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# Finite Element Approximation of the Cahn-Hilliard Equation on Surfaces

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**Abstract**

In this paper, we consider the phase separation on general surfaces by solving the nonlinear Cahn-Hilliard equation using a finite element method. A fully discrete approximation scheme is introduced, and we establish a priori estimates for the discrete solution that does not rely on any knowledge of the exact solution beyond the initial time. This in turn leads to convergence and optimal error estimates of the discretization scheme. Numerical examples are also provided to demonstrate how the scheme can be effectively implemented.

## 1 Introduction

The Cahn-Hilliard equation introduced in [8] is a very general mathematical model that describes phase separations. It has found many applications in various fields, such as foams modeling, solidification processes, dendritic flow, image processing, planet formation and so on [5, 9, 17, 25, 27, 30, 34, 36, 40]. The phase separation processes have been successfully investigated with the Cahn-Hilliard equation in a wide variety of non-equilibrium systems. There have been many algorithms and simulations performed using a variety of discretization methods including finite difference, finite volume, finite element and spectral methods, see, e.g., [2, 3, 9, 10, 21, 23, 24, 28, 30, 31, 38, 39] and the references cited therein.

Various experimental studies have shown that interesting phase separations could occur on static or dynamic surfaces, such as phase separation on lipid bilayer membranes, crystal growth on curved surfaces, and phase separations within thin films, see [1, 4, 20, 35]. Thus, theoretical analysis and numerical implementation of the phase transition models on general surfaces are attracting more and more attentions. For instance, a finite volume method for Cahn-Hilliard equations on the sphere was studied in [37], the numerical approximations of the Ginzburg-Landau model for a superconducting hollow sphere were studied using a gauge invariant finite volume discretization on a spherical centroidal Voronoi tesselation [15]. The finite element method has been used For the discretization of partial differential equations (PDEs) defined on surfaces including the Cahn-Hilliard equation, finite element methods have been studied in [11, 12, 20].

Development of fully discrete approximation schemes for nonlinear PDEs is important because these schemes not only directly reduce differential equations to systems of algebraic equations, but also suggest what kinds of ordinary differential equation solvers are needed for the semi-discrete approximation schemes. For fully discrete approximations of the Cahn-Hilliard equation that are

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most relevant to the work presented here, let us mention that in [16], Du and Nicolaides proposed and analyzed a fully discrete approximation scheme for one dimensional Cahn-Hilliard equations. One of its features is the existence of a Lyapunov functional associated with the approximation scheme. This leads to some estimates for the discrete solution in a certain Sobolev space. Combining these with Sobolev imbedding theorem in one space dimension, they are able to prove the pointwise boundedness of the discrete solution, and consequently, they obtain the Lipschitz property of the nonlinear term, which guarantees the existence and uniqueness of discrete solution and make the error analysis possible. This idea was studied further in [26]. We note that a pointwise estimate for solutions of fully discrete schemes is important for the mathematical analysis, since, as pointed out in [33], the linear part of the equation does not always control the nonlinear term automatically. Nevertheless, in higher dimensional spaces, the imbedding theorem used on in [16] is no longer valid and pointwise boundedness does not follow directly from the existence of the Lyapunov functional. Thus, new a priori estimates for the discrete solution are needed.

In this paper, we demonstrate well-posedness and convergence of a fully discrete finite element approximation scheme of the Cahn-Hilliard equation defined on a general surface. Our approach requires a combination of both standard and nonstandard techniques due to the lack of maximum principle for fourth-order equations. On one hand, the approach is similar to the usual arguments for establishing a priori estimates of discrete solutions of PDEs. For example, some estimates for the discrete potential function are provided first, then we apply the idea of elliptic regularity to get high order estimates. On the other hand, we present a more delicate analysis of the initial approximation in our derivation. This type of analysis is an important part of the estimation, but it appears rarely in the literature on the approximation of non-linear or semi-linear parabolic type equations. It is done here through some technical discussions on the approximation space.

We now give an outline of this paper: In Sections 2 and 3, we present the model problem on general surfaces, and the fully discrete finite element approximation scheme. Section 4 contains some properties of the initial approximation, some estimates on the discrete chemical potential $p$, and then the desired pointwise boundedness of the discrete solution. Then we give the error analysis in Section 5. Finally, some numerical experiments are presented in Section 6.

## 2 Model Problem

Before setting up the model, we introduce some basic notations first. Given an open connected and bounded $C^{k,\alpha}$ surface $\mathbf{S}$ in $\mathbb{R}^3$ with $k \in \mathcal{N} \cup \{0\}$ and $0 \leq \alpha < 1$, we assume that it can be represented globally by some oriented distance function (level set function) $d = d(\mathbf{x})$ defined in an open subset $\Omega$ in $\mathbb{R}^3$, such that $\mathbf{S} = \{\mathbf{x} \in \Omega \mid d(\mathbf{x}) = 0\}$ with $d \in C^{k,\alpha}$ and $\nabla d \neq 0$ in $\Omega$ with $\nabla$ being the standard gradient operator in $\mathbb{R}^3$. Moreover, we assume that on a strip (band)

$$\mathbf{U} = \{\mathbf{x} \in \mathbb{R}^3 \quad | \quad d(\mathbf{x}) < \delta\}, \qquad \text{for some } \delta > 0,$$

around $\mathbf{S}$, there is a unique decomposition for any $\mathbf{x} \in \mathbf{U}$,

$$\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{p}(\mathbf{x})) \tag{2.1}$$

with $\mathbf{p}(\mathbf{x}) \in \mathbf{S}$ and $\vec{\mathbf{n}}(\mathbf{p}(\mathbf{x}))$ being the unit outward normal to the surface $\mathbf{S}$ at $\mathbf{p}(\mathbf{x})$. The parameter $\delta$ is usually determined by the surface curvatures if $\mathbf{S}$ is sufficiently smooth. Without loss of generality, we assume that $|\nabla d| \equiv 1$ in $\mathbf{U}$. Let $\nabla_s = (\nabla_{s,1}, \nabla_{s,2}, \nabla_{s,3}) = \nabla - (\vec{\mathbf{n}} \cdot \nabla)\vec{\mathbf{n}}$ denote the tangential (surface) gradient operator, and $\Delta_s = \nabla_s \cdot \nabla_s$ be the so-called Laplace-Beltrami operator on $\mathbf{S}$. We use the standard notation for $L^q(\mathbf{S})$ on $\mathbf{S}$, and we define the Sobolev spaces as follows:

$$W^{m,q}(\mathbf{S}) = \{u \in L^q(\mathbf{S}) \mid u \text{ possesses weak tangential derivatives up to order } m$$

which are in $L^q(\mathbf{S})\}$ .

We denote, in addition, that $H^m(\mathbf{S}) = W^{m,2}(\mathbf{S})$ on $\mathbf{S}$. To make the space $H^m(\mathbf{S})$ well defined, it is customary to assume $k + \alpha \geq \max\{1, m\}$ [29]. To avoid technical complication, we further assume that $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth (say with $k = 4$) and $\partial\mathbf{S} \neq \varnothing$ for the rest of the paper unless stated otherwise.

To introduce the Cahn-Hilliard equation, we begin with the free energy functional

$$I(u) = \int_{\mathbf{S}} \left\{ \mathcal{H}(u(\mathbf{x})) + \frac{\sigma}{2} |\nabla_s u(\mathbf{x})|^2 \right\} ds, \tag{2.2}$$

for any $u = u(x) \in H^1(\mathbf{S})$ with $\mathcal{H}$ being the bulk free energy density, and a positive constant $\sigma > 0$ which symbolizes the so-called diffuse interfacial width. Let $u = u(\mathbf{x}, t)$ be a function for $\mathbf{x}$ on $\mathbf{S}$ at time $t$ which, in the original work of [7, 8], denotes the concentration of one species of the binary mixture. The chemical potential of the system is then of the form

$$p = \frac{\partial I}{\partial u} = \mathcal{H}'(u) - \sigma \Delta_s u = \phi(u) - \sigma \Delta_s u. \tag{2.3}$$

By assuming a constant unit mobility, we get the dynamic equation

$$u_t = \Delta_s p, \tag{2.4}$$

which leads to a simple form of the Cahn-Hilliard equation as

$$u_t = \Delta_s(\phi(u) - \sigma \Delta_s u). \tag{2.5}$$

As a model case, the function $\phi$ is assumed to be of the form:

$$\phi(u) = \mathcal{H}'(u) = \gamma_2 u^3 + \gamma_1 u^2 + \gamma_0 u, \tag{2.6}$$

where $\gamma_0$, $\gamma_1$ and $\gamma_2$ are given constants and $\gamma_2 > 0$ is assumed. Here we study the homogenous Dirichlet type boundary value problem for both the concentration and the chemical potential, that is,

$$u(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \partial\mathbf{S}, \tag{2.7}$$

and

$$p(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \partial\mathbf{S}. \tag{2.8}$$

The initial condition for $u$ is given by

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in \mathbf{S}. \tag{2.9}$$

In order to introduce numerical discretization of the above equations, we first uniquely extend the functions defined on $\mathbf{S}$ to $\mathbf{U}$. That is, given a function $u$ defined on $\mathbf{S}$, its extension in $\mathbf{U}$ is given by

$$u^l(\mathbf{x}) = u(\mathbf{p}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{U}. \tag{2.10}$$

The same extension can be done for the unit normal. For simplicity, we still use the same notation for the extension, that is, for $\mathbf{x} \in \mathbf{U}$, we simply let $\vec{\mathbf{n}}(\mathbf{x}) = \vec{\mathbf{n}}(\mathbf{p}(\mathbf{x}))$.

