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FIBONACCI SETS ARE GOOD FOR DISCREPANCY AND NUMERICAL INTEGRATION

DMITRIY BILYK, V.N. TEMLYAKOV, AND RUI YU*

Abstract. We study the Fibonacci sets from the point of view of their quality for numerical integration and discrepancy. Let $\{b_n\}_{n=0}^{\infty}$ be the sequence of Fibonacci numbers. The b_n-point Fibonacci set $\mathcal{F}_n \subset [0,1]^2$ is defined as $\mathcal{F}_n := \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n}$, where $\{x\}$ is the fractional part of a number $x \in \mathbb{R}$. It is known that cubature formulas based on Fibonacci sets $\mathcal{F}_n$ give optimal in the sense of order rate of error of numerical integration for classes of functions with mixed smoothness.

We prove that the Fibonacci sets have optimal in the sense of order L_∞ discrepancy. We establish that the symmetrized Fibonacci set $\mathcal{F}'_n$ has minimal in the sense of order L_2 discrepancy and provide an exact formula for this quantity. We also introduce centered L_p discrepancy which is a modification of the L_p discrepancy to make it symmetric with respect to the center of the unit square. We prove that the Fibonacci set $\mathcal{F}_n$ has minimal in the sense of order centered L_p discrepancy for all $p \in (1,\infty)$. We apply the Fourier method to prove the results.

Keywords: Discrepancy, Fibonacci Numbers, Numerical Integration, Fourier Coefficients.

AMS-classification numbers: 11K38, 11B39, 65D30, 42A16.

1. Introduction

Let $\mathcal{P}_N$ be a set of N points in the unit cube $[0,1]^d$ in dimension d, then the extent of uniform distribution of $\mathcal{P}_N$ can be measured by the discrepancy function:

$$D(\mathcal{P}_N, \mathbf{x}) := \#\{\mathcal{P}_N \cap [\mathbf{0}, \mathbf{x})\} - N \cdot |[\mathbf{0}, \mathbf{x})|, \tag{1.1}$$

where $\mathbf{x} = (x_1, \dots, x_d), [\mathbf{0}, \mathbf{x}) = \prod_{j=1}^{d} [0, x_j)$, and $|\cdot|$ denotes the Lebesgue measure. The L_p norm of the above discrepancy function, usually called the L_p discrepancy, is a benchmark that one uses to evaluate quality of a particular set of N points. The fundamental problem of the discrepancy theory is to construct sets with small L_p discrepancy.

The main principle of discrepancy theory, or theory of irregularities of distribution, states that the quantity

$$D(N,d)_p := \inf_{\mathcal{P}_N} \|D(\mathcal{P}_N, \mathbf{x})\|_p$$

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must necessarily increase with N in the case $d \geq 2$. We refer to Kuipers and Niederreiter [12], Beck and Chen [2], Matoušek [14], and Chazelle [4] for a detailed survey. The principal results in estimating $D(N,d)_p$ from below are:
K. Roth's Theorem. ([16], 1954) In all dimensions $d \geq 2$, we have

$$D(N,d)_2 \geq C(d)(\log N)^{\frac{d-1}{2}}, \tag{1.2}$$

where $C(d)$ is a positive constant that may depend on d.

W. Schmidt's Theorem. ([19], 1972) In dimension $d = 2$,

$$D(N,2)_\infty \geq C \log N, \tag{1.3}$$

where C is a positive absolute constant.

Both bounds (1.2) and (1.3) are known to be sharp in the sense of order, see [7], [8], [17] and [9] for more detail. One of the most famous examples showing sharpness of (1.3) is the van der Corput "digit-reversing" set, which has been well studied in [7]. The reader is referred to [14] for a geometric proof of the fact that the L_∞ discrepancy of the N-point van der Corput set is of order $\log N$. Another important example, described in several books, e.g. [14], [11], is the irrational lattice. Consider $\{(\frac{i}{N}, \{i\alpha\})\}_{i=1}^N$, where α is an irrational number and $\{x\}$ is the fractional part of the number x. If the partial quotients of the continued fraction of α are bounded, then the L_∞ discrepancy of this set is of the order of $\log N$. The idea of this example goes back to Lerch [13].

