaces and derive some preliminary estimates in Section 3. Condition number estimates are established in Sections 4 and 5, followed by numerical results in Section 6. We conclude the paper with some remarks in Section 7.

We note in passing that domain decomposition methods for discontinuous Galerkin methods for second order problems were studied in [20, 25], and two-level additive Schwarz preconditioners for the discontinuous interior penalty method [4] for fourth order problems was investigated in [21].

2. Two-level Additive Schwarz Preconditioners

We will use quadrilateral meshes in this paper in view of the potential of their three dimensional counterparts for future investigation. For simplicity we will also focus on the case where $\mathcal{T}_h$ is a rectangular mesh. The extension to general convex quadrilateral meshes will be discussed in Section 7.

Let $V_h \subset H_0^1(\Omega)$ be the Q_2 finite element space [14, 11] associated with $\mathcal{T}_h$ and the operator $A_h : V_h \longrightarrow V_h'$ be defined by

$$\langle A_h v_1, v_2\rangle = \mathcal{A}_h(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_h, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual, and $\mathcal{A}_h$ is the variational form defined by (1.3). We can then rewrite (1.9) as $A_h u_h = \phi_h$, where $\langle \phi_h, v\rangle = \int_\Omega f v\, dx$ for all $v \in V_h$. Therefore A_h is the operator that needs to be preconditioned.

Note that, for η sufficiently large (which is assumed to be the case), the following relation [12] holds:

$$C_1 |v|^2_{H^2(\Omega,\mathcal{T}_h)} \le \langle A_h v, v\rangle = \mathcal{A}_h(v,v) \le C_2 |v|^2_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{2.2}$$

where

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| [\![\partial v/\partial n]\!] \|^2_{L_2(e)} \tag{2.3}$$

and the constants C_1 and C_2 depend only on the shape regularity of $\mathcal{T}_h$. Here and throughout this paper we follow the standard notation for L_2-based Sobolev spaces [1, 14, 11].

The two-level additive Schwarz preconditioner is constructed in terms of subdomain solves and a coarse grid solve.

Let $\mathcal{T}_H$ be a coarse rectangular mesh for Ω and $V_H \subset H_0^1(\Omega)$ be a finite element space associated with $\mathcal{T}_H$. (The choice of the finite element for the coarse space will be discussed in Section 3.) We can then define $A_0 : V_H \longrightarrow V_H'$ by

$$\langle A_0 v_1, v_2\rangle = \mathcal{A}_H(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_H, \tag{2.4}$$

where $\mathcal{A}_H$ is the analog of $\mathcal{A}_h$ for the coarse grid $\mathcal{T}_H$.

Let Ω_j, $1 \le j \le J$, be overlapping subdomains of Ω such that $\Omega = \cup_{j=1}^J \Omega_j$ and the boundaries of Ω_j are aligned with the edges of $\mathcal{T}_h$. We assume that there exist $\theta_j \in C^\infty(\bar\Omega)$ for $1 \le j \le J$ such that

$$\theta_j = 0 \qquad \text{on} \quad \Omega \setminus \Omega_j, \tag{2.5}$$

$$\sum_{j=1}^J \theta_j = 1 \qquad \text{on} \quad \bar\Omega, \tag{2.6}$$

$$\|\nabla\theta_j\|_{L_\infty(\Omega)} \le \frac{C}{\delta}, \qquad \|\nabla^2\theta_j\|_{L_\infty(\Omega)} \le \frac{C}{\delta^2}, \tag{2.7}$$

where $\nabla^2\theta_j$ is the Hessian of θ_j, $\delta > 0$ is a parameter that measures the overlap among the subdomains and C is a positive constant independent of h, H and J.

Remark 2.1. Suppose $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. We can construct Ω_j by enlarging the subdomains of $\mathcal{T}_H$ by the amount δ so that each Ω_j is the union of rectangles in $\mathcal{T}_h$ (cf. Figure 1). The construction of θ_j satisfying (2.5)–(2.7) is then standard [28].

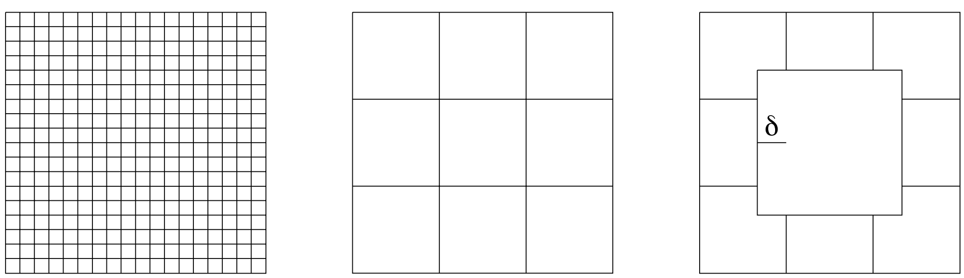

FIGURE 1. $\mathcal{T}_h$, $\mathcal{T}_H$ and Ω_j

Let $V_j \subset H_0^1(\Omega_j)$ ($\subset H_0^1(\Omega)$) be the Q_2 finite element space associated with the fine grid $\mathcal{T}_h$ on Ω_j. The following variational form is the analog of (1.3):

$$\begin{aligned}\mathcal{A}_j(w,v) = &\sum_{\substack{D\in\mathcal{T}_h\\ D\subset\Omega_j}} \sum_{i,j=1}^2 \int_D \frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}dx\\ &+ \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\bar\Omega_j}} \int_e \left(\left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] + \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right) ds \end{aligned} \tag{2.8}$$

$$+\eta \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\bar{\Omega}_j}} \frac{1}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds \qquad \forall\, v,w\in V_j,$$

where the jumps $[\![\cdot]\!]$ and averages $\{\!\{\cdot\}\!\}$ are defined by (1.7) if $e\subset\Omega_j$ and (1.8) if $e\subset\partial\Omega_j$.

The bilinear form $\mathcal{A}_j(\cdot,\cdot)$ can be represented by the operator $A_j : V_j \longrightarrow V_j'$ defined by

$$\langle A_j v_1, v_2\rangle = \mathcal{A}_j(v_1,v_2) \qquad \forall\, v_1,v_2\in V_j, \tag{2.9}$$

and we have the following analog of (2.2):

$$C_1|v|^2_{H^2(\Omega_j,\mathcal{T}_h)} \le \langle A_j v,v\rangle = \mathcal{A}_j(v,v) \le C_2|v|^2_{H^2(\Omega_j,\mathcal{T}_h)} \qquad \forall\, v\in V_j, \tag{2.10}$$

where

$$|v|^2_{H^2(\Omega_j,\mathcal{T}_h)} = \sum_{\substack{D\in\mathcal{T}_h\\ D\subset\Omega_j}} |v|^2_{H^2(D)} + \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\bar{\Omega}_j}} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}. \tag{2.11}$$

Let $I_j : V_j \longrightarrow V_h$ be the natural injection for $1\le j\le J$. Note that (2.3) and (2.11) imply immediately

$$|I_j v|_{H^2(\Omega,\mathcal{T}_h)} \le C_3|v|_{H^2(\Omega_j,\mathcal{T}_h)} \qquad \forall\, v\in V_j, \tag{2.12}$$

where the positive constant C_3 is independent of h, H and J. Furthermore, under the condition $\delta\le H$ (which is assumed to be the case),

(2.13) the relation $\mathcal{A}_h(v_j,v_\ell)=0\ \forall\, v_j\in V_j$ and $v_\ell\in V_\ell$ holds for all but N_c many ℓ's,

where the positive integer N_c is independent of h, H, J and δ, and in particular,

(2.14) each point of Ω belongs to less than N_c many subdomains.