Let $\mathbf{S}^h$ be a polyhedral approximation to $\mathbf{S}$ having triangular faces, we assume that for each point $\mathbf{y} \in \mathbf{S}$ there is at most one point $\mathbf{x} \in \mathbf{S}^h$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{y}$ as suggested in [20], and vertices of each triangular face of $\mathbf{S}^h$ are on $\mathbf{S}$.

We then do the similar extension from $\mathbf{S}^h$ to $\mathbf{U}$. Given a function $u^h$ defined on $\mathbf{S}^h$, first project it onto $\mathbf{S}$ by $\tilde{u}^h(\mathbf{p}(\mathbf{y})) = u^h(\mathbf{y})$ for $\mathbf{y} \in \mathbf{S}^h$, then we apply (2.10) again to extend $\tilde{u}^h$ to $\mathbf{U}$, i.e.,

$$u^{h,l}(\mathbf{x}) = \tilde{u}^h(\mathbf{p}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{U}. \tag{2.11}$$

We may equivalently write $u^{h,l}(\mathbf{x}) = u^h(\mathbf{y})$ for any pair of $\mathbf{x} \in \mathbf{U}$ and $\mathbf{y} \in \mathbf{S}^h$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{p}(\mathbf{y}) \in \mathbf{S}$. Note that all extensions of functions to $\mathbf{U}$ are constant along normals to $\mathbf{S}$, thus, extensions of functions defined on $\mathbf{S}$ and those on $\mathbf{S}^h$ share much the same properties.

We use $ds(\mathbf{x})$ and $ds_h(\mathbf{y})$ to denote the area elements of $\mathbf{S}$ and $\mathbf{S}^h$ respectively at the points $\mathbf{x} \in \mathbf{S}$ and $\mathbf{y} \in \mathbf{S}^h$. Let

$$\mu_h(\mathbf{p}(\mathbf{x})) = \frac{ds(\mathbf{p}(\mathbf{x}))}{ds_h(\mathbf{x})}, \tag{2.12}$$

for any $\mathbf{x} \in \mathbf{S}^h$. We then assume, since $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth, that

$$|1 - \mu_h(\mathbf{x})| \le ch^2, \tag{2.13}$$

where $h$ is the mesh size parameter. Moreover, here and in the sequel, $c$ is used to denote a generic postive constant which is independent of $h$ as $h \to 0$, that is, may take different values but remain uniformly bounded as the discretization gets refined. We note that the assumption (2.13) is generally true for regular and quasi-uniform triangulations of a smooth surface $\mathbf{S}$ [19].

# 3 The Fully Discrete Approximation Scheme

Let $\mathcal{U}$ denote a finite dimensional subspace of $H^1(\mathbf{S}^h)$. In the context of finite element approximations, we take $\mathcal{U}$ to be a continuous finite element space with respect to certain triangulation of the general surface $\mathbf{S}$ with mesh size parameter $h$. In this paper, we take $\mathcal{U}$ to be piecewise linear function space for simplicity. Let $(0, T)$ be the time interval of interest, which is discretized into $N$ subintervals, each with a step size $\Delta t = T/N$. The choice of a uniform time step size is not essential to our following discussion.

For notational convenience, for any $v, w \in H_0^1(\mathbf{S})$, and any $V^h, W^h \in H_0^1(\mathbf{S})$, we let

$$(v, w)_s = \int_{\mathbf{S}} v(\mathbf{x}) \cdot w(\mathbf{x}) ds\ , \quad (V^h, W^h)_{s_h} = \int_{\mathbf{S}^h} V^h(\mathbf{x}) W^h(\mathbf{x}) ds_h\ .$$

To approximate the nonlinear term in the equation, we define the function $\tilde{\phi} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{\phi}(x, y) = \begin{cases} (\mathcal{H}(x) - \mathcal{H}(y))/(x - y), & \text{if } x \neq y\ , \\ \phi(x), & \text{if } x = y\ , \end{cases} \tag{3.1}$$

where $\mathcal{H}$ is as in equation (2.2). With this notation, we obtain the following fully discrete scheme: find $(U_n^h, P_{n-1}^h) \in \mathcal{U} \times \mathcal{U}$, $n = 1, 2, \ldots, N$, such that for any $V^h, W^h \in \mathcal{U}$:

$$U_0^h = u_0^{h,l}, \tag{3.2}$$
$$(\delta_t U_n^h, V^h)_{s_h} + (\nabla_{s_h} P_n^h, \nabla_{s_h} V^h)_{s_h} = 0, \tag{3.3}$$
$$-(P_n^h, W^h)_{s_h} + \sigma(\nabla_{s_h} U_{n+1/2}^h, \nabla_s W^h)_{s_h} + (\tilde{\phi}(U_n^h, U_{n+1}^h), W^h)_{s_h} = 0, \tag{3.4}$$

where $u_0^{h,l}$ is an approximation to the initial condition $u_0$ onto $\mathbf{S}^h$ given by the $H^1$ projection,

$$U_{n+1/2}^h = \frac{U_n^h + U_{n+1}^h}{2}, \quad \text{and} \quad \delta_t U_n^h = \frac{U_{n+1}^h - U_n^h}{\Delta t}.$$

Similar notations are also used for $P_n^h$.

For (3.2)-(3.4), we can define the Lyapunov functional: $\forall\, U^h \in \mathcal{U}$, let

$$I^h(U^h) = \int_{\mathbf{S}^h} \left\{ \mathcal{H}(U^h) + \frac{\sigma}{2} |\nabla_{s_h} U^h|^2 \right\} ds_h,$$

thus we have the following lemma:

**Lemma 1** *For $n = 1, 2, \ldots, N$, we have*

$$\frac{I^h(U_{n+1}^h) - I^h(U_n^h)}{\Delta t} + \|\nabla_{s_h} P_n^h\|_0^2 = 0. \tag{3.5}$$

**Proof:** For equations (2.2), (3.3) and (3.4), we have

$$\begin{aligned} I^h(U_{n+1}^h) - I^h(U_n^h) &= \sigma \Delta t (\nabla_{s_h} U_{n+1/2}^h, \nabla_{s_h} \delta_t U_n^h)_{s_h} \\ &\quad + \Delta t (\tilde{\phi}(U_n^h, U_{n+1}^h), \delta_t U_n^h)_{s_h} \\ &= \Delta t (P_n^h, \delta_t U_n^h)_{s_h} \\ &= -\Delta t (\nabla_{s_h} P_n^h, \nabla_{s_h} P_n^h)_{s_h} \\ &= -\Delta t \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2. \end{aligned}$$

Hence, we obtain (3.5) which proves the lemma. □

One can easily verify that Lemma 1 implies the following theorem:

**Theorem 1** *Let $u_0(\mathbf{x}) \in H_0^1(\mathbf{S})$. Assume that there is a constant $c > 0$, independent of $h$, such that $\|u_0^{h,l}\|_{H^1(\mathbf{S})} \le c \|u_0\|_{H^1(\mathbf{S})}$. Then, the solution $(U_n^h, P_{n-1}^h)$ of (3.3) and (3.4) satisfies for $n = 1, 2, \ldots, N$,*

$$\|U_n^h\|_{H^1(\mathbf{S}^h)} \le c, \tag{3.6}$$

*and*

$$\sum_{j=0}^{n} \Delta t \|\nabla_{s_h} P_j^h\|_{L^2(\mathbf{S}^h)}^2 \le c, \tag{3.7}$$

*where the generic constant $c$ in the above two equations is independent of $h$, $n$, $\Delta t$ and $N$.*

**Proof:** By summing up (3.5) over $n$, we get (3.7) and also that $I^h(U_n^h) \le c$ for some generic constant $c$. It is easy to establish the coercivity of the functional $I^h$ in $H^1(\mathbf{S}^h)$ which then gives (3.6). □

# 4 Pointwise Boundedness of Discrete Solutions

As stated in [16], one needs to prove the pointwise boundness of $U_n^h$ for any $n$, such that $\tilde{\phi}(U_n^h, U_{n+1}^h)$ becomes Lipschitz continuous with some Lipschitz constant independent of $h$, $n$ and $\Delta t$, then the error estimates of the proposed fully discrete finite element scheme can be analyzed in a standard manner. Since the imbedding theorem used in [16] is no longer valid in the manifold case, we see that the pointwise boundedness does not follow directly from the existence of the Lyapunov functional. Thus, some further estimates are needed.

### 4.1 Some technical lemmas

We first present some technical results that are of later use.