Unfortunately, the L_2 discrepancy of these "classical" examples either fails to be of optimal order (the L_2 discrepancy of the N-point van der Corput set is of order $\log N$, [10]), or is unknown. However, there are standard ways to modify these sets in order to achieve the smallest possible order of the L_2 discrepancy:

1. *Cyclic Shifts.* The translation idea originated in K. Roth's paper [17], who later applied it to the van der Corput set probabilistically, [18]. A deterministic example of such a shift was recently constructed by Bilyk [3].
2. *Random Digital Scrambling.* This approach is introduced in [5] and one may refer to [14] for a comprehensive discussion and interesting constructive examples.
3. *Davenport's Reflection Principle.* Roughly speaking, if a finite set $\mathcal{P}_N$ has low L_∞ discrepancy, then symmetrizing this set produces a new set of low L_2 discrepancy. Davenport proved this in the case of irrational lattice, see [8]; Chen and Skriganov [6] established this for the van der Corput set (also Proinov [15] employed the symmetrization idea for the generalized van der Corput sequences). In the present paper, we apply this principle to the Fibonacci set.

We study the Fibonacci sets from the point of view of their quality for numerical integration and discrepancy. Let $\{b_n\}_{n=0}^\infty$ be the sequence of Fibonacci numbers that is defined as follows

$$b_0 = b_1 = 1, \quad b_n = b_{n-1} + b_{n-2}, \quad \text{for} \quad n \geq 2.$$

The b_n-point Fibonacci set $\mathcal{F}_n \subset [0,1]^2$ is defined as

$$\mathcal{F}_n := \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n},$$

where $\{x\}$ is the fractional part of a number $x \in \mathbb{R}$.

In Section 2 we prove that

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \leq C \log b_n.$$

This bound, combined with Schmidt's lower bound (1.3), shows that the Fibonacci sets have optimal in the sense of order L_∞ discrepancy. We do not know if $\mathcal{F}_n$ has minimal in the sense of order L_p discrepancy for $p < \infty$. However, in Sections 3 and 4 we give arguments that the Fibonacci set $\mathcal{F}_n$ is also good in the sense of L_p for $p \in (1, \infty)$. In Section 3 we prove that the symmetrized set $\mathcal{F}'_n$ has minimal in the sense of order L_2 discrepancy. We also derive a formula, which allows one to compute the exact value of this quantity. In Section 4 we introduce centered L_p discrepancy which is a modification of the L_p discrepancy to make it symmetric with respect to the center of the unit square. We prove that the Fibonacci set $\mathcal{F}_n$ has minimal in the sense of order centered L_p discrepancy for all $p \in (1, \infty)$. In both Section 3 and Section 4 we apply the Fourier method to prove the results. On the base of these results we make a conclusion that the Fibonacci set $\mathcal{F}_n$ is good from the point of view of discrepancy. This set is related to another low discrepancy sequence – the aforementioned irrational lattice. In particular, the set

$$\mathcal{A}_n(\alpha) := \left\{ \left(\frac{\mu}{b_n}, \{\mu\alpha\} \right) \right\}_{\mu=1}^{b_n},$$

where $\alpha = \dfrac{\sqrt{5}-1}{2}$ is the *golden section*, is close to the set $\mathcal{F}_n$ for large n.