Suppose the space V_H is connected to V_h by an operator $I_0 : V_H \longrightarrow V_h$. (The construction of I_0 will be given in Section 3.) We can now define the two-level additive Schwarz preconditioner $B : V_h' \longrightarrow V_h$ by

$$B = \sum_{j=0}^{J} I_j A_j^{-1} I_j^t, \tag{2.15}$$

where $I_j^t : V_h' \longrightarrow V_j'$ is the transpose of $I_j : V_j \longrightarrow V_h$, i.e.,

$$\langle I_j^t\psi, v\rangle = \langle \psi, I_j v\rangle \qquad \forall\, v\in V_j. \tag{2.16}$$

(We take V_0 to be V_H.)

Remark 2.2. The operators A_j $(0\le j\le J)$, I_j $(0\le j\le J)$ and A_h can be represented by matrices with respect to the natural nodal bases of V_j and V_h and the canonical dual bases of V_j' and V_h'. The matrix for I_j^t is then the transpose of the matrix for I_j. In other words, the matrix form for the preconditioner B is also given by (2.15).

Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator. Then every $v \in V_h$ can be written as $v = \sum_{j=1}^{J} v_j$ where $v_j = \Pi_h(\theta_j v)$ and the θ_j's are the partition of unity functions that appear in (2.5)–(2.7). From (2.5) we see that $v_j \in V_j$ and therefore the condition

$$V_h = \sum_{j=0}^{J} I_j V_j \tag{2.17}$$

is satisfied. It follows from the additive Schwarz theory [16, 17, 32, 5, 31, 23, 30, 11] that B is symmetric positive definite and therefore the eigenvalues of BA_h are positive. Furthermore, the maximum and minimum eigenvalues of BA_h are characterized by the following formulas:

$$\lambda_{\max}(BA_h) = \max_{\substack{v \in V_h \\ v \neq 0}} \frac{\langle A_h v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{j=0}^{J} I_j v_j \\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle}, \tag{2.18}$$

$$\lambda_{\min}(BA_h) = \min_{\substack{v \in V_h \\ v \neq 0}} \frac{\langle A_h v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{j=0}^{J} I_j v_j \\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle}. \tag{2.19}$$

Remark 2.3. It is clear from (2.18) and (2.19) that the results of this paper would not be affected if the exact solve A_j^{-1} is replaced by an inexact solve B_j^{-1} as long as $\langle B_j v, v\rangle \approx \langle A_j v, v\rangle$ for all $v \in V_j$.

3. Coarse Spaces and Preliminary Estimates

In this section, we define the coarse spaces and derive some preliminary estimates which will be used in the analysis of the condition number of BA_h.

Our first choice of the coarse space $V_H \subset H_0^1(\Omega)$ is the Q_1 Lagrange tensor product finite element space associated with $\mathcal{T}_H$. The Q_1 Lagrange element is depicted in Figure 2 together with the Q_3 Bogner-Fox-Schmit element [6], where we use the solid dot $\bullet$ to denote pointwise evaluation of the shape functions, the circle $\circ$ to denote pointwise evaluation of all the first order derivatives of the shape functions and the arrow $\nearrow$ to denote pointwise evaluation of the mixed second order derivative.

Let $\tilde{V}_H \subset H_0^2(\Omega)$ be the Bogner-Fox-Schmit finite element space associated with $\mathcal{T}_H$. The two spaces V_H and $\tilde{V}_H$ are connected by a map $E_H : V_H \longrightarrow \tilde{V}_H$ defined by

$$(E_H v)(p) = v(p), \tag{3.1}$$

$$\nabla(E_H v)(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D \in \mathcal{T}_p} \nabla v_{_D}(p), \tag{3.2}$$

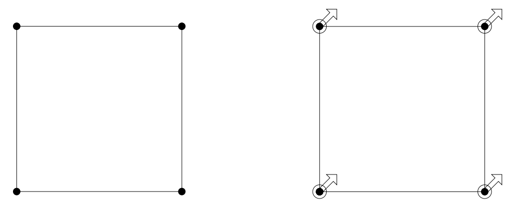

FIGURE 2. Q_1 element and Q_3 Bogner-Fox-Schmit element

$$\frac{\partial^2(E_H v)}{\partial x_1 \partial x_2}(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D \in \mathcal{T}_p} \frac{\partial^2 v_D}{\partial x_1 \partial x_2}(p), \tag{3.3}$$

where p is any interior vertex of the rectangular mesh, $\mathcal{T}_p$ is the set of the rectangles sharing p as a vertex, $|\mathcal{T}_p|$ is the number of elements in $\mathcal{T}_p$ and $v_D = v\big|_D$. The operator $I_0 : V_H \longrightarrow V_h$ appearing in (2.15) is then defined by

$$I_0 = \Pi_h \circ E_H, \tag{3.4}$$

where $\Pi_h : C^0(\bar{\Omega}) \longrightarrow V_h$ is the nodal interpolation operator.

Remark 3.1. The Q_3 Bogner-Fox-Schmit element is a C^1 relative of the Q_1 Lagrange tensor product element. The *enriching* map E_H that connects a C^0 element to a C^1 element enables us to derive estimates that are independent of the mesh sizes (cf. (3.22) below).

The following lemma gives the properties of the operator E_H. To avoid the proliferation of constants, we henceforth use the notation $A \lesssim B$ to represent the statement $A \leq$ constant $\times$ B, where the positive constant depends only on the shape regularity of the meshes and not h, H, J nor δ. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Lemma 3.2. *The following estimates hold*:

$$\|v - E_H v\|_{L_2(\Omega)} \lesssim H^2 |v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H, \tag{3.5}$$

$$|v - E_H v|_{H^1(\Omega)} \lesssim H |v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H, \tag{3.6}$$

$$|E_H v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H, \tag{3.7}$$

where $|\cdot|_{H^2(\Omega,\mathcal{T}_H)}$ *is the analog of* $|\cdot|_{H^2(\Omega,\mathcal{T}_h)}$ (*cf.* (2.3)) *for* $v \in V_H$.

Proof. Let $v \in V_H$ be arbitrary. Let D be a subdomain in $\mathcal{T}_H$ and p be a vertex of D that is interior to Ω. By (3.2), the Cauchy-Schwarz inequality and a standard inverse estimate [14, 11], we have

$$\begin{aligned} |\nabla v_D(p) - \nabla(E_H v)(p)|^2 &= \Big|\frac{1}{|\mathcal{T}_p|} \sum_{D' \in \mathcal{T}_p} [\nabla v_D(p) - \nabla v_{D'}(p)]\Big|^2 \\ &\lesssim \sum_{D' \in \mathcal{T}_p} |\nabla v_D(p) - \nabla v_{D'}(p)|^2 \end{aligned} \tag{3.8}$$

$$\lesssim \sum_{e\in\mathcal{E}_p} \frac{1}{|e|}\|[\![\nabla v]\!]\|_{L_2(e)}^2 = \sum_{e\in\mathcal{E}_p} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2,$$

where $\mathcal{E}_p$ is the set of edges sharing p as a common vertex. Note that in the derivation of (3.8) we have used the fact that for any two subdomains D_1 and D_2 sharing e as a common edge, the tangential derivatives of v_{D_1} and v_{D_2} agree on e.