For any $u \in H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})$, we define the projection of $u$, say $\Pi^h u$, onto the finite element space $\mathcal{U}$ as follows: let $f = -\Delta_s u \in L^2(\mathbf{S})$, then $\Pi^h u$ is defined by

$$(\nabla_{s_h} \Pi^h u, \nabla_{s_h} W^h)_{s_h} = (f^l, W^h)_{s_h}, \quad \forall\, W^h \in \mathcal{U}.$$

As discussed in [19], one can easily deduce that if $u \in H^2(\mathbf{S})$, $\Pi^h u$ actually is the discrete solution of $-\Delta_s u = f$ over triangulation $\mathbf{S}^h$ and one has the following energy norm error estimate: for $u \in H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})$, there exists a generic constant $c > 0$, such that

$$\|u - (\Pi^h u)^l\|_{H^1(\mathbf{S})} \le ch\|u\|_{H^2(\mathbf{S})}. \tag{4.1}$$

Furthermore, concerning the extension defined earlier, we have

**Lemma 2** *For any $u, v \in H^1(\mathbf{S})$, there exist generic constants $c_1, c_2 > 0$ such that*

$$c_1\|u - v\|_{H^1(\mathbf{S})} \le \|u^l - v^l\|_{H^1(\mathbf{S}^h)} \le c_2\|u - v\|_{H^1(\mathbf{S})}. \tag{4.2}$$

**Proof:** From the definition of $\|\cdot\|_{H^1(S^h)}$, we have

$$\begin{aligned} \|u^l - v^l\|_{H^1(S^h)} &= \left(\int_{\mathbf{S}^h} |u^l - v^l|^2 ds_h\right)^{1/2} + \left(\int_{\mathbf{S}^h} |\nabla_{s_h}(u^l - v^l)|^2 ds_h\right)^{1/2} \\ &= \left(\int_{\mathbf{S}} \frac{1}{\mu_h}|u - v|^2 ds\right)^{1/2} + \left(\int_{\mathbf{S}} \frac{1}{\mu_h}|\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\nabla_s(u - v)|^2 ds\right)^{1/2}, \end{aligned} \tag{4.3}$$

where $\mathbf{P}_h(\mathbf{x}) = \mathbf{I} - \vec{\mathbf{n}}_h(\mathbf{x}) \otimes \vec{\mathbf{n}}_h(\mathbf{x})$, $\mathbf{H} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the Weingarten map [11].

From the discussions in [19], we know that when $h$ is efficiently small, there always exists a generic constant $c_0 >$ such that

$$\frac{1}{c_0} \le \mu_h \le c_0, \qquad \frac{1}{c_0} \le \mathbf{P}_h(\mathbf{I} - d\mathbf{H}) \le c_0,$$

which implies the existence of generic constants $c_1, c_2 > 0$ satisfying (4.2). $\square$

**Lemma 3** *For $\epsilon \in (0, 1)$, there exist some generic constants $c_1, c_2 > 0$ such that for any $U \in H^1(\mathbf{S}^h)$,*

$$c_1\|U^l\|_{W^{1,2+\epsilon}(\mathbf{S})} \le \|U\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \le c_2\|U^l\|_{W^{1,2+\epsilon}(\mathbf{S})}\ . \tag{4.4}$$

**Proof:** Since the surface is sufficiently smooth, we have $U^l \in H^1(\mathbf{S})$, therefore $U^l \in W^{1,2+\epsilon}(\mathbf{S})$. By using the fact (see [19]) that when $h$ is efficiently small, there always exists a generic constant $c > 0$ such that

$$\frac{1}{c}|\nabla_s U^l| \le |\nabla_{s_h} U| \le c|\nabla_s U^l|,$$

we can easily obtain the conclusion. $\square$

**Lemma 4** *For $\epsilon \in (0, 1)$, there exists a generic constant $c > 0$, such that for any $u \in H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})$,*

$$\|\Pi^h u\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \le c\|u\|_{H^2(\mathbf{S})}. \tag{4.5}$$

**Proof:** For $u \in H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})$, by the inverse inequality for finite element functions, the lemmas 3, 2 and the inequality (4.1), we can find generic constant $c > 0$ satisfying

$$\begin{aligned}\|\Pi^h u\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} &\le \|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + \|u^l - \Pi^h u\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \\ &\le \|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + ch^{\frac{2}{2+\epsilon}-\frac{2}{2}}\|u^l - \Pi^h u\|_{H^1(\mathbf{S}^h)} \\ &\le \|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + ch^{\frac{2}{2+\epsilon}-1}\|u - (\Pi^h u)^l\|_{H^1(\mathbf{S})} \\ &\le c\|u\|_{W^{1,2+\epsilon}(\mathbf{S})} + ch^{\frac{2}{2+\epsilon}}\|u\|_{H^2(\mathbf{S})}. \end{aligned} \tag{4.6}$$

Now, using the Sobolev imbedding theorem $H^2(\mathbf{S}) \hookrightarrow W^{1,2+\epsilon}(\mathbf{S})$, we have

$$\|\Pi^h u\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \le c\|u\|_{H^2(\mathbf{S})}$$

for some constant $c > 0$. □

Similarly, we define another projection $\Lambda^h$, onto $\mathbf{S}^h$, as follows: for any $u \in H_0^1(\mathbf{S})$,

$$(\Lambda^h u, V^h)_{s_h} = (u, V^{h,l})_s, \quad \forall\, V^h \in \mathcal{U}, \tag{4.7}$$

i.e., equivalently,

$$(\Lambda^h u, V^h)_{s_h} = (\mu_h u^l, V^h)_{s_h}, \quad \forall\, V^h \in \mathcal{U}, \tag{4.8}$$

then for any $V^h \in \mathcal{U}$, we have

$$\|\Lambda^h u - V^h\|_{L^2(\mathbf{S}^h)} \le \|\mu_h u^l - V^h\|_{L^2(\mathbf{S}^h)}.$$

Then we can prove the following lemma:

**Lemma 5** *For $\epsilon \in (0, 1)$, there exists a generic constant $c > 0$ such that for any $u \in H_0^1(\mathbf{S}) \cap W^{1,2+\epsilon}(\mathbf{S})$,*

$$\|\Lambda^h u\|_{H^1(\mathbf{S}^h)} \le c\|u\|_{W^{1,2+\epsilon}(\mathbf{S})}. \tag{4.9}$$

**Proof:** By the best approximation property and the inverse theorem in finite element space [6], we can always find $U^h \in \mathcal{U}$ satisfying the following inequality:

$$\begin{aligned}\|\Lambda^h u\|_{H^1(\mathbf{S}^h)} &\le \|u^l\|_{H^1(\mathbf{S}^h)} + \|u^l - U^h\|_{H^1(\mathbf{S}^h)} + \|\Lambda^h u - U^h\|_{H^1(\mathbf{S}^h)} \\ &\le c\|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + c\|u^l - U^h\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + ch^{-1}\|\Lambda^h u - U^h\|_{L^2(\mathbf{S}^h)} \\ &\le c\|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + ch^{-1}\|\mu_h u^l - U^h\|_{L^2(\mathbf{S}^h)} \\ &\le c\|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + ch^{-1}\|u^l - U^h\|_{L^2(\mathbf{S}^h)} + ch^{-1}\|(1-\mu_h)u^l\|_{L^2(\mathbf{S}^h)} \\ &\le c\|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} + c\|u^l\|_{H^1(\mathbf{S}^h)} \\ &\le c\|u^l\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \\ &\le c\|u\|_{W^{1,2+\epsilon}(\mathbf{S})}. \end{aligned}$$

□

At last, we state a property of function $\mathcal{H}$ in the following lemma.

**Lemma 6** *There exists a constant $k > 0$ such that*

$$\mathcal{H}(x) - \mathcal{H}(y) - \phi(y)(x - y) \ge -k(x - y)^2, \quad \forall\, x, y \in \mathbb{R}.$$

**Proof:** For any $x \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{H}''(x) &= \phi'(x) \\ &= 3\gamma_2 x^2 + 2\gamma_1 x + \gamma_0 \\ &\geq \gamma_0 - \frac{1}{3\gamma_2}\gamma_1^2 \,.\end{aligned}$$

By the mean value theorem, we can always find a positive constant $k > 0$ such that

$$\mathcal{H}(x) - \mathcal{H}(y) - \phi(y)(x-y) \geq -k(x-y)^2 \tag{4.10}$$

for any $x, y \in \mathbb{R}$. □

### 4.2 Estimates on the initial approximation

In order to get energy type estimates for the discretization scheme, first we consider the approximation of the initial condition, especially the initial chemical potential.

Let $U_0^h = \Pi^h u_0$, $p_0^h = \Lambda^h p_0$ where $p_0 = \phi(U_0^{h,l}) - \sigma\Delta_s u_0$. Then we have

**Lemma 7** *There exists a constant $c > 0$ such that*

$$\|p_0^h\|_{H^1(\mathbf{S}^h)} \leq c.$$

**Proof:** Under the assumption on $u_0(\mathbf{x})$, we have from Lemma 5

$$\|U_0^h\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \leq c, \qquad \|\Lambda^h \Delta_s u_0\|_{H^1(\mathbf{S}^h)} \leq c,$$

where the constant $c$ is independent of $h$. Since

$$W^{1,2+\epsilon}(\mathbf{S}^h) \hookrightarrow L^\infty(\mathbf{S}^h),$$

we then get

$$\|U_0^h\|_{W^{0,\infty}(\mathbf{S}^h)} \leq c.$$

This implies that $\|\phi'(U_0^h)\|_{W^{0,\infty}(\mathbf{S}^h)}$ is bounded, and so is $\|\phi'(U_0^h)\nabla_{s_h} U_0^h\|_{W^{0,2+\epsilon}(\mathbf{S}^h)}$ for arbitrary $\epsilon \in (0,1)$. Thus,

$$\|\phi(U_0^h)\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \leq c,$$

where $c$ does not depend on $h$.