It is well known (see, for instance, [21]) that the L_∞ discrepancy of a finite set is closely related to the error of numerical integration with knots at the given points. We shall discuss this topic in more detail here. The quality of a set of N points for numerical integration can be measured in the following standard way. For a certain function class W compare the error of numerical integration with knots from the given set with optimal error for cubature formulas with N knots. We give a precise formulation of the problem. Numerical integration seeks good ways of approximating an integral

$$\int_\Omega f(\mathbf{x}) d\mu$$

by an expression of the form

$$\Lambda_N(f, \xi) := \sum_{j=1}^{N} \lambda_j f(\xi^j), \quad \xi = (\xi^1, \dots, \xi^N), \quad \xi^j \in \Omega, \quad j = 1, \dots, N. \tag{1.4}$$

It is clear that we must assume that f is integrable and defined at the points $\xi^1, \dots, \xi^N$. The expression (1.4) is called a cubature formula (Λ, ξ) (in our case $\Omega \subset \mathbb{R}^2$) with knots $\xi = (\xi^1, \dots, \xi^N)$ and weights $\Lambda = (\lambda_1, \dots, \lambda_N)$. For a function class W we introduce a concept of error of the cubature formula $\Lambda_N(\cdot, \xi)$ by

$$\Lambda_N(W, \xi) := \sup_{f \in W} \left| \int_\Omega f d\mu - \Lambda_N(f, \xi) \right|. \tag{1.5}$$

In the case of equal weights $\lambda_j = 1/N$ we denote this error by $\Lambda^e_N(W, \xi)$. The following errors are the best we can achieve with cubature formulas with N knots

$$\delta_N(W) := \inf_{\substack{\lambda_1, \dots, \lambda_N \\ \xi^1, \dots, \xi^N}} \Lambda_N(W, \xi); \quad \delta^e_N(W) := \inf_{\xi^1, \dots, \xi^N} \Lambda^e_N(W, \xi).$$

With these definitions at hand, the relation between the L_∞ discrepancy of a set $\mathcal{P}_N \subset [0,1]^2$ and the error of numerical integration with knots at $\mathcal{P}_N$ is straightforward. Define the following class of functions

$$\chi^d := \{\chi_{[\mathbf{0},\mathbf{x}]}(\mathbf{y}) := \prod_{j=1}^{d} \chi_{[0,x_j]}(y_j), \quad x_j \in [0,1], \quad j = 1, \dots, d\},$$

where $\chi_{[0,u]}(v)$ is a characteristic function of the interval $[0,u]$. Then it is clear that

$$\Lambda_N^e(\chi^d, \mathcal{P}_N) = N^{-1}\|D(\mathcal{P}_N, \mathbf{x})\|_\infty. \tag{1.6}$$

Thus, the results of the paper, discussed above, show that the Fibonacci sets $\mathcal{F}_n$ are good for numerical integration of functions in this class.

We now define classes of functions with bounded mixed derivative. These classes are important in numerical integration of multivariate functions. Let for $r > 0$

$$F_r(t) := 1 + 2\sum_{k=1}^{\infty} k^{-r}\cos(2\pi kt - r\pi/2). \tag{1.7}$$

For $\mathbf{x} = (x_1, x_2)$ denote

$$F_r(\mathbf{x}) := F_r(x_1)F_r(x_2)$$

and

$$MW_p^r := \{f : f = \varphi * F_r, \quad \|\varphi\|_p \le 1\},$$

where $*$ means convolution and $\|\cdot\|_p$ is the standard L_p norm.

It is known (see, for instance, survey [21]) that the Fibonacci sets $\mathcal{F}_n$ are also good for numerical integration of functions from the classes MW_p^r. The following known result gives the order of $\Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n)$ for all parameters $1 \le p \le \infty$, $r > 1/p$. In our paper, "$\asymp$" stands for "of the same order of magnitude as" and "$\ll$" stands for "less than a constant multiple of".

Theorem 1.1. *We have*

$$\Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n) \asymp \begin{cases} b_n^{-r}(\log b_n)^{1/2}, & 1 < p \le \infty, r > \max\left(\frac{1}{p}, \frac{1}{2}\right); \\ b_n^{-r}\log b_n, & p = 1, r > 1; \\ b_n^{-r}(\log b_n)^{1-r}, & 2 < p \le \infty, \frac{1}{p} < r < \frac{1}{2}; \\ b_n^{-r}((\log b_n)(\log\log b_n))^{1/2}, & 2 < p \le \infty, r = 1/2. \end{cases} \tag{1.8}$$

The following theorem gives the lower bounds for optimal rates of numerical integration (again, see survey [21]).