For a vertex p of D that belongs to $\partial\Omega$, we also have, by (1.8),

$$|\nabla v_D(p) - \nabla(E_H v)(p)|^2 = |\nabla v_D(p)|^2 \lesssim \frac{1}{|e|}\|[\![\partial v/\partial n]\!]_e\|_{L_2(e)}^2, \tag{3.9}$$

where $e \subset \partial\Omega$ is an edge of $\mathcal{T}_h$ with p as an endpoint and we have used the fact that the tangential derivative of v along $\partial\Omega$ vanishes.

Furthermore, the second order nodal values of $E_h v$ can be easily estimated through a standard inverse estimate:

$$\Big|\frac{\partial^2(E_H v)}{\partial x_1\partial x_2}(p)\Big|^2 \lesssim \sum_{D\in\mathcal{T}_p}\Big|\frac{\partial^2 v_D}{\partial x_1\partial x_2}(p)\Big|^2 = \sum_{D\in\mathcal{T}_p}(\operatorname{diam} D)^{-2}|v|_{H^2(D)}^2. \tag{3.10}$$

Finally we observe that, by scaling,

$$\begin{aligned}\|w\|_{L_2(D)}^2 \approx (\operatorname{diam} D)^2 \sum_{p\in\mathcal{V}_D}(w(p))^2 + (\operatorname{diam} D)^4 \sum_{p\in\mathcal{V}_D}(\nabla w(p))^2 \\ + (\operatorname{diam} D)^6 \sum_{p\in\mathcal{V}_D}\Big[\frac{\partial^2 w}{\partial x_1\partial x_2}(p)\Big]^2 \qquad \forall\, w\in\tilde{V}_H,\end{aligned} \tag{3.11}$$

where $\mathcal{V}_D$ is the set of the vertices of D.

Since $v - E_H v \in \tilde{V}_H$, it follows from (3.1) and (3.8)–(3.11) that

$$\|v - E_H v\|_{L_2(D)}^2 \lesssim (\operatorname{diam} D)^4\Big(\sum_{p\in\mathcal{V}_D}\sum_{e\in\mathcal{E}_p}\frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{p\in\mathcal{V}_D}\sum_{D'\in\mathcal{T}_p}|v|_{H^2(D')}^2\Big), \tag{3.12}$$

which together with standard inverse estimates also implies

$$|v - E_H v|_{H^1(D)}^2 \lesssim (\operatorname{diam} D)^2\Big(\sum_{p\in\mathcal{V}_D}\sum_{e\in\mathcal{E}_p}\frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{p\in\mathcal{V}_D}\sum_{D'\in\mathcal{T}_p}|v|_{H^2(D')}^2\Big), \tag{3.13}$$

$$|v - E_H v|_{H^2(D)}^2 \lesssim \sum_{p\in\mathcal{V}_D}\sum_{e\in\mathcal{E}_p}\frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \sum_{p\in\mathcal{V}_D}\sum_{D'\in\mathcal{T}_p}|v|_{H^2(D')}^2. \tag{3.14}$$

Summing up (3.12)–(3.13) over all the subdomains of $\mathcal{T}_H$, we obtain (3.5) and (3.6).

Finally, from (3.14) we have

$$|E_H v|_{H^2(\Omega)}^2 \lesssim \sum_{D\in\mathcal{T}_H}\big(|v - E_H v|_{H^2(D)}^2 + |v|_{H^2(D)}^2\big) \lesssim |v|_{H^2(\Omega,\mathcal{T}_H)}^2.$$

$\square$

The next lemma gives the relevant properties of Π_h.

Lemma 3.3. *The following estimates on* Π_h *are valid*:

$$\|\zeta - \Pi_h\zeta\|_{L_2(\Omega)} + h|\zeta - \Pi_h\zeta|_{H^1(\Omega)} \lesssim h^2|\zeta|_{H^2(\Omega)} \qquad \forall\, \zeta \in H_0^2(\Omega), \tag{3.15}$$

$$|\Pi_h\zeta|_{H^2(\Omega,\mathcal{T}_h)} \lesssim |\zeta|_{H^2(\Omega)} \qquad \forall\, \zeta \in H_0^2(\Omega). \tag{3.16}$$

Proof. Let $\zeta \in H_0^2(\Omega)$ be arbitrary. On each $D \in \mathcal{T}_h$ we have the standard estimates [14, 11]:

$$\|\zeta - \Pi_h\zeta\|_{L_2(D)} + (\operatorname{diam} D)|\zeta - \Pi_h\zeta|_{H^1(D)} \lesssim (\operatorname{diam} D)^2|\zeta|_{H^2(D)}, \tag{3.17}$$

$$|\Pi_h\zeta|_{H^2(D)} \lesssim |\zeta|_{H^2(D)}, \tag{3.18}$$

which implies (3.15) immediately.

For any $e \in \mathcal{E}_h$, by the trace theorem (with scaling), (3.17) and (3.18), we have

$$\begin{aligned}
\frac{1}{|e|}\left\|\left[\!\left[\frac{\partial(\Pi_h\zeta)}{\partial n}\right]\!\right]\right\|_{L_2(e)}^2 &= \frac{1}{|e|}\left\|\left[\!\left[\frac{\partial(\zeta - \Pi_h\zeta)}{\partial n}\right]\!\right]\right\|_{L_2(e)}^2 \\
&\lesssim \frac{1}{|e|}\sum_{D\in\mathcal{T}_e}\left\|\frac{\partial(\zeta - \Pi_h\zeta)_D}{\partial n_e}\right\|_{L_2(e)}^2 \\
&\lesssim \sum_{D\in\mathcal{T}_e}\left[(\operatorname{diam} D)^{-2}\left\|\frac{\partial(\zeta - \Pi_h\zeta)}{\partial n_e}\right\|_{L_2(D)}^2 + \left|\frac{\partial(\zeta - \Pi_h\zeta)}{\partial n_e}\right|_{H^1(D)}^2\right] \\
&\lesssim \sum_{D\in\mathcal{T}_e}\left[(\operatorname{diam} D)^{-2}|\zeta - \Pi_h\zeta|_{H^1(D)}^2 + |\zeta - \Pi_h\zeta|_{H^2(D)}^2\right] \\
&\lesssim \sum_{D\in\mathcal{T}_e}|\zeta|_{H^2(D)}^2,
\end{aligned} \tag{3.19}$$

where $\mathcal{T}_e$ is the set of the subdomains in $\mathcal{T}_h$ sharing e as a common edge.

The estimate (3.16) follows by summing up (3.18) and (3.19) over all the subdomains in $\mathcal{T}_h$ and all the edges in $\mathcal{E}_h$. $\square$

We can now derive the key estimates for the operator I_0.