Using Lemma 5 again, we also have

$$\|\Lambda^h \phi(U_0^h)\|_{H^1(\mathbf{S}^h)} \leq c,$$

and by the definition of $p_0^h$, we get that

$$\|p_0^h\|_{H^1(\mathbf{S}^h)} \leq c,$$

for a generic constant $c$ independent of $h$. □

By the definition of $p_0$, for any $v \in H^1(\mathbf{S})$, we have

$$-(p_0, v)_s + \sigma(\nabla_s u_0, \nabla_s v)_s + (\phi(U_0^{h,l}), v)_s = 0, \tag{4.11}$$

Using the definition of $\Pi^h$ and under the assumption that $u_0 \in H^2(\mathbf{S})$, we have that for any $V^h \in \mathcal{U}$,

$$\sigma(\nabla_{s_h} U_0^h, \nabla_{s_h} V^h)_{s_h} = ((p_0 - \phi(U_0^{h,l}))^l, V^h)_{s_h} = (p_0^l, V^h)_{s_h} - (\phi(U_0^h), V^h)_{s_h}, \tag{4.12}$$

Using the definition of $\Lambda_h$, the following equation also holds

$$-(p_0^h, V^h)_{s_h} + \sigma(\nabla_{s_h} U_0^h, \nabla_{s_h} V^h)_{s_h} + (\phi(U_0^h), V^h)_{s_h} = 0. \tag{4.13}$$

Next we want to show the boundedness of $P_0^h$, which is a important component of the discrete solution to the fully discrete scheme (3.2)-(3.4) when $n = 1$.

**Theorem 2** *Let $\Delta t$ be sufficiently small, i.e. $\Delta t < \sigma/k^2$, where $k$ is as stated in Lemma 6, then*

$$\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} \le \left(1 - \frac{k\Delta t}{2\sigma}\right)^{-1} \|\nabla_{s_h} p_0^h\|^2_{L^2(\mathbf{S}^h)}.$$

**Proof:** Let $\eta = (U_1^h - U_0^h)/\Delta t$. By (3.4), it holds that for any $V^h \in \mathcal{U}$,

$$-(P_0^h, V^h)_{s_h} + \frac{\sigma}{2}(\nabla_{s_h}(U_1^h + U_0^h), \nabla_{s_h} V^h)_{s_h} + (\tilde{\phi}(U_0^h, U_1^h), V^h)_{s_h} = 0.$$

Subtracting the above equation from (4.13) and setting $V^h = \eta$, we then obtain

$$\begin{aligned}
(P_0^h - p_0^h, \eta)_{s_h} &= \frac{\sigma}{2}(\nabla_{s_h}(U_1^h - U_0^h), \nabla_{s_h}\eta)_{s_h} + (\tilde{\phi}(U_0^h, U_1^h) - \phi(U_0^h), \eta)_{s_h} \\
&= \frac{\Delta t}{2}\sigma(\nabla_{s_h}\eta, \nabla_{s_h}\eta)_{s_h} + \frac{1}{\Delta t}\int_{\mathbf{S}^h} [\mathcal{H}(U_1^h) - \mathcal{H}(U_0^h) - \phi(U_0^h)(U_1^h - U_0^h)] ds_h \\
&\ge \frac{\Delta t \sigma}{2}\|\nabla_{s_h}\eta\|^2_{L^2(\mathbf{S}^h)} - k\Delta t\|\eta\|^2_{L^2(\mathbf{S}^h)},
\end{aligned}$$

where the last step is a result of Lemma 6.

From equation (3.3), it follows that for any $V^h \in \mathcal{U}$,

$$(\eta, V^h)_{s_h} = -(\nabla_{s_h} P_0^h, \nabla_{s_h} V^h)_{s_h}.$$

Letting $V^h = \eta$ and using Cauchy's inequality, we have

$$\|\eta\|^2_{L^2(\mathbf{S}^h)} \le \frac{\sigma}{2k}\|\nabla_{s_h}\eta\|^2_{L^2(\mathbf{S}^h)} + \frac{k}{2\sigma}\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)}.$$

Thus, we obtain

$$\begin{aligned}
(P_0^h - p_0^h, \eta)_{s_h} &\ge \frac{\Delta t\sigma}{2}\|\nabla_{s_h}\eta\|^2_{L^2(\mathbf{S}^h)} - k\Delta t\frac{\sigma}{2k}\|\nabla_{s_h}\eta\|^2_{L^2(\mathbf{S}^h)} - k\Delta t\frac{k}{2\sigma}\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} \\
&= -\frac{k^2\Delta t}{2\sigma}\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)}.
\end{aligned}$$

With (3.3), we get

$$(\eta, P_0^h - p_0^h)_{s_h} = -(\nabla_{s_h} P_0^h, \nabla_{s_h}[P_0^h - p_0^h])_{s_h},$$

and so

$$\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} = (\nabla_{s_h} P_0^h, \nabla_{s_h} p_0^h)_{s_h} + (\nabla_{s_h} P_0^h, \nabla_{s_h}[P_0^h - p_0^h])_{s_h}$$

$$\leq \|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)} \|\nabla_{s_h} p_0^h\|_{L^2(\mathbf{S}^h)} - (\eta, P_0^h - p_0^h)_{s_h}$$
$$\leq \|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)} \|\nabla_{s_h} p_0^h\|_{L^2(\mathbf{S}^h)} + \frac{k^2 \Delta t}{2\sigma} \|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)}.$$

Since $\Delta t < \sigma/k^2$, we then get for $a = 1/2 - k^2 \Delta t/(4\sigma) > 0$,

$$\|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)} \|\nabla_{s_h} p_0^h\|_{L^2(\mathbf{S}^h)} \leq a \|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} + \frac{1}{4a} \|\nabla_{s_h} p_0^h\|^2_{L^2(\mathbf{S}^h)},$$

which leads to

$$\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} \leq a \|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} + \frac{1}{4a} \|\nabla_{s_h} p_0^h\|^2_{L^2(\mathbf{S}^h)} + \frac{k^2 \Delta t}{2\sigma} \|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)}.$$

We then obtain

$$\|\nabla_{s_h} P_0^h\|^2_{L^2(\mathbf{S}^h)} \leq \frac{1}{4a^2} \|\nabla_{s_h} p_0^h\|^2_{L^2(\mathbf{S}^h)},$$

or equivalently,

$$\|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)} \leq \left(1 - \frac{k^2 \Delta t}{2\sigma}\right)^{-1} \|\nabla_{s_h} p_0^h\|_{L^2(\mathbf{S}^h)}. \tag{4.14}$$

This completes the proof of the theorem. □

Finally, utilizing the boundary condition, we may apply the Poincare inequality, Lemma 7 and Theorem 2 to obtain:

**Corollary 1** *There exists a generic constant $c > 0$, independent of $h$, $\Delta t$ and $N$, such that for sufficiently small $\Delta t$,*

$$\|P_0^h\|_{H^1(\mathbf{S}^h)} \leq c. \tag{4.15}$$

### 4.3 Estimates on the discrete chemical potential

In this section, we derive estimates for the discrete chemical potential function $P_n^h$ when $n \geq 1$.

Let us use the notation

$$\delta_{2t} U_n^h = \frac{U_{n+2}^h - U_n^h}{2\Delta t} = \frac{\delta_t U_{n+1}^h + \delta_t U_n^h}{2}, \quad \forall\, n \geq 0\ .$$

From the discrete approximation scheme (3.3) and (3.4), it holds that for $n = 0, 1, 2, \ldots, N-1$,

$$(\delta_{2t} U_n^h, V^h)_{s_h} + (\nabla_{s_h} P_{n+1/2}^h, \nabla_{s_h} V^h)_{s_h} = 0, \quad \forall\, V^h \in \mathcal{U}, \tag{4.16}$$

$$-(\delta_t P_n^h, W^h)_{s_h} + \sigma(\nabla_{s_h} \delta_{2t} U_n^h, \nabla_{s_h} W^h)_{s_h} + (\delta_t \tilde{\phi}_n, W^h)_{s_h} = 0, \quad \forall\, W^h \in \mathcal{U}, \tag{4.17}$$

where $\tilde{\phi}_h = \tilde{\phi}(U_{n+1}^h, U_n^h)$ and

$$\delta_t \tilde{\phi}_n = \frac{\tilde{\phi}(U_{n+1}^h, U_n^h) - \tilde{\phi}(U_n^h, U_{n-1}^h)}{\Delta t}.$$

**Theorem 3** *There exists a constant $c > 0$, independent of $h$, $\Delta t$, $n$ and $N$, such that when $\Delta t$ is sufficiently small, it holds*

$$\|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)} \leq c, \quad \forall\, n = 1, 2, \ldots, N-1. \tag{4.18}$$

**Proof:** Take $V^h = \delta_t P_n^h$, $W^h = \delta_{2t} U_n^h$ in (4.16) and (4.17), then it holds

$$\frac{\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 - \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2}{2\Delta t} = -(\delta_{2t} U_n^h, \delta_t P_n^h)_{s_h}$$
$$= -\sigma(\nabla_{s_h}\delta_{2t}U_n^h, \nabla_{s_h}\delta_{2t}U_n^h)_{s_h} - (\delta_t\tilde{\phi}_n, \delta_{2t}U_n^h)_{s_h}.$$

For the last term of the above equation,

$$\delta_t\tilde{\phi}_n = \delta_{2t}U_n^h \cdot [\frac{\gamma_2}{2}(U_{n+1}^h + U_n^h + U_{n-1}^h)^2 + \frac{\gamma_2}{2}((U_{n+1}^h)^2 + (U_n^h)^2 + (U_{n-1}^h)^2) + \gamma_1(U_{n+1}^h + U_n^h + U_{n-1}^h) + \gamma_0], \tag{4.19}$$

so we get

$$\delta_t\tilde{\phi}_n \cdot \delta_{2t}U_n^h \geq (\gamma_0 - \frac{\gamma_1^2}{2\gamma_2}) \cdot (\delta_{2t}U_n^h)^2,$$

which leads us to

$$\frac{\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 - \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2}{2\Delta t} \leq -\sigma\|\nabla_{s_h}\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2 + c\|\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2, \tag{4.20}$$

for some constant $c > 0$.