Theorem 1.2. *The following lower bound is valid for any cubature formula (Λ, ξ) with N knots $(r > 1/p)$*

$$\Lambda_N(MW_p^r, \xi) \ge C(r,p)N^{-r}(\log N)^{\frac{1}{2}}, \qquad 1 \le p < \infty.$$

The lower bounds provided by Theorem 1.2 and the upper bounds from Theorem 1.1 show that the Fibonacci cubature formulas $\Lambda_{b_n}^e(\cdot, \mathcal{F}_n)$ are optimal (in the sense of order) among all cubature formulas in the case $1 < p < \infty$, $r > \max(1/p, 1/2)$:

$$\delta_{b_n}(MW_p^r) \asymp \Lambda_{b_n}^e(MW_p^r, \mathcal{F}_n) \asymp b_n^{-r}(\log b_n)^{1/2}.$$

We shall also make a remark in Section 2 which shows that the sets $\mathcal{F}_n$ are much better than their siblings $\mathcal{A}_n(\alpha)$ from the point of view of numerical integration of smooth functions.

These results motivate us to conduct a thorough study of the Fibonacci sets. It is well known (see, e.g., [21], Proposition 1.2) that the L_∞ discrepancy governs integration errors not only for characteristic functions, but also for the class MW_1^1:

$$c_1(d)\Lambda_N^e(\chi^d,\xi) \le \Lambda_N^e(MW_1^1,\xi) \le c_2(d)\Lambda_N^e(\chi^d,\xi). \tag{1.9}$$

This allows us to prove the relation

$$\Lambda_{b_n}^e(MW_1^1,\mathcal{F}_n) \asymp b_n^{-1}\log b_n, \tag{1.10}$$

that is not covered by Theorem 1.1.

2. L_∞ discrepancy of the Fibonacci set

Discrepancy was introduced as a quantitative measure of non-uniformity of distribution for infinite sequences. The difference between the discrepancy of a finite point set and discrepancy of an infinite sequence is not radical, but rather, it distinguishes a "static" and a "dynamic" setting, see [14] for more information. The discrepancy of an infinite sequence $u = \{u_1, u_2, \ldots\}$ in $[0,1]$ is defined as

$$\Delta(u,N) := \sup_{0\le a\le 1}\left|\#\{\{u_1,\ldots,u_N\}\cap[0,a)\} - Na\right|, \tag{2.1}$$

which can be viewed as a well-defined function of the natural number N.

In the definition of the discrepancy function for rectangles (1.1), one deals with the behavior of the whole set, whereas in the definition of the discrepancy for infinite sequence (2.1), one looks at all the initial j-segments with $j \le N$ simultaneously. It has been shown by Roth [16] that the dynamic problem in dimension one is equivalent to the static problem in dimension two, and similar reduction is possible between dynamic settings in dimension d and static settings in dimension $d+1$. The following relationship is easily verified, see [14] for details,

$$\|D(\mathcal{F}_n,\mathbf{x})\|_\infty \le 2\max\{\Delta(u,N) : N \in \{1,2,\ldots,b_n\}\} + 2, \tag{2.2}$$

where in our case $\Delta(u,N)$ denotes the discrepancy for the sequence $u = \left\{\{\frac{\mu b_{n-1}}{b_n}\}\right\}_{n=1}^{\infty}$ up to the first N terms.

It is well-known that the sequence of Fibonacci numbers $\{b_n\}_{n=0}^\infty$ gives the denominators of the partial quotients of the golden ratio $\dfrac{\sqrt{5}-1}{2}$. Based on the property of the Fibonacci numbers, we prove the following lemmas.