Lemma 3.4. *The following estimates on* I_0 *hold*:

$$\|v - I_0v\|_{L_2(\Omega)} \lesssim H^2|v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H, \tag{3.20}$$

$$|v - I_0v|_{H^1(\Omega)} \lesssim H|v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H, \tag{3.21}$$

$$|I_0v|_{H^2(\Omega,\mathcal{T}_h)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_H)} \qquad \forall\, v \in V_H. \tag{3.22}$$

Proof. Let $v \in V_H$ be arbitrary. Since $E_Hv \in H_0^2(\Omega)$, using (3.4), the triangle inequality, Lemma 3.2, Lemma 3.3 and the fact that $h \le H$, we have

$$\begin{aligned}
\|v - I_0v\|_{L_2(\Omega)} &= \|v - \Pi_hE_Hv\|_{L_2(\Omega)} \\
&\lesssim \|v - E_Hv\|_{L_2(\Omega)} + \|E_Hv - \Pi_hE_Hv\|_{L_2(\Omega)} \\
&\lesssim H^2|v|_{H^2(\Omega,\mathcal{T}_H)} + h^2|E_Hv|_{H^2(\Omega)} \\
&\lesssim H^2|v|_{H^2(\Omega,\mathcal{T}_H)}.
\end{aligned}$$

Similarly we can obtain (3.21). Finally, it follows from (3.4), (3.7) and (3.16) that

$$|I_0 v|_{H^2(\Omega,\mathcal{T}_h)} = |\Pi_h E_H v|_{H^2(\Omega,\mathcal{T}_h)} \lesssim |E_H v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_H)}.$$

□

Remark 3.5. If we replace the operator I_0 defined by (3.4) by the natural injection from V_H to V_h, then the estimate (3.22) is not valid.

Recall that V_h is the Q_2 Lagrange tensor product finite element space associated with $\mathcal{T}_h$. The Q_2 element and the Q_4 Bogner-Fox-Schmit element are depicted in Figure 3, where we use the arrow $\uparrow$ to denote pointwise evaluation of the normal derivative of the shape functions.

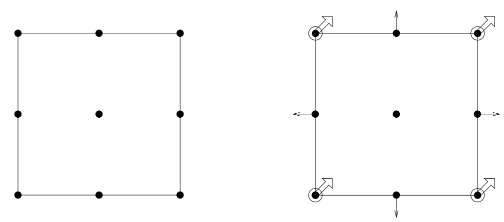

FIGURE 3. Q_2 element and Q_4 Bogner-Fox-Schmit element

Let $\tilde{V}_h(\subset H_0^2(\Omega))$ be the Q_4 Bogner-Fox-Schmit finite element space associated with $\mathcal{T}_h$. We can define a map $E_h : V_h \longrightarrow \tilde{V}_h$ analogous to E_H by

$$(E_h v)(p) = v(p), \tag{3.23}$$

$$\nabla(E_h v)(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D\in\mathcal{T}_p} \nabla v_{_D}(p), \tag{3.24}$$

$$\frac{\partial(E_h v)}{\partial n_e}(m_e) = \frac{1}{|\mathcal{T}_e|} \sum_{D\in\mathcal{T}_e} \frac{\partial v_{_D}}{\partial n_e}(m_e), \tag{3.25}$$

$$\frac{\partial^2(E_h v)}{\partial x_1 \partial x_2}(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D\in\mathcal{T}_p} \frac{\partial^2 v_{_D}}{\partial x_1 \partial x_2}(p), \tag{3.26}$$

where (3.23) is defined for any interior node p associated with $\mathcal{T}_h$, (3.24) and (3.26) are defined for any interior vertex p of $\mathcal{T}_h$, and (3.25) is defined for any interior edge e of $\mathcal{T}_h$ with midpoint m_e. Note that (3.23) implies

$$\Pi_h E_h v = v \qquad \forall\, v \in V_h. \tag{3.27}$$

Remark 3.6. The Q_4 Bogner-Fox-Schmit element is a C^1 relative of the Q_2 Lagrange tensor product element. The map E_h was introduced in [12] for the post-processing of the solutions obtained by the C^0 interior penalty methods.

The following lemma is the analog of Lemma 3.2.

Lemma 3.7. *The following estimates hold*:

$$\|v - E_h v\|_{L_2(\Omega)} \lesssim h^2 |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{3.28}$$

$$|v - E_h v|_{H^1(\Omega)} \lesssim h |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{3.29}$$

$$|E_h v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h. \tag{3.30}$$

Proof. Let $v \in V_h$ be arbitrary. First of all the analogs of (3.8)–(3.10) are valid for E_h and we also have the following analog of (3.11) on each $D \in \mathcal{T}_h$:

$$\begin{aligned} \|w\|^2_{L_2(D)} &\approx (\operatorname{diam} D)^2 \sum_{p\in\mathcal{V}_D} (w(p))^2 + (\operatorname{diam} D)^4 \sum_{p\in\mathcal{V}_D} (\nabla w(p))^2 \\ &\quad + (\operatorname{diam} D)^4 \sum_{e\in\mathcal{E}(D)} \left[\frac{\partial w}{\partial n_e}(m_e)\right]^2 + (\operatorname{diam} D)^6 \sum_{p\in\mathcal{V}_D} \left[\frac{\partial^2 w}{\partial x_1 \partial x_2}(p)\right]^2 \qquad \forall\, w \in \tilde{V}_h, \end{aligned} \tag{3.31}$$

where $\mathcal{E}(D)$ is the set of the edges of D.

Let $e \subset \Omega$ be an edge of $D \in \mathcal{T}_h$. It follows from (3.25) and a standard inverse estimate that

$$\begin{aligned} \left|\frac{\partial v_{\scriptscriptstyle D}}{\partial n_e}(m_e) - \frac{\partial (E_h v)}{\partial n_e}(m_e)\right|^2 &= \left|\frac{1}{2}\sum_{D'\in\mathcal{T}_e} \left(\frac{\partial v_{\scriptscriptstyle D}}{\partial n_e}(m_e) - \frac{\partial v_{\scriptscriptstyle D'}}{\partial n_e}(m_e)\right)\right|^2 \\ &\lesssim \frac{1}{|e|}\, \|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\,. \end{aligned} \tag{3.32}$$

On the other hand, if $e \subset \partial\Omega$ is an edge of $\mathcal{T}_h$, then we have, by (1.8),

$$\left|\frac{\partial v}{\partial n_e}(m_e) - \frac{\partial (E_h v)}{\partial n_e}(m_e)\right|^2 = \left|\frac{\partial v}{\partial n_e}(m_e)\right|^2 \lesssim \frac{1}{|e|}\, \|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}\,. \tag{3.33}$$

The estimates (3.28)–(3.30) can then be obtained from (3.31)–(3.33) and the analogs of (3.8)–(3.10), as in the proof of Lemma 3.2. □

Let $\Pi_H : C^0(\bar{\Omega}) \longrightarrow V_H$ be the nodal interpolation operator and $J_h^H : V_h \longrightarrow V_H$ be the restriction of Π_H to V_h. The operator J_h^H will play a role in the analysis of the two-level additive Schwarz preconditioner. Note that (3.23) implies

$$J_h^H v = \Pi_H E_h v \qquad \forall\, v \in V_h,$$

and the lemma below is an analog of Lemma 3.4 that follows from Lemma 3.7 and the analog of Lemma 3.3 for Π_H.