Let $V^h = \delta_{2t}U_n^h$ in (4.16), and take use of the Cauchy inequality, we get that for some $\lambda > 0$,

$$\|\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2 \leq \lambda\|\nabla_{s_h}\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2 + \frac{1}{4\lambda}\|\nabla_{s_h}P_{n+1/2}^h\|_{L^2(\mathbf{S}^h)}^2 .$$

Combining the above two inequalities, we easily obtain

$$\frac{\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 - \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2}{2\Delta t} \leq (c\lambda - \sigma)\|\nabla_{s_h}\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2 + \frac{c}{4\lambda}\|\nabla_{s_h}P_{n+1/2}^h\|_{L^2(\mathbf{S}^h)}^2. \tag{4.21}$$

Taking $\lambda = \sigma/2c$, the above inequality then becomes

$$\frac{1}{\Delta t}(\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 - \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2) + \sigma\|\nabla_{s_h}\delta_{2t}U_n^h\|_{L^2(\mathbf{S}^h)}^2$$
$$\leq \frac{c^2}{2\sigma}(\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 + \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2).$$

Thus,

$$\frac{1}{\Delta t}(\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 - \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2) \leq \frac{c^2}{2\sigma}(\|\nabla_{s_h} P_{n+1}^h\|_{L^2(\mathbf{S}^h)}^2 + \|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2).$$

Multiplying $\Delta t$ to both sides of the above inequality, and summing the results over $n$ from 0 to $m-1$ for any integer $m > 1$, we get

$$\|\nabla_{s_h} P_m^h\|_{L^2(\mathbf{S}^h)}^2 \leq \|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)}^2 + \frac{c^2}{\sigma}\Big(\Delta t\sum_{n=0}^{m}\|\nabla_{s_h} P_n^h\|_{L^2(\mathbf{S}^h)}^2\Big).$$

Using Theorem 1, it holds that there is a constant $c > 0$, independent of $h$, $n$, $\Delta t$ and $N$, such that

$$\|\nabla_{s_h} P_m^h\|_{L^2(\mathbf{S}^h)}^2 \leq \|\nabla_{s_h} P_0^h\|_{L^2(\mathbf{S}^h)}^2 + c.$$

Combining this with Corollary 1, the proof of theorem is then complete. □

### 4.4 Pointwise boundedness of the discrete solution

In this section, we aim to prove the pointwise boundedness for the discrete solution $\{U_n^h,\ n = 1, 2, \ldots, N\}$.

**Theorem 4** *For $\epsilon \in (0,1)$, there exists a generic constant $c > 0$, independent of $h$, $n$, $\Delta t$ and $N$, such that*

$$\|U^h_{n+1/2}\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \leq c, \quad \forall n = 0, 1, \ldots, N-1\,.$$

**Proof:** With (3.4), we have that for $n = 0, 1, \ldots, N-1$ and any $W^h \in \mathcal{U}$,

$$\sigma(\nabla_{s_h} U^h_{n+1/2}, \nabla_{s_h} W^h)_{s_h} = (P^h_n - \tilde{\phi}(U^h_n, U^h_{n+1}), W^h)_{s_h} = (F_n, W^h)_{s_h},$$

where $F_n = P^h_n - \tilde{\phi}(U^h_n, U^h_{n+1})$.

Using Theorem 3 and the Poincare inequality, we get

$$\|P^h_n\|_{L^2(\mathbf{S}^h)} \leq c.$$

Moreover, (3.6) indicates that

$$\|U^h_n\|_{L^2(\mathbf{S}^h)} \leq c,$$

which leads to

$$\|\tilde{\phi}(U^h_n, U^h_{n+1})\|_{L^2(\mathbf{S}^h)} \leq c, \quad \forall\, n = 0, 1, \ldots, N-1.$$

Then we have for $n = 0, 1, \ldots, N-1$,

$$\|F_n\|_{L^2(\mathbf{S}^h)} \leq c\,,$$

which is equivalent to

$$\|F^l_n\|_{L^2(\mathbf{S})} \leq c.$$

For a fixed $n$, let $\tilde{u}$ be the solution of the equation

$$-\Delta_s \tilde{u} = F^l_n/\sigma$$

over $\mathbf{S}$ with the homogeneous Dirichlet boundary condition, we can show that such $\tilde{u}$ exists and satisfies the following property

$$\|\tilde{u}\|_{H^2(\mathbf{S})} \leq c\|F^l_n\|_{L^2(\mathbf{S})}.$$

Using the weak form of the above equation

$$\sigma(\nabla_s \tilde{u}, \nabla_s w)_s = (F^l_n, w)_s, \quad \forall w \in H^1(\mathbf{S}),$$

as well as the definition of $\Pi^h(\cdot)$, we can find that

$$U^h_{n+1/2} = \Pi^h \tilde{u}.$$

Therefore, by Lemma 4, we get for $n = 0, 1, \ldots, N-1$,

$$\begin{aligned}\|U^h_{n+1/2}\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} &= \|\Pi^h \tilde{u}\|_{W^{1,2+\epsilon}(\mathbf{S}^h)} \\ &\leq c\|\tilde{u}\|_{H^2(\mathbf{S})} \\ &\leq c\|F^l_n\|_{L^2(\mathbf{S})} \\ &\leq c,\end{aligned}$$

where $c$ is independent of $h$, $n$, $\Delta t$ and $N$. □

Based on the above theorem, we will prove the pointwise boundedness of $U^h_{n+1/2}$.

**Corollary 2** *There exists a constant $c > 0$, independent of $h$, $n$, $\Delta t$ and $N$, such that*

$$\|U^h_{n+1/2}\|_{L^\infty(\mathbf{S}^h)} \le c, \quad \forall\, n = 0, 1, \ldots, N-1. \tag{4.22}$$

**Proof:** This is straight from the Sobolev imbedding Theorem $(W^{1,2+\epsilon}(\mathbf{S}^h) \hookrightarrow L^\infty(\mathbf{S}^h), \ \text{for } \epsilon > 0)$ and Theorem 4. □

Now, let us prove the the pointwise boundedness for the discrete solution under some stability conditions.

**Theorem 5** *Let $\Delta t$ be sufficiently small, i.e.*

$$\Delta t < \sigma/k^2, \tag{4.23}$$

*and*

$$\Delta t/h^2 \le c_0, \tag{4.24}$$

*where $c_0$ is a certain constant. Then, there exists a constant $c > 0$ which depends on the initial condition $u_0$ but is independent of $h$, $n$, $\Delta t$ and $N$, such that*

$$\|U^h_n\|_{W^{0,\infty}(\mathbf{S}^h)} \le c, \quad \forall\, n = 1, 2, \ldots, N. \tag{4.25}$$

**Proof:** In (3.3), set $V^h = \partial_t U^h_n$, we have

$$\|\delta_t U^h_n\|^2_{L^2(\mathbf{S}^h)} \le \|\nabla_{s_h} \delta_t U^h_n\|_{L^2(\mathbf{S}^h)} \|\nabla_{s_h} P^h_n\|_{L^2(\mathbf{S}^h)}. \tag{4.26}$$

By Theorem 3, we get the following inequality

$$\|\delta_t U^h_n\|^2_{L^2(\mathbf{S}^h)} \le c\|\nabla_{s_h} \delta_t U^h_n\|_{L^2(\mathbf{S}^h)}. \tag{4.27}$$

Using the inverse inequality on the last term, we have

$$\|\delta_t U^h_n\|_{L^2(\mathbf{S}^h)} \le ch^{-1}.$$

Applying the inverse inequality again, we further obtain

$$\|\delta_t U^h_n\|_{W^{0,\infty}(\mathbf{S}^h)} \le ch^{-2},$$

which leads to

$$\|U^h_{n+1} - U^h_n\|_{W^{0,\infty}(\mathbf{S}^h)} \le c\Delta t h^{-2} \le cc_0, \quad \forall n = 0, 1, \ldots, N-1. \tag{4.28}$$

Combining (4.28) and Theorem 4, we have for any $n = 0, 1, \ldots, N-1$,

$$\|U^h_{n+1}\|_{W^{0,\infty}(\mathbf{S}^h)} \le \|U^h_{n+1/2}\|_{W^{0,\infty}(\mathbf{S}^h)} + \left\|\frac{U^h_{n+1} - U^h_n}{2}\right\|_{W^{0,\infty}(\mathbf{S}^h)} \le c,$$

which proves the theorem. □

The stability condition (4.24) needed for proving the above theorem requires the time step increment be refined at a faster rate than the spatial discretization parameter $h$, when refinement of the discretization is used. Since the scheme is implicit, this may not be essential to the stability of the approximation scheme. In fact, the stability condition is not necessary for the one dimensional problem. On the other hand, the stability condition for a typical fully explicit finite difference scheme for fourth-order problems requires $\Delta t \le ch^4$, which is considerably more restrictive than the one specified here.

## 5 Error Estimates for the Approximation Scheme

We have derived some nice properties of the discrete solutions in previous sections, and we also note note that existence and uniqueness of the discrete solutions for our scheme can be shown using the approach very similar to that of [16]. On the other hand, with all the previous arguments the error estimate becomes kind of standard, except for some consideration of the projections between the surface and its planar triangulation.