Lemma 2.1. *For any Fibonacci numbers b_i and b_j, with $i > j \ge 1$, there exists an $\varepsilon_j \in \mathbb{R}$ with $|\varepsilon_j| < 1$, satisfying*

$$\frac{b_{i-1}}{b_i} = \frac{b_{j-1}}{b_j} + \frac{\varepsilon_j}{b_j^2}.$$

Proof. The identity $b_n = b_{n-1} + b_{n-2}$ gives

$$\begin{aligned} |b_{i-1}b_j - b_i b_{j-1}| &= |b_{i-1}b_j - (b_{i-1}+b_{i-2})b_{j-1}| = |b_{i-1}(b_j - b_{j-1}) - b_{i-2}b_{j-1}| \\ &= |b_{i-1}b_{j-2} - b_{i-2}b_{j-1}| = |(b_{i-2}+b_{i-3})b_{j-2} - b_{i-2}(b_{j-2}+b_{j-3})| \\ &= |-b_{i-2}b_{j-3} + b_{i-3}b_{j-2}| = ... \\ &= |b_{i-j+1}b_0 - b_{t-j}b_1| = b_{i-j-1} \end{aligned}$$

and

$$\begin{aligned} b_i &= b_{i-1} + b_{i-2} = b_{i-2} + b_{i-3} + b_{i-3} = 2b_{i-2} + b_{i-3} \\ &= \ldots = b_j b_{i-j} + b_{j-1}b_{i-j-1}. \end{aligned}$$

Thus $b_i > b_j b_{i-j} \geq b_j b_{i-j-1}$, which gives

$$\begin{aligned} \left| \frac{b_{i-1}}{b_i} - \frac{b_{j-1}}{b_j} \right| &= \left| \frac{b_{i-1}b_j - b_i b_{j-1}}{b_i b_j} \right| = \frac{b_{i-j-1}}{b_i b_j} \\ &< \frac{b_{i-j-1}}{b_j b_{i-j-1} b_j} = \frac{1}{b_j^2}. \end{aligned}$$

Therefore there exists some ε_j with $|\varepsilon_j| < 1$ such that $\frac{b_{i-1}}{b_i} - \frac{b_{j-1}}{b_j} = \frac{\varepsilon_j}{b_j^2}$. $\square$

Now for any interval $[0, a) \subset [0, 1]$, defining

$$Z(n_j, b_j, [0,a)) := \# \left\{ \left\{ \frac{(n_j + r)b_{n-1}}{b_n} \right\}_{r=1}^{b_j} \cap [0, a) \right\}, \quad j = 0, 1, \ldots, \tag{2.3}$$

we prove that

Lemma 2.2. $\big|Z(n_j, b_j, [0, a)) - b_j a\big| \leq 4.$

Proof. When $b_j \leq 2$, both $Z(n_j, b_j, [0, a))$ and $b_j a$ are at most 2, and thus it is obvious to see $\big|Z(n_j, b_j, [0, a)) - b_j a\big| \leq 4$.

When $b_j > 2$, It follows from Lemma 2.1 that

$$\frac{(n_j + r)b_{n-1}}{b_n} = \frac{n_j b_{n-1}}{b_n} + r\left(\frac{b_{j-1}}{b_j} + \frac{\varepsilon_j}{b_j^2}\right) = \frac{n_j b_{n-1}}{b_n} + r\frac{b_{j-1}}{b_j} + r\frac{\varepsilon_j}{b_j^2}, \tag{2.4}$$

for some ε_j with $|\varepsilon_j| < 1$.
We want to investigate when $\frac{(n_j + r)b_{n-1}}{b_n}$ modulo 1 lies in $[0, a)$.

If $n_j = 0$ and $\varepsilon_j = 0$ in (2.4),

$$\left\{ \left\{ \frac{(n_j + r)b_{n-1}}{b_n} \right\} \right\}_{r=1}^{b_j} = \left\{ \left\{ \frac{r b_{j-1}}{b_j} \right\} \right\}_{r=1}^{b_j} = \left\{ \frac{r}{b_j} \right\}_{r=1}^{b_j},$$

since b_j and b_{j-1} are relatively prime. Thus it is easy to see that

$$\big|Z(n_j, b_j, [0, a)) - b_j a\big| = \left| \left[\frac{a}{1/b_j} \right] - b_j a \right| = b_j a - [b_j a] \leq 1,$$

where $[x]$ is the integer part of a real number x.