Lemma 3.8. *The following estimates on J_h^H are valid*:

$$\|v - J_h^H v\|_{L_2(\Omega)} \lesssim H^2 |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{3.34}$$

$$|v - J_h^H v|_{H^1(\Omega)} \lesssim H |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{3.35}$$

$$|J_h^H v|_{H^2(\Omega,\mathcal{T}_h)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h. \tag{3.36}$$

Our second choice for the coarse space $V_H \subset H_0^1(\Omega)$ is the Q_2 Lagrange tensor product finite element space associated with $\mathcal{T}_H$. In this case we take $\tilde{V}_H$ to be the Q_4 Bogner-Fox-Schmit space associated with $\mathcal{T}_H$ and define $E_H : V_H \longrightarrow \tilde{V}_H$ by the analogs of (3.23)–(3.26). We can then define $I_0 : V_H \longrightarrow V_h$ by (3.4). In view of Lemma 3.7 (which also holds for E_H), the results in Lemma 3.4 remain valid for this I_0.

4. A Condition Number Estimate

In this section we derive an estimate for the condition number of BA_h. We begin with an upper bound for the eigenvalues of BA_h.

Lemma 4.1. *The following upper bound for the eigenvalues of BA_h holds*:

$$\lambda_{\max}(BA_h) \lesssim 1. \tag{4.1}$$

Proof. Let $v \in V_h$ be arbitrary. For any $v_j \in V_j$ such that $v = \sum_{j=0}^J I_j v_j$, we have, by (2.2) (and its analog for $\mathcal{A}_H$), (2.10), (2.12), (2.13), (3.22) and the Cauchy-Schwarz inequality,

$$\begin{aligned}\langle A_h v, v\rangle \approx |v|^2_{H^2(\Omega,\mathcal{T}_h)} &\lesssim |I_0 v_0|^2_{H^2(\Omega,\mathcal{T}_h)} + \Big|\sum_{j=1}^J I_j v_j\Big|^2_{H^2(\Omega,\mathcal{T}_h)}\\ &\lesssim |v_0|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{j=1}^J |v_j|^2_{H^2(\Omega,\mathcal{T}_h)} \approx \sum_{j=0}^J \langle A_j v_j, v_j\rangle,\end{aligned}$$

which implies

$$\langle A_h v, v\rangle \lesssim \min_{\substack{v=\sum_{j=0}^J I_j v_j\\ v_j\in V_j}} \sum_{j=0}^J \langle A_j v_j, v_j\rangle. \tag{4.2}$$

The estimate (4.1) follows from (2.18) and (4.2). □

We now turn our attention to a lower bound for the eigenvalues of BA_h.

Lemma 4.2. *The following lower bound for the eigenvalues of BA_h holds*:

$$\lambda_{\min}(BA_h) \gtrsim \left(1 + \frac{H^4}{\delta^4}\right)^{-1}. \tag{4.3}$$

Proof. Let $v \in V_h$ be arbitrary,

$$v_0 = J_h^H v, \tag{4.4}$$

$$v_j = \Pi_h(\theta_j(v - I_0 v_0)) \qquad \text{for} \quad 1 \le j \le J. \tag{4.5}$$

From (2.6) and the fact that $v - I_0 v_0 \in V_h$, it is clear that

$$\sum_{j=0}^J I_j v_j = v. \tag{4.6}$$

Below we will carefully estimate the terms $\langle A_j v_j, v_j\rangle$ for $0 \le j \le J$.

First we consider v_0. From (3.36), (4.4) and the analogs of (2.2) and (2.3) for $|\cdot|_{H^2(\Omega,\mathcal{T}_H)}$, we have

$$\langle A_0 v_0, v_0\rangle = \langle A_H v_0, v_0\rangle \approx |J_h^H v|^2_{H^2(\Omega,\mathcal{T}_H)} \lesssim |v|^2_{H^2(\Omega,\mathcal{T}_h)} \approx \langle A_h v, v\rangle. \tag{4.7}$$

Next we consider v_j for $1 \le j \le J$. Let

$$w = v - I_0 v_0 \quad \text{and} \quad w_j = \theta_j w. \tag{4.8}$$

From Lemma 3.4, Lemma 3.8, (4.4) and (4.8) we have

$$\begin{aligned} \|w\|_{L_2(\Omega)} &= \|v - I_0 J_h^H v\|_{L_2(\Omega)} \\ &\lesssim \|v - J_h^H v\|_{L_2(\Omega)} + \|I_0 J_h^H v - J_h^H v\|_{L_2(\Omega)} \\ &\lesssim H^2 |v|_{H^2(\Omega,\mathcal{T}_h)} + H^2 |J_h^H v|_{H^2(\Omega,\mathcal{T}_H)} \lesssim H^2 |v|_{H^2(\Omega,\mathcal{T}_h)}, \end{aligned} \tag{4.9}$$

and similarly,

$$|w|_{H^1(\Omega)} \lesssim H |v|_{H^2(\Omega,\mathcal{T}_h)}, \tag{4.10}$$

$$|w|_{H^2(\Omega,\mathcal{T}_h)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_h)}. \tag{4.11}$$

We can also rewrite (4.5) as

$$v_j = \Pi_h(\theta_j w) = \Pi_h w_j. \tag{4.12}$$

Let $D \subset \Omega_j$ be an arbitrary subdomain in $\mathcal{T}_h$ and $\tilde{\theta}_{j,D}$ be the bilinear interpolant of θ_j on D, i.e., $\tilde{\theta}_{j,D} \in Q_1(D)$ and $\tilde{\theta}_{j,D} = \theta_j$ at the vertices of D. We have the following standard interpolation error estimates [14, 11]:

$$\|\tilde{\theta}_{j,D}\|_{L_\infty(D)} \le \|\theta_j\|_{L_\infty(D)}, \tag{4.13}$$

$$\|\nabla\tilde{\theta}_{j,D}\|_{L_\infty(D)} \lesssim \|\nabla\theta_j\|_{L_\infty(D)}, \tag{4.14}$$

$$\|\nabla^2\tilde{\theta}_{j,D}\|_{L_\infty(D)} \lesssim \|\nabla^2\theta_j\|_{L_\infty(D)}, \tag{4.15}$$

$$\|\tilde{\theta}_{j,D} - \theta_j\|_{L_\infty(D)} \lesssim (\operatorname{diam} D)^2 \|\nabla^2\theta_j\|_{L_\infty(D)}. \tag{4.16}$$

It follows from (2.6), (2.7), (3.18), (4.12)–(4.16), a standard inverse estimate and the chain rule that

$$\begin{aligned} |v_j|^2_{H^2(D)} &\lesssim |\Pi_h(\tilde{\theta}_{j,D} w)|^2_{H^2(D)} + |\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)|^2_{H^2(D)} \\ &\lesssim |\tilde{\theta}_{j,D} w|^2_{H^2(D)} + (\operatorname{diam} D)^{-4} \|\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)\|^2_{L_2(D)} \\ &\lesssim \|\tilde{\theta}_{j,D}\|^2_{L_\infty(D)} |w|^2_{H^2(D)} + \|\nabla\tilde{\theta}_{j,D}\|^2_{L_\infty(D)} |w|^2_{H^1(D)} \\ &\qquad + \|\nabla^2\tilde{\theta}_{j,D}\|^2_{L_\infty(D)} \|w\|^2_{L_2(D)} + (\operatorname{diam} D)^{-4} \|\theta_j - \tilde{\theta}_{j,D}\|^2_{L_\infty(D)} \|w\|^2_{L_2(D)} \\ &\lesssim |w|^2_{H^2(D)} + \frac{1}{\delta^2} |w|^2_{H^1(D)} + \frac{1}{\delta^4} \|w\|^2_{L_2(D)}. \end{aligned} \tag{4.17}$$