To simplify the notation, we use the abbreviation $u - u(t)$ and $u = p(t)$ to denote the exact solution $u = u(\cdot, t)$ and the corresponding chemical potential at time $t$, both of which are assumed to be sufficiently smooth. We let $u_t$, $u_{tt}$ be the time derivatives of $u$, $p_t$ be that of $p$.

**Lemma 8** *Let $u \in L^\infty(\mathbf{S} \times (0,T))$. Assume there exists a constant $c > 0$ such that*

$$\|U_n^h\|_{W^{0,\infty}(\mathbf{S}^h)} \le c, \quad \forall\, n = 0, 1, ..., N. \tag{5.1}$$

*Then there exists a generic constant $c > 0$ such that*

$$\begin{aligned}&\|\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_n))\|_{L^2(\mathbf{S}^h)}\\ &\quad\le c\left(\|U_{n+1}^h - u^l(t_{n+1})\|_{L^2(\mathbf{S}^h)} + \|U_n^h - u^l(t_n)\|_{L^2(\mathbf{S}^h)} + \|u^l(t_n) - u^l(t_{n+1})\|^2_{L^4(\mathbf{S}^h)}\right)\end{aligned}$$

**Proof:** By the triangle inequality,

$$\begin{aligned}&\|\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_n))\|_{L^2(\mathbf{S}^h)}\\ \le\ &\|\tilde{\phi}(U_n^h, U_{n+1}^h) - \tilde{\phi}(U_n^h, u^l(t_{n+1}))\|_{L^2(\mathbf{S}^h)}\\ &+ \|\tilde{\phi}(U_n^h, u^l(t_{n+1})) - \tilde{\phi}(u^l(t_n), u^l(t_{n+1}))\|_{L^2(\mathbf{S}^h)}\\ &+ \|\tilde{\phi}(u^l(t_n), u^l(t_{n+1})) - \phi(u^l(t_n))\|_{L^2(\mathbf{S}^h)}\\ =\ &I_1 + I_2 + I_3,\end{aligned}$$

where $\{I_i\}$ $(i = 1, 2, 3)$ denote the terms in the previous summation in their corresponding orders.

Due to the uniform boundedness of $U_n^h$ and $u(t)$, it easily follows that

$$\begin{aligned}I_1 &\le c\|U_{n+1}^h - u^l(t_{n+1})\|_{L^2(\mathbf{S}^h)},\\ I_2 &\le c\|U_n^h - u^l(t_n)\|_{L^2(\mathbf{S}^h)},\end{aligned}$$

for some constant $c > 0$.

For the term $I_3$, recall the algebraic identities

$$\begin{aligned}\frac{u^2 + uv + v^2}{3} - \Big(\frac{u+v}{2}\Big)^2 &= \frac{1}{12}(u-v)^2,\\ \frac{u^3 + u^2 v + uv^2 + v^3}{4} - \Big(\frac{u+v}{2}\Big)^3 &= \frac{1}{8}(u+v)(u-v)^2,\end{aligned}$$

then we have

$$\tilde{\phi}(u^l(t_n), u^l(t_{n+1})) - \phi(u^l(t_n)) = \Big(\frac{\gamma_2}{4}u^l(t_n) + \frac{\gamma_1}{12}\Big)[u^l(t_n) - u^l(t_{n+1})]^2,$$

thus

$$I_3 \le c\|u^l(t_n) - u^l(t_{n+1})\|^2_{L^4(\mathbf{S}^h)},$$

which finishes the proof. □

We remark that in Lemma 8, if we change $\phi(u^l(t_n))$ to $\phi(u^l(t_{n+1}))$, a similar result follows. Then we have the following error estimate for our fully discrete finite element scheme.

**Theorem 6** *For $n = 1, 2, ..., N$, adopt the assumptions about $u$ and $U_n^h$ in Lemma 8, and assume*

$$u \in C^1([0,T], H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})) \cap C^3([0,T], L^2(\mathbf{S})) \ , , \tag{5.2}$$

$$p \in L^\infty([0,T], H_0^1(\mathbf{S}) \cap H^2(\mathbf{S})) \cap C^2([0,T], L^2(\mathbf{S})) \ , \tag{5.3}$$

*then it holds that there exists a constant $c > 0$ independent of $h$, $\Delta t$ and $n$, such that*

$$\begin{aligned} \|U_n^h - u^l(t_n)\|^2_{L^2(\mathbf{S}^h)} \le \|u^l(t_n) - \Pi^h u(t_n)\|^2_{L^2(\mathbf{S}^h)} + c\Delta t \sum_{i=0}^{n-1} \Big[ \|\delta_t(u^l(t_i) - \Pi^h u(t_i))\|^2_{L^2(\mathbf{S}^h)} \\ + \|u^l(t_i) - \Pi^h u(t_i)\|^2_{L^2(\mathbf{S}^h)} + \|p^l(t_i) - \Pi^h p(t_i)\|^2_{L^2(\mathbf{S}^h)} \\ + \Delta t^4 (\|u^l_{ttt}(t_{i+\theta_i})\|^2_{L^2(\mathbf{S}^h)} + \|u^l_{ttt}(t_{i+\kappa_i})\|^2_{L^2(\mathbf{S}^h)} \\ + \|p^l_{tt}(t_{n+\tau_i})\|^2_{L^2(\mathbf{S}^h)} + \|p^l_{tt}(t_{n+\nu_i})\|^2_{L^2(\mathbf{S}^h)} + \|u^l_t(t_{i+\gamma_i})\|^4_{L^4(\mathbf{S}^h)} ) \Big] \ , \end{aligned} \tag{5.4}$$

*where $t_j = j\Delta t$, $t_{j+\theta_j} = (j + \theta_j)\Delta t$ for $0 < \theta_j < 1$, $j = 1, 2, ..., n - 1$, adopt the same definition for $\kappa_j$, $\tau_j$, $\nu_j$ and $\gamma_j$.*

**Proof:** For $n = 0, 1, 2, ...N$, let us define

$$E_n = U_n^h - \Pi^h u(t_n), \qquad F_n = P_n^h - \Pi^h p(t_n), \qquad F_{n+(1/2)} = P_n^h - \Pi^h p(t_{n+(1/2)}),$$

and

$$\xi(t) = u^l(t) - \Pi^h u(t), \qquad \eta(t) = p^l(t) - \Pi^h p(t).$$

By the definition of the fully discrete scheme we see that the above quantities satisfy the following equations:

$$\begin{aligned} &\Big(\frac{E_{n+1} - E_n}{\Delta t}, V^h)_{s_h} + (\nabla_{s_h} F_{n+(1/2)}, \nabla_{s_h} V^h\Big)_{s_h} \\ &\quad = \Big(\frac{\xi(t_{n+1}) - \xi(t_n)}{\Delta t}, V^h\Big)_{s_h} - (\delta_t u^l(t_n), V^h)_{s_h} - (\nabla_{s_h} \Pi^h p(t_{n+(1/2)}), \nabla_{s_h} V^h)_{s_h}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &- (F_{n+(1/2)}, W^h)_{s_h} + \sigma\Big(\nabla_{s_h} \frac{E_{n+1} + E_n}{2}, \nabla_{s_h} W^h\Big)_{s_h} \\ &\quad = -(\eta(t_{n+(1/2)}), W^h)_{s_h} + (p^l(t_{n+(1/2)}), W^h)_{s_h} - \sigma\Big(\nabla_{s_h} \frac{\Pi^h u(t_{n+1}) + \Pi^h u(t_n)}{2}, \nabla_{s_h} W^h\Big)_{s_h} \\ &\quad\quad - (\tilde{\phi}(U_n^h, U_{n+1}^h), W^h)_{s_h}. \end{aligned} \tag{5.6}$$

Set $V^h = \frac{E_{n+1}+E_n}{2}$ in (5.5), $W^h = F_{n+(1/2)}$ in (5.6), multiply the first result by $\sigma$ and subtract by the second result, we then obtain

$$\begin{aligned} &\sigma\Big(\frac{E_{n+1} - E_n}{\Delta t}, \frac{E_{n+1} + E_n}{2}\Big)_{s_h} + (F_{n+(1/2)}, F_{n+(1/2)})_{s_h} \\ &= \sigma\Big(\frac{\xi(t_{n+1}) - \xi(t_n)}{\Delta t}, V^h\Big)_{s_h} - \sigma(\delta_t u^l(t_n), V^h)_{s_h} - \sigma(\nabla_{s_h} \Pi^h p(t_{n+(1/2)}), \nabla_{s_h} V^h)_{s_h} + (\eta(t_{n+(1/2)}), W^h)_{s_h} \\ &\quad - (p^l(t_{n+(1/2)}), W^h)_{s_h} + \sigma\Big(\nabla_{s_h} \frac{\Pi^h u(t_{n+1}) + \Pi^h u(t_n)}{2}, \nabla_{s_h} W^h\Big)_{s_h} + (\tilde{\phi}(U_n^h, U_{n+1}^h), W^h)_{s_h} \\ &= T_1 + T_2 + T_3 + T_4, \end{aligned} \tag{5.7}$$

where

$$T_1 = \sigma\Big(\frac{\xi(t_{n+1}) - \xi(t_n)}{\Delta t}, V^h\Big)_{s_h},$$

$$
\begin{aligned}
&T_2 = -\sigma(\delta_t u^l(t_n), V^h)_{s_h} - \sigma(\nabla_{s_h}\Pi^h p(t_{n+(1/2)}), \nabla_{s_h} V^h)_{s_h},\\
&T_3 = (\eta(t_{n+(1/2)}), W^h)_{s_h},\\
&T_4 = -(p^l(t_{n+(1/2)}), W^h)_{s_h} + \sigma\Big(\nabla_{s_h}\frac{\Pi^h u(t_{n+1}) + \Pi^h u(t_n)}{2}, \nabla_{s_h} W^h\Big)_{s_h} + (\tilde{\phi}(U_n^h, U_{n+1}^h), W^h)_{s_h}.
\end{aligned}
$$