If $n_j \neq 0$ and $\varepsilon_j = 0$ in (2.4),

$$\frac{(n_j + r)b_{n-1}}{b_n} = \frac{n_j b_{n-1}}{b_n} + r\frac{b_{j-1}}{b_j},$$

the sequence $\left\{\left\{\frac{(n_j + r)b_{n-1}}{b_n}\right\}\right\}_{r=1}^{b_j}$ is merely shifted by the value of $\frac{n_j b_{n-1}}{b_n}$, and the distance of every two elements is unchanged from the case when $n_j = 0$ and $\varepsilon_j = 0$. Thus the change of $Z(n_j, b_j, [0, a))$ is at most 1, therefore we have

$$\left|Z(n_j, b_j, [0, a)) - b_j a\right| \leq 2.$$

If $n_j \neq 0$ and $\varepsilon_j \neq 0$, the change from $\frac{rb_{j-1}}{b_j}$ to $\frac{rb_{j-1}}{b_j} + \frac{r\varepsilon_j}{b_j^2}$ only causes the terms of the sequence modulo 1 to deviate slightly from points in case when $n_j \neq 0$ and $\varepsilon_j = 0$. However the deviation can be at most $\frac{1}{b_j}$, because of

$$|\varepsilon_j| < 1 \quad \text{and} \quad \left|\frac{r\varepsilon_j}{b_j}\right| < \frac{1}{b_j}.$$

In other words, there will be at most two elements of the sequence that will be deviated outward or into the interval $[0, a)$. Therefore

$$\left|Z(n_j, b_j, [0, a)) - b_j a\right| \leq 4$$

which proves the lemma.

□

Now we have the following theorem which is one of our main results.

Theorem 2.3. *For* $\mathcal{F}_n = \{(\mu/b_n, \{\mu b_{n-1}/b_n\})\}_{\mu=1}^{b_n}$, *we have*

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \ll \log b_n. \tag{2.5}$$

Proof. We reduce the problem to the study of the discrepancy $\Delta(u, N)$ defined in (2.1) for the sequence $u = \left\{\left\{\frac{\mu b_{n-1}}{b_n}\right\}\right\}_{\mu=1}^{\infty}$ and use the relation (2.2) to deduce our result. For each natural number $2 \leq N \leq b_n$, we divide it into c_{t-1} blocks of b_{t-1}, c_{t-2} blocks of b_{t-2} consecutive numbers and so on. Then if $b_{t-1} \leq N < b_t$, $1 \leq t \leq n$, let

$$c_{t-1} := \left[\frac{N}{b_{t-1}}\right],$$

where $[x]$ denotes the integer part of a positive real number x and $[x] \leq x < [x] + 1$. We write

$$N = c_{t-1}b_{t-1} + N^{'}$$

with $0 \leq N^{'} < b_{t-1}$.

Let $c_{t-2} := \left[\frac{N^{'}}{b_{t-2}}\right]$, then

$$N = c_{t-1}b_{t-1} + c_{t-2}b_{t-2} + N^{''}$$

with $0 \le N^{''} < b_{t-2}$. This process gives

$$N = c_{t-1}b_{t-1} + c_{t-2}b_{t-2} + \ldots + c_1 b_1 + c_0 \tag{2.6}$$

with

$$1 \le c_{t-1} \le \frac{N}{b_{t-1}} < 2, \quad 0 \le c_{j-1} < \frac{b_j}{b_{j-1}}, j = 1, \ldots, t-1. \tag{2.7}$$

In fact, by the way they are defined, c_{t-1} must be 1, and c_0 through c_{t-2} could be either 0 or 1.