Let $e \in \mathcal{E}_h$ be arbitrary. We have, from (4.12),

$$
\begin{aligned}
\frac{1}{|e|}\,\|[\![\partial v_j/\partial n]\!]\|_{L_2(e)}^2 &= \frac{1}{|e|}\,\|[\![\partial(\Pi_h w_j)/\partial n]\!]\|_{L_2(e)}^2 \\
&\lesssim \frac{1}{|e|}\,\|[\![\partial w_j/\partial n]\!]\|_{L_2(e)}^2 + \frac{1}{|e|}\,\|[\![\partial(\Pi_h w_j - w_j)/\partial n]\!]\|_{L_2(e)}^2 .
\end{aligned}
\tag{4.18}
$$

Using (4.8) we can estimate the first term on the right-hand side of (4.18) as follows:

$$
\begin{aligned}
\frac{1}{|e|}\,\|[\![\partial w_j/\partial n]\!]\|_{L_2(e)}^2 &= \frac{1}{|e|}\,\|[\![\partial(\theta_j w)/\partial n]\!]\|_{L_2(e)}^2 \\
&= \frac{1}{|e|}\,\|[\![\theta_j(\partial w/\partial n)]\!]\|_{L_2(e)}^2 \\
&\leq \frac{1}{|e|}\,\|[\![\partial w/\partial n]\!]\|_{L_2(e)}^2 \\
&\lesssim \frac{1}{|e|}\,\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \frac{1}{|e|}\,\|[\![\partial(I_0 v_0)/\partial n]\!]\|_{L_2(e)}^2 .
\end{aligned}
\tag{4.19}
$$

For the second term on the right-hand side of (4.18), we find from (2.6), (2.7), (3.17), (3.18), (4.8), the trace theorem (with scaling) and the chain rule that

$$
\begin{aligned}
&\frac{1}{|e|}\,\|[\![\partial(\Pi_h w_j - w_j)/\partial n]\!]\|_{L_2(e)}^2 \lesssim \sum_{D\in\mathcal{T}_e} \frac{1}{|e|}\,\|\partial(\Pi_h (w_j)_{{}_D} - (w_j)_{{}_D})/\partial n_e\|_{L_2(e)}^2 \\
&\quad\lesssim \sum_{D\in\mathcal{T}_e} (\operatorname{diam} D)^{-2}\,|\Pi_h w_j - w_j|_{H^1(D)}^2 + \sum_{D\in\mathcal{T}_e} |\Pi_h w_j - w_j|_{H^2(D)}^2 \\
&\quad\lesssim \sum_{D\in\mathcal{T}_e} |w_j|_{H^2(D)}^2 \\
&\quad\lesssim \sum_{D\in\mathcal{T}_e} \|\theta_j\|_{L_\infty(D)}^2 |w|_{H^2(D)}^2 + \sum_{D\in\mathcal{T}_e} \|\nabla\theta_j\|_{L_\infty(D)}^2 |w|_{H^1(D)}^2 \\
&\qquad\quad + \sum_{D\in\mathcal{T}_e} \|\nabla^2\theta_j\|_{L_\infty(D)}^2 \|w\|_{L_2(D)}^2 \\
&\quad\lesssim \sum_{D\in\mathcal{T}_e} \Big[|w|_{H^2(D)}^2 + \frac{1}{\delta^2}|w|_{H^1(D)}^2 + \frac{1}{\delta^4}\|w\|_{L_2(D)}^2 \Big] .
\end{aligned}
\tag{4.20}
$$

Combining (4.18)–(4.20), we have

$$
\begin{aligned}
\frac{1}{|e|}\,\|[\![\partial v_j/\partial n]\!]\|_{L_2(e)}^2 &\lesssim \frac{1}{|e|}\,\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 + \frac{1}{|e|}\,\|[\![\partial(I_0 v_0)/\partial n]\!]\|_{L_2(e)}^2 \\
&\qquad + \sum_{D\in\mathcal{T}_e} \Big[|w|_{H^2(D)}^2 + \frac{1}{\delta^2}|w|_{H^1(D)}^2 + \frac{1}{\delta^4}\|w\|_{L_2(D)}^2 \Big] .
\end{aligned}
\tag{4.21}
$$

We can now conclude from (2.3), (2.11), (2.14), (3.20), (4.7), (4.9)–(4.11), (4.17) and (4.21) that

$$
\begin{aligned}
\sum_{j=1}^{J} |v_j|^2_{H^2(\Omega_j,\mathcal{T}_h)} &\approx \sum_{j=1}^{J} \Big(\sum_{\substack{D\in\mathcal{T}_h\\ D\subset\Omega_j}} |v_j|^2_{H^2(D)} + \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\bar{\Omega}_j}} \frac{1}{|e|} \|[\![\partial v_j/\partial n]\!]\|^2_{L_2(e)} \Big) \\
&\lesssim |v|^2_{H^2(\Omega,\mathcal{T}_h)} + |I_0 v_0|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{D\in\mathcal{T}_h} \left[|w|^2_{H^2(D)} + \frac{1}{\delta^2}|w|^2_{H^1(D)} + \frac{1}{\delta^4}\|w\|^2_{L_2(D)} \right] \\
&\lesssim |v|^2_{H^2(\Omega,\mathcal{T}_h)} + \frac{H^2}{\delta^2}|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \frac{H^4}{\delta^4}|v|^2_{H^2(\Omega,\mathcal{T}_h)} \\
&\lesssim \left(1 + \frac{H^4}{\delta^4}\right) |v|^2_{H^2(\Omega,\mathcal{T}_h)}.
\end{aligned}
\tag{4.22}
$$

Combining (2.2), (2.10), (4.7) and (4.22) we arrive at the estimate

$$
\langle A_0 v_0, v_0\rangle + \sum_{j=1}^{J} \langle A_j v_j, v_j\rangle \lesssim \left(1 + \frac{H^4}{\delta^4}\right) \langle A_h v, v\rangle,
$$

which together with (4.6) implies

$$
\min_{\substack{v=\sum_{j=0}^{J} v_j\\ v_j\in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle \lesssim \left(1 + \frac{H^4}{\delta^4}\right) \langle A_h v, v\rangle. \tag{4.23}
$$

Since $v \in V_h$ is arbitrary, the estimate (4.3) follows from (2.19) and (4.23). □

From Lemma 4.1 and Lemma 4.2 we have the following condition number estimate for the two-level additive Schwarz preconditioner.