Now let us estimate $T_2$ and $T_4$. To avoid complexity, we omit most details of our analysis and give the following results:

$$
\begin{aligned}
T_2 &= -\sigma\Big(\frac{1}{\mu_h}(\frac{u(t_{n+1}) - u(t_n)}{\Delta t} - u_t(t_{n+(1/2)})), V^{h,l}\Big)_s\\
&= -\frac{\sigma\Delta t^2}{48}\Big(u_{ttt}(t_{n+\theta_n}) + u_{ttt}(t_{n+\kappa_n}), V^h\Big)_{s_h}
\end{aligned}
$$

$$
\begin{aligned}
T_4 &= -(p^l(t_{n+(1/2)}), W^h)_{s_h} + (\tilde{\phi}(U_n^h, U_{n+1}^h), W^h)_{s_h}\\
&\quad + \Big(\frac{p^l(t_{n+1}) + p^l(t_n)}{2} - \frac{\phi(u^l(t_{n+1})) + \phi(u^l(t_n))}{2}, W^h\Big)_{s_h}\\
&= \Big(\frac{p^l(t_{n+1}) + p^l(t_n)}{2} - p^l(t_{n+(1/2)}), W^h\Big)_{s_h} + \Big(\frac{\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_{n+1}))}{2}, W^h\Big)_{s_h}\\
&\quad + (\frac{\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_n))}{2}, W^h)_{s_h}.\\
&= \frac{\Delta t^2}{16}(p_{tt}^l(t_{n+\tau_n}) + p_{tt}^l(t_{n+\nu_n}), W^h)_{s_h} + \Big(\frac{\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_{n+1}))}{2}, W^h\Big)_{s_h}\\
&\quad + (\frac{\tilde{\phi}(U_n^h, U_{n+1}^h) - \phi(u^l(t_n))}{2}, W^h)_{s_h}.
\end{aligned}
$$

With all the above approximations and Lemma 8, we sum both sides of (5.7) with $n$ ranging from 0 to $n-1$. It follows that

$$
\begin{aligned}
&\frac{\sigma}{2\Delta t}\|E_n\|_{L^2(\mathbf{S}^h)}^2 + \sum_{i=0}^{n-1}\|F_{i+(1/2)}\|_{L^2(\mathbf{S}^h)}^2 - \frac{\sigma}{2\Delta t}\|E_0\|_{L^2(\mathbf{S}^h)}^2\\
&\quad \le \frac{\sigma}{2}\sum_{i=0}^{n-1}\|\delta_t\xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + \frac{\sigma\Delta t^4}{1304}\sum_{i=0}^{n-1}(\|u_{ttt}^l(t_{i+\theta_i})\|_{L^2(\mathbf{S}^h)}^2 + \|u_{ttt}^l(t_{i+\kappa_i})\|_{L^2(\mathbf{S}^h)}^2) + \frac{1}{2}\sum_{i=0}^{n-1}\|\eta(t_i)\|_{L^2(\mathbf{S}^h)}^2\\
&\qquad + \frac{\sigma}{4}\sum_{i=0}^{n-1}\|E_{i+1} + E_i\|_{L^2(\mathbf{S}^h)}^2 + \frac{\Delta t^4}{512}\sum_{i=0}^{n-1}(\|p_{tt}^l(t_{i+\tau_i})\|_{L^2(\mathbf{S}^h)}^2 + \|p_{tt}^l(t_{i+\nu_i})\|_{L^2(\mathbf{S}^h)}^2)\\
&\qquad + c\sum_{i=0}^{n-1}(\|E_i\|_{L^2(\mathbf{S}^h)}^2 + \|\xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + \|u^l(t_i) - u^l(t_{i+1})\|_{L^4(\mathbf{S}^h)}^4) + \frac{1}{2}\sum_{i=0}^{n-1}\|F_{i+(1/2)}\|_{L^2(\mathbf{S}^h)}^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|E_n\|_{L^2(\mathbf{S}^h)}^2 &\le \|E_0\|_{L^2(\mathbf{S}^h)}^2 + \Delta t\sum_{i=0}^{n-1}\|\delta_t\xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + \frac{\Delta t^5}{652}\sum_{i=0}^{n-1}(\|u_{ttt}^l(t_{i+\theta_i})\|_{L^2(\mathbf{S}^h)}^2 + \|u_{ttt}^l(t_{i+\kappa_i})\|_{L^2(\mathbf{S}^h)}^2)\\
&\quad + \Delta t\sum_{i=0}^{n-1}\|E_i\|_{L^2(\mathbf{S}^h)}^2 + \frac{\Delta t}{2}\|E_n\|_{L^2(\mathbf{S}^h)}^2 + \frac{\Delta t}{\sigma}\sum_{i=0}^{n-1}\|\eta(t_i)\|_{L^2(\mathbf{S}^h)}^2
\end{aligned}
$$

$$+ \frac{\Delta t^5}{256\sigma} \sum_{i=0}^{n-1} (\|p_{tt}^l(t_{i+\tau_i})\|_{L^2(\mathbf{S}^h)}^2 + \|p_{tt}^l(t_{i+\nu_i})\|_{L^2(\mathbf{S}^h)}^2)$$
$$+ c\Delta t \sum_{i=0}^{n-1} \|E_i\|_{L^2(\mathbf{S}^h)}^2 + c\Delta t \sum_{i=0}^{n-1} \|\xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + c\Delta t^5 \sum_{i=0}^{n-1} \|u_t^l(t_{i+\gamma_i})\|_{L^4(\mathbf{S}^h)}^4.$$

Apply the discrete Gronwall inequality and consider the definition of $E_0$ and $U_0^h$, then there exists a constant $c > 0$ such that

$$\begin{aligned} \|E_n\|_{L^2(\mathbf{S}^h)}^2 \le c\Delta t \sum_{i=0}^{n-1} \Big[ & \|\delta_t \xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + \|\xi(t_i)\|_{L^2(\mathbf{S}^h)}^2 + \|\eta(t_i)\|_{L^2(\mathbf{S}^h)}^2 \\ & + \Delta t^4 (\|u_{ttt}^l(t_{i+\theta_i})\|_{L^2(\mathbf{S}^h)}^2 + \|u_{ttt}^l(t_{i+\kappa_i})\|_{L^2(\mathbf{S}^h)}^2 \\ & + \|p_{tt}^l(t_{i+\tau_i})\|_{L^2(\mathbf{S}^h)}^2 + \|p_{tt}^l(t_{i+\nu_i})\|_{L^2(\mathbf{S}^h)}^2 + \|u_t^l(t_{i+\gamma_i})\|_{L^4(\mathbf{S}^h)}^4) \Big]. \end{aligned}$$

Thus the conclusion (5.4) follows. □

By the definition of $\mathcal{U}$ and the conclusions in [19], it can be seen that if $u \in C^1((0,T), H_0^1(\mathbf{S}) \cap H^2(\mathbf{S}))$ and $p \in L^\infty((0,T), H_0^1(\mathbf{S}) \cap H^2(\mathbf{S}))$, then there exists some constant $c > 0$ which satisfies

$$\begin{aligned} \|u^l(t_i) - \Pi^h u(t_i)\|_{L^2(\mathbf{S}^h)} &\le ch^2, \\ \|p^l(t_i) - \Pi^h p(t_i)\|_{L^2(\mathbf{S}^h)} &\le ch^2, \\ \|\delta_t(u^l(t_i) - \Pi^h u(t_i))\|_{L^2(\mathbf{S}^h)} &\le ch^2, \end{aligned}$$

and under some regularity assumptions for time derivatives of $u$ and $p$, we can deduce the following corollary from Theorem 6,

**Corollary 3** *Under the assumptions in Theorem 6, it holds that for $n = 1, 2, ..., N$, there exists a constant $c > 0$ independent of $h$, $\Delta t$ and $n$, such that*

$$\|U_n^h - u^l(t_n)\|_{L^2(\mathbf{S}^h)} \le c(h^2 + \Delta t^2).$$

From the discussion in previous sections, we know that the condition (5.1) is automatically satisfied under the conditions specified in Theorem 5. The error estimate is of optimal order, with respect to the approximation space.

## 6 Numerical Experiments

We now present some numerical simulations using themethod developed here. To ensure the accurate finite element solution, the meshes of the surface $\mathbf{S}$ to be used in our numerical experiments for discretization are generated by the so-called *constrained centroidal Voronoi Delaunay triangulation* (CCVDT) algorithm [14]. We now give a brief description below.