For the sequence $u = \{u_1, u_2, \ldots\} = \{\{\frac{\mu b_{n-1}}{b_n}\}\}_{\mu=1}^{\infty}$, consider the case that all $c_j' s$ are nonzero and one can easily see that general case will have the same result. Its first N elements $\{u_1, u_2, \ldots, u_N\}$ can be divided into t disjoint subesets S_0, S_1, $\ldots$, S_{t-1} of cardinaties b_0, b_1, $\ldots$, b_{t-1} respectively. We denote these sets by $S_j = \left\{\left\{\frac{(n_j + r)b_{n-1}}{b_n}\right\}\right\}_{r=1}^{b_j}$, where n_j is some nonnegative integer, $j = 0, 1, \ldots, t-1$. Therefore

$$\begin{aligned}
|\#\{\{u_1, \ldots, u_N\} \cap [0, a)\} - Na| &= \left|\#\{\bigcup_{j=0}^{t-1} S_j \cap [0, a)\} - \sum_{j=0}^{t-1} c_j b_j a\right| \\
&= \left|\sum_{j=0}^{t-1} c_j \#\{S_j \cap [0, a)\} - \sum_{j=0}^{t-1} c_j b_j a\right| \\
&\le \sum_{j=0}^{t-1} c_j |Z(n_j, b_j, [0, a) - b_j a| \\
&\le 4 \sum_{j=0}^{t-1} c_j \le 4t.
\end{aligned}$$

Note that $\frac{b_{t-1}}{b_{t-3}} \ge 2$ which gives $b_{t-1} \ge 2b_{t-3} \ge .. \ge 2^{[\frac{t}{2}]}$, we have t is of order $\log N$ at most. This implies

$$\big|\#\{\{u_1, \ldots, u_N\} \cap [0, a)\} - Na\big| \ll \log N.$$

Thus by definition (2.1),

$$\Delta(u, N) = \sup_{0 \le a < 1} \big|\#\{\{u_1, \ldots, u_N\} \cap [0, a)\} - Na\big| \ll \log N.$$

and hence the relation (2.2) gives

$$\begin{aligned}
\|D(\mathcal{F}_n, \mathbf{x})\|_\infty &\le 2 \max\{\Delta(u, N) : N \in \{1, 2, \ldots, b_n\}\} + 2 \\
&\ll \log b_n.
\end{aligned}$$

□

As we stated in the introduction the irrational lattice $\left\{\left(\frac{\mu}{b_n}, \{\mu\alpha\}\right)\right\}_{\mu=1}^{b_n}$ has sharp L_∞ norm if partial quotients of the continued fraction of α are bounded. Now consider $\alpha = \frac{\sqrt{5}-1}{2}$, then the set

$$\mathcal{A}_n(\alpha) := \left\{\left(\frac{\mu}{b_n}, \{\mu\alpha\}\right)\right\}_{\mu=1}^{b_n}$$

is closely related to the set $\mathcal{F}_n$ and it is known that

$$\|D(\mathcal{A}_n(\alpha), \mathbf{x})\|_\infty \ll \log b_n. \tag{2.8}$$

The sets $\mathcal{F}_n$ and $\mathcal{A}_n(\alpha)$ are close to each other in the following sense. For $\mu \in [1, b_n]$ the x-coordinates of the μth points of $\mathcal{F}_n$ and $\mathcal{A}_n(\alpha)$ are the same and the difference between the y-coordinates of these points is small. This follows from the well-known inequality

$$\left|\alpha - \frac{b_{n-1}}{b_n}\right| \le \frac{1}{2b_n^2}. \tag{2.9}$$

For completeness we give a simple proof of the above inequality. Consider $P(x) = x^2 + x - 1$. Then $P(\alpha) = 0$ and $|P(b_{n-1}/b_n)| = b_n^{-2}$. We have

$$|P(b_{n-1}/b_n) - P(\alpha)| = P'(\xi)|b_{n-1}/b_n - \alpha|,$$

$\xi \in \left(\frac{b_{n-1}}{b_n}, \alpha\right)$. It is easy to see that $\frac{1}{2} \le \frac{b_{n-1}}{b_n} \le \frac{2}{3}$ and $\frac{1}{2} \le \alpha \le \frac{2}{3}$. Therefore,

$$2 \le |P'(\xi)| \le \frac{7}{3}. \tag{2.10}$$

This implies (2.9).
By (2.9) we obtain

$$|\{\mu b_{n-1}/b_n\} - \{\mu\alpha\}| \le |\mu b_{n-1}/b_n - \mu\alpha| \le \frac{\mu}{2b_n^2} \le \frac{1}{2b_n}.$$

This inequality and the following simple known lemma show that the bound

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \ll \log b_n, \quad n \ge 2 \tag{2.11}$$

can also be derived as a perturbation of (2.8).