Theorem 4.3. *The condition number of BA_h satisfies the estimate*

$$
\kappa(BA_h) = \frac{\lambda_{\max}(BA_h)}{\lambda_{\min}(BA_h)} \le C \left(1 + \frac{H^4}{\delta^4}\right), \tag{4.24}
$$

where the positive constant C depends on the shape regularity of $\mathcal{T}_h$ and $\mathcal{T}_H$ but not h, H, δ nor J.

5. The Case of Small Overlap

Theorem 4.3 implies in particular that the two-level additive Schwarz preconditioner is an optimal preconditioner when δ is comparable to H (the case of generous overlap). In the case of a small overlap, i.e. when $\delta << H$, the factor $[1 + (H/\delta)^4]$ in Theorem 4.3 becomes significant. In this section we show that it can be improved to $[1 + (H/\delta)^3]$ provided that we have more information on the subdomains Ω_j. The arguments we use follow those in [18] for conforming finite elements.

More precisely, we assume, in addition to (2.5)–(2.7), that

(5.1) the subdomains are rectangles,

(5.2) $h \leq \delta \ll H$ and in particular 2δ is less than the length of any edge of the subdomains,

$$\Omega_i \cap \Omega_j \subset \{(x_1,x_2) \in \Omega_i : \mathrm{dist}\,((x_1,x_2),\partial\Omega_i) < 2\delta\} \cap \{(x_1,x_2) \in \Omega_j : \mathrm{dist}\,((x_1,x_2),\partial\Omega_j) < 2\delta\}. \tag{5.3}$$

Remark 5.1. These additional assumptions on the subdomains are valid if the subdomains are constructed according to Remark 2.1.

The proof of the following lemma can be found in [9].

Lemma 5.2. *Let $d < l$ be two positive numbers, $G = \{(x_1,x_2) : 0 < x_1 < l,\ 0 < x_2 < d\}$ and $Q = \{(x_1,x_2) : 0 < x_1 < l,\ 0 < x_2 < l\}$. Then there exists a positive constant C independent of d and l such that*

$$\|\zeta\|^2_{L_2(G)} \leq C\frac{d}{l}\left(\|\zeta\|^2_{L_2(Q)} + l^4|\zeta|^2_{H^2(Q)}\right) \qquad \forall\,\zeta \in H^2(Q). \tag{5.4}$$

We now return to the proof of Lemma 4.2.

Let $\Omega_{j,\epsilon}$ and $\Omega_{j,\epsilon,h}$ be defined by

$$\begin{aligned}\Omega_{j,\epsilon} &= \{(x_1,x_2) \in \Omega_j : \mathrm{dist}((x_1,x_2),\partial\Omega_j) < \epsilon\},\\ \Omega_{j,\epsilon,h} &= \text{ the union of all } D \in \mathcal{T}_h \text{ such that } D \subset \Omega_{j,\epsilon}.\end{aligned}$$

Conditions (2.5), (2.6) and (5.3) imply that θ_j is identically one in the rectangle $\{(x_1,x_2) \in \Omega_j : \mathrm{dist}((x_1,x_2),\partial\Omega_j) \geq 2\delta\}$. Thus the terms $\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)$, $\nabla\tilde{\theta}_{j,D}$, $\nabla^2\tilde{\theta}_{j,D}$, $\nabla\theta_j$ and $\nabla^2\theta_j$ in (4.17) vanish except for those D which are subsets of $\Omega_{j,2\delta}$.

Therefore, derivations similar to (4.17) and (4.20) yield

$$\begin{aligned}\sum_{\substack{D\subset\Omega_j\\ D\in\mathcal{T}_h}} |v_j|^2_{H^2(D)} &= \sum_{\substack{D\subset\Omega_j\\ D\in\mathcal{T}_h}} \|\tilde{\theta}_{j,D}\|^2_{L_\infty(D)}|w|^2_{H^2(D)} + \sum_{\substack{D\subset\Omega_{j,2\delta}\\ D\in\mathcal{T}_h}} \Big[\|\nabla\tilde{\theta}_{j,D}\|^2_{L_\infty(D)}|w|^2_{H^1(D)}\\ &\qquad + \|\nabla^2\tilde{\theta}_{j,D}\|^2_{L_\infty(D)}\|w\|^2_{L_2(D)} + (\mathrm{diam}\,D)^{-4}\|\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)\|^2_{L_2(D)}\Big]\\ &\lesssim |w|^2_{H^2(\Omega_j,\mathcal{T}_h)} + \frac{1}{\delta^2}|w|^2_{H^1(\Omega_{j,2\delta})} + \frac{1}{\delta^4}\|w\|^2_{L_2(\Omega_{j,2\delta})},\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}&\sum_{\substack{e\in\mathcal{T}_h\\ e\subset\bar{\Omega}_j}} \frac{1}{|e|}\|[\![\partial(\Pi_h w_j - w_j)/\partial n]\!]\|^2_{L_2(e)}\\ &\quad\lesssim \sum_{\substack{D\subset\Omega_j\\ D\in\mathcal{T}_h}} \|\theta_j\|^2_{L_\infty(D)}|w|^2_{H^2(D)}\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&+ \sum_{\substack{D\subset\Omega_{j,2\delta}\\ D\in\mathcal{T}_h}} \Big[\|\nabla\theta_j\|^2_{L_\infty(D)}|w|^2_{H^1(D)} + \|\nabla^2\theta_j\|^2_{L_\infty(D)}\|w\|^2_{L_2(D)}\Big]\\
&\lesssim |w|^2_{H^2(\Omega_j,\mathcal{T}_h)} + \frac{1}{\delta^2}|w|^2_{H^1(\Omega_{j,2\delta})} + \frac{1}{\delta^4}\|w\|^2_{L_2(\Omega_{j,2\delta})}.
\end{aligned}
$$

Recall that $E_h w \in \tilde{V}_h \subseteq H^2(\Omega_j)$. Hence it follows from Lemma 3.3, (3.27), (5.1), (5.2), Lemma 5.2 (with scaling) and a standard inverse estimate that

$$
\begin{aligned}
\|w\|^2_{L_2(\Omega_{j,2\delta})} &= \|\Pi_h E_h w\|^2_{L_2(\Omega_{j,2\delta})}\\
&\lesssim \|\Pi_h E_h w - E_h w\|^2_{L_2(\Omega_{j,2\delta})} + \|E_h w\|^2_{L_2(\Omega_{j,2\delta})}\\
&\lesssim h^4|E_h w|^2_{H^2(\Omega_{j,2\delta})} + \|E_h w\|^2_{L_2(\Omega_{j,2\delta})}\\
&\lesssim \|E_h w\|^2_{L_2(\Omega_{j,2\delta})}\\
&\lesssim \frac{\delta}{H}\big[\|E_h w\|^2_{L_2(\Omega_j)} + H^4|E_h w|^2_{H^2(\Omega_j)}\big].
\end{aligned}
\tag{5.7}
$$

In view of (5.5)–(5.7), the estimate (4.22) becomes

$$
\sum_{j=1}^J |v_j|^2_{H^2(\Omega_j,\mathcal{T}_h)} \lesssim |v|^2_{H^2(\Omega,\mathcal{T}_h)} + \frac{H^2}{\delta^2}|v|^2_{H^2(\Omega,\mathcal{T}_h)} + \frac{1}{\delta^3 H}\big[\|E_h w\|^2_{L_2(\Omega)} + H^4|E_h w|^2_{H^2(\Omega)}\big]. \tag{5.8}
$$