Given a density function $\rho(\mathbf{x})$ defined on $\mathbf{S}$, for any region $V \subset \mathbf{S}$, we call $\mathbf{x}^c$ the 'constrained mass centroid of $V$ on $\mathbf{S}$' if

$$\mathbf{x}^c = \arg\min_{\mathbf{x} \in V} F(\mathbf{x}), \qquad \text{where} \qquad F(\mathbf{x}) = \int_V \rho(\mathbf{y}) \|\mathbf{y} - \mathbf{x}\|^2 \, ds(\mathbf{y}). \tag{6.1}$$

The existence of solutions of (6.1) can be easily obtained by using the continuity and compactness of $F$; however, solutions may not be unique. In general, given a Voronoi tessellation $\mathcal{W} = \{\mathbf{x}_i, V_i\}_{i=1}^n$

of $\mathbf{S}$, the generators $\{\mathbf{x}_i\}_{i=1}^n$ do not coincide with $\{\mathbf{x}_i^c\}_{i=1}^n$, where $\mathbf{x}_i^c$ denotes the constrained mass centroid of $V_i$ for $i = 1, \ldots, n$. We refer to a Voronoi tessellation of $\mathbf{S}$ as a *constrained centroidal Voronoi tessellation*(CCVT) if and only if the points $\{\mathbf{x}_i\}_{i=1}^n$ which serve as the generators of the associated Voronoi tessellation $\{V_i\}_{i=1}^n$ are also the constrained mass centroids of those regions [14], i.e., if and only if we have that

$$\mathbf{x}_i = \mathbf{x}_i^c \quad \text{for } i = 1, \ldots, n\,.$$

The CCVT is a generalization of the standard centroidal Voronoi tessellation [13] which is a concept with many applications including mesh generation and optimization. The dual tessellation of CCVT of $\mathbf{S}$ is then called a CCVDT. Constrained centroidal Voronoi meshes on surfaces in $\mathbb{R}^3$ have many good geometric properties, see [14, 18] for detailed studies as well as efficient algorithms for constructing CCVT/CCVDT meshes.

For all the experiments we are going to show, the meshes are all generated by CCVDT algorithm (see previous chapters for details), and we always set $\phi(u) = u^3 - u$, and numerical tests will be done on different surfaces, with different $\sigma$. Firstly, we observe the approximate solutions $u$ on half unit sphere with $\sigma = 0.008$, $\Delta t = 0.002$. Figure 1 shows the results on meshes with 1219 and 4777 nodes, respectively. The initial condition for the coarse mesh is randomly generated, then we project the values onto the finer mesh, such that the two experiments have the same initial condition. We can see the excellent agreement between these two cases.

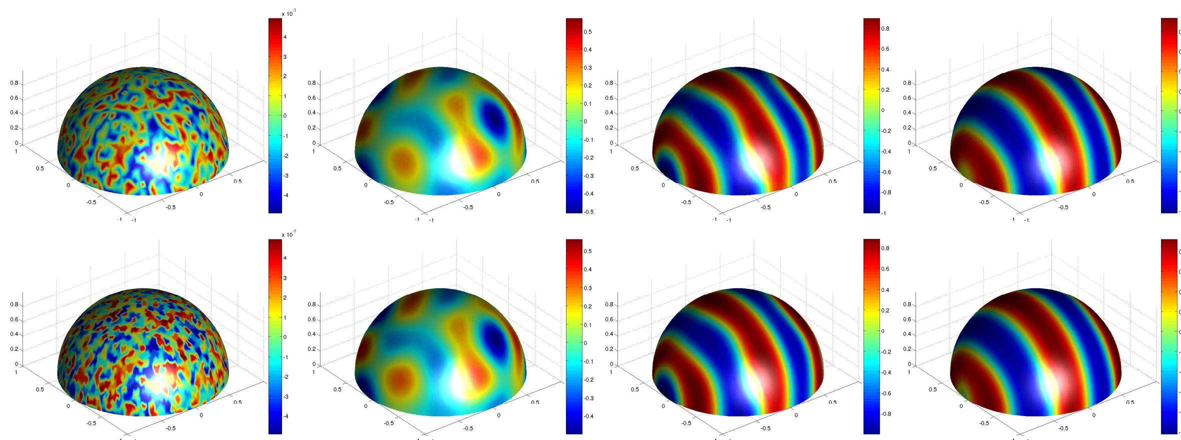

Figure 1: Numerical solutions of the concentration $u$ at $t = 0,\ 0.2,\ 0.4,\ 0.6$ (from left to right), on meshes with 1219 (top row) and 4777 (bottom row) nodes.

Our second experiment is performed on a saddle-like surface defined by

$$\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3 \mid (x_3 - x_2^2)^2 + x_1^2 + x_2^2 = 1,\ x_3 \geq x_2^2,\ x_1 \geq 0\}.$$

We set $\sigma = 0.006$, $\Delta t = 0.002$ and Figure 2 shows the numerical results at different time with 3420 and 13493 nodes. The initial conditions for different meshes are set by using the same trick as in Experiment 1. We can observe the solution finally converges to the steady state and the solutions agree well on the two meshes.

Finally, we test our scheme on a closed surface. The surface $\mathbf{S}$ is chosen to be the unit sphere, defined by $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, take $\sigma = 0.01$, $\Delta t = 0.005$. We solve the equation on mesh with 2014 and 8050 nodes, initial conditions on two meshes are set as in previous two experiments to guarantee the consistency, and results at different time steps are presented in Figure 3. As we can see, our scheme also works well for closed surface, though the theoretical analysis is done for an open surface.

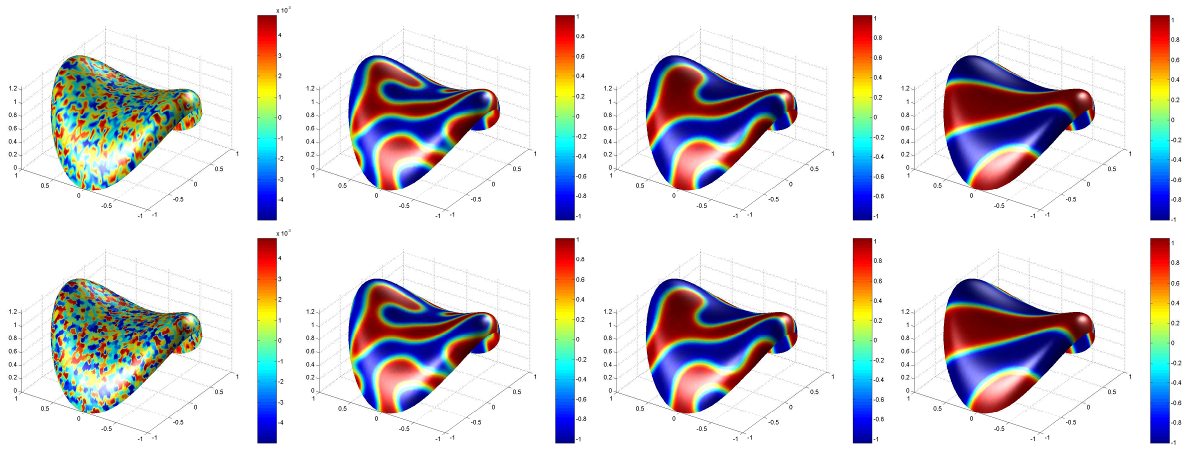

Figure 2: Numerical solution of the concentration $u$ at $t = 0,\ 1,\ 2,\ 10$ (left to right), on meshes with 3420 (top row) and 13493 (bottom row) nodes.

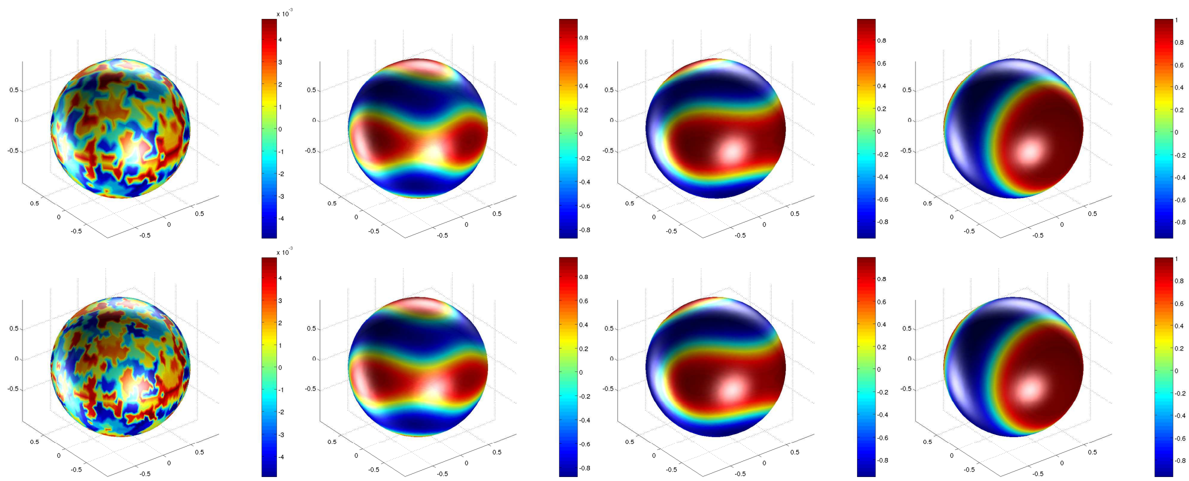

Figure 3: Numerical solution of the concentration $u$ at $t = 0,\ 1,\ 2,\ 5$ (from left to right), on meshes with 2014 (top row) and 8050 (bottom row) nodes.

# 7 Conclusions

We conclude our discussions by noting that while a uniform time discretization is used in the analysis given here for the purpose of simplifying notations, much of the conclusions remain valid for nonuniform and adaptive time steps. The approximation scheme as well as the analysis presented here can be easily modified to deal with the Neumann type boundary value problems for the Cahn-Hilliard equation where the reductions to coupled systems are again allowed. In the future, it will be interesting to study the extensions to more complex situation where the surfaces evolve together with the phase field variables.

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