Furthermore, from Lemma 3.7, (4.9), (4.11), (5.2) and a standard inverse estimate, we have

$$
\begin{aligned}
\|E_h w\|^2_{L_2(\Omega)} + H^4|E_h w|^2_{H^2(\Omega)} &\lesssim \|E_h w - w\|^2_{L_2(\Omega)} + \|w\|^2_{L_2(\Omega)} + H^4|E_h w|^2_{H^2(\Omega)}\\
&\lesssim h^4|w|^2_{H^2(\Omega,\mathcal{T}_h)} + \|w\|^2_{L_2(\Omega)} + H^4|w|^2_{H^2(\Omega,\mathcal{T}_h)}\\
&\lesssim H^4|v|^2_{H^2(\Omega,\mathcal{T}_h)}.
\end{aligned}
\tag{5.9}
$$

Combining (5.8) and (5.9) we find

$$
\sum_{j=1}^J |v_j|^2_{H^2(\Omega_j,\mathcal{T}_h)} \lesssim \left(1+\frac{H^3}{\delta^3}\right)|v|^2_{H^2(\Omega,\mathcal{T}_h)},
$$

and (4.23) becomes

$$
\min_{\substack{v=\sum_{j=0}^J I_j v_j\\ v_j\in V_j}} \sum_{j=0}^J \langle A_j v_j, v_j\rangle \lesssim \left(1+\frac{H^3}{\delta^3}\right)\langle A_h v, v\rangle. \tag{5.10}
$$

Finally (2.19) and (5.10) yield a new lower bound for $\lambda_{\min}(BA_h)$:

$$
\lambda_{\min}(BA_h) \gtrsim \left(1+\frac{H^3}{\delta^3}\right)^{-1}. \tag{5.11}
$$

The second condition number estimate for the two-level additive Schwarz preconditioner now follows from (4.1) and (5.11).

Theorem 5.3. *Under the additional assumptions* (5.1)–(5.3), *we have the following improved bound on the condition number of* BA_h :

$$\kappa(BA_h) \leq C\left(1 + \frac{H^3}{\delta^3}\right),$$

where the positive constant C *depends on the shape regularity of* $\mathcal{T}_h$ *and* $\mathcal{T}_H$ *but not* h, H, δ *nor* J.

6. Numerical Experiments

In this section we present the results of some numerical experiments for the biharmonic problem on the unit square. The penalty parameter η in $\mathcal{A}_h$, $\mathcal{A}_H$ and $\mathcal{A}_j$ is taken to be 5.

In the first set of experiments we take the coarse space V_H to be the Q_1 finite element space associated with $\mathcal{T}_H$, for $H = 2^{-1}$, 2^{-2} and 2^{-3}. The corresponding overlapping domain decomposition has $J = 4$, 16 and 64 subdomains (cf. Remark 2.1 for the construction of the domain decomposition). For each choice of H and h, we generate a vector $v_h \in V_h$ randomly, compute the right-hand side vector $g = A_h v_h$ and apply the preconditioned conjugate gradient algorithm to the equation $A_h z = g$ using the two-level additive Schwarz preconditioner. The number of iterations needed for reducing the energy norm error by a factor of 10^{-2} is computed for 5 random choices of v_h and then averaged. The results are reported in Table 1, Table 2 and Table 3. They demonstrate that the condition number $\kappa(BA_h)$ is independent of h.

δ \ h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-2}	5	4.8	4	4	4	4
2^{-3}	—	5	4	4	4	4
2^{-4}	—	—	5.2	4	4	4
2^{-5}	—	—	—	4	4	4

Table 1. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_1 coarse space, $J = 4$, $H = 1/2$)

In Table 4 we collect the number of iterations for $h = 2^{-6}$ according to J and H/δ. They demonstrate the independence of $\kappa(BA_h)$ on J and at the same time the adverse effect of the increase in H/δ.

δ \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-3}	6	5.6	5	5	5
2^{-4}	—	5.4	5	4	4
2^{-5}	—	—	5.2	4	4

TABLE 2. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_1 coarse space, $J = 16$, $H = 1/4$)

δ \ h	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-4}	6	6	5	5
2^{-5}	—	6	5	4

TABLE 3. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_1 coarse space, $J = 64$, $H = 1/16$)

In the second set of experiments we take the coarse space V_H to be the Q_2 finite element space associated with $\mathcal{T}_H$, and carry out similar computations. The results are reported in Table 5, Table 6, Table 7 and Table 8. They are very similar to the results for the first set of experiments.

In the last experiment we take H to be $1/4$, V_H to be the Q_1 finite element space associated with $\mathcal{T}_H$ and replace the operator I_0 by the natural injection operator from V_H into V_h. The results are reported in Table 9. They show that the performance of the two-level preconditioner suffers from the absence of the enriching operator (cf. Remark 3.5).

7. CONCLUDING REMARKS

We have demonstrated that the two-level additive Schwarz preconditioner can be extended to C^0 interior penalty methods for fourth order elliptic boundary value problems. The preconditioned system behaves in the same way as the preconditioned system for classical conforming and nonconforming finite element methods [8, 9, 10, 7]. The novelty of the

H/ δ \ J	4	16	64	256
2	4	5	5	5
4	4	4	5	6
8	4	4	5.4	—

TABLE 4. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_1 coarse space, $h = 2^{-6}$)

δ \ h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-2}	5.2	5	4	4	4	4
2^{-3}	—	5	4	4	4	4
2^{-4}	—	—	4.6	4	4	4
2^{-5}	—	—	—	4	4	4

TABLE 5. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_2 coarse space, $J = 4$, $H = 1/2$)

analysis is in the role played by the jumps of the normal derivatives across the edges (see for example (3.12)–(3.14), (3.19), (4.18)–(4.21) and (5.6)), and the enriching map developed in [12] for the post-processing of the solutions of C^0 interior penalty methods is useful for both the construction and the analysis of the preconditioners.

The results of this paper can be extended to C^0 interior penalty methods that are based on triangular or convex quadrilateral finite elements [12]. The key again is to use the enriching maps that connect C^0 finite elements to their C^1 relatives, which are the triangular Argyris elements [2] or the quadrilateral generalized Bogner-Fox-Schmit elements [12].

δ \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-3}	6	6	5	5	5
2^{-4}	—	5	5	4	4
2^{-5}	—	—	5	4	4

Table 6. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_2 coarse space, $J = 16$, $H = 1/4$)

δ \ h	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-4}	6	6	5	5
2^{-5}	—	5	5	4

Table 7. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_2 coarse space, $J = 64$, $H = 1/16$)

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H/ δ \ J	4	16	64	256
2	4	5	5	5
4	4	4	5	5
8	4	4	5	—

TABLE 8. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_2 coarse space, $h = 2^{-6}$)

δ \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-3}	8	9	9.8	8	9
2^{-4}	—	10.4	10.4	9.6	7.8
2^{-5}	—	—	11.8	11.2	7.8

TABLE 9. Average number of iterations for reducing the energy norm error by a factor of 10^{-2} (Q_1 coarse space without enriching operator, $J = 16$, $H = 1/4$)

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