Lemma 2.4. *Let $P_N = \{p_k\}_{k=1}^N \subset [0,1]^d$ and $Q_N = \{q_k\}_{k=1}^N \subset [0,1]^d$ be such that $\|p_k - q_k\|_\infty \le \delta$, $k = 1, \dots, N$. Then*

$$\big|\|D(P_N, \mathbf{x})\|_\infty - \|D(Q_N, \mathbf{x})\|_\infty\big| \le N\delta d.$$

The bounds (2.8) and (2.11) show that the sets $\mathcal{F}_n$ and $\mathcal{A}_n(\alpha)$ are equally good from the point of view of the L_∞ discrepancy. Theorem 1.1 from the introduction shows that the sets $\mathcal{F}_n$ are good for numerical integration. We now demonstrate by a simple example that sets $\mathcal{A}_n(\alpha)$ are not good for numerical integration. Indeed, consider a function

$$f(x_1, x_2) := e^{2\pi i x_2}.$$

It is easy to check that $f \in MW_p^r$ for all r and $1 \le p \le \infty$. The error of numerical integration of f using $\mathcal{A}_n(\alpha)$ with equal weights $\dfrac{1}{b_n}$ is

$$\left|\frac{1}{b_n}\sum_{\mu=1}^{b_n} e^{2\pi i\mu\alpha}\right| = \frac{1}{b_n}\left|\frac{1-e^{2\pi i b_n\alpha}}{1-e^{2\pi i\alpha}}\right|.$$

Using (2.10) we get

$$\frac{3}{7}\cdot\frac{1}{b_n^2} \le \left|\alpha - \frac{b_{n-1}}{b_n}\right| \le \frac{1}{2b_n^2}.$$

This implies for $n \ge 3$

$$|1-e^{2\pi i b_n\alpha}| \ge |\sin 2\pi\{b_n\alpha\}| \ge \frac{2}{\pi}\cdot 2\pi b_n\cdot\frac{3}{7}\cdot\frac{1}{b_n^2} = \frac{12}{7}\cdot\frac{1}{b_n}.$$

Therefore, the error of numerical integration of f is bounded from below by cb_n^{-2}. It means that the cubature formula

$$Q_{n,\alpha}(g) := \frac{1}{b_n}\sum_{q\in\mathcal{A}_n(\alpha)} g(q)$$

has a saturation property for $r > 2$.

We now prove (1.10) which is the result of numerical integration by using the results we got for discrepancy. Therorem 2.3 and Schmidt's lower bound (1.3) imply

$$\|D(\mathcal{F}_n, \mathbf{x})\|_\infty \asymp \log b_n.$$

This and relation (1.6) give

$$\Lambda_{b_n}^e(\chi^2, \mathcal{F}_n) \asymp b_n^{-1}\log b_n. \tag{2.12}$$

The relation (1.10) now follows from (2.12) and the well-known inequalities (1.9).

3. L_2 discrepancy of Modified Fibonacci Set

Inspired by the *Davenport's Reflection Principle*, mentioned in the first section, we now symmetrize $\mathcal{F}_n$ to a $2b_n$-point set

$$\mathcal{F}_n' := \{(p_1,p_2)\cup(p_1,1-p_2) : (p_1,p_2)\in\mathcal{F}_n\}. \tag{3.1}$$

Its discrepancy function is

$$D(\mathcal{F}_n', \mathbf{x}) := \#\{\mathcal{F}_n' \cap [0,x_1)\times[0,x_2)\} - 2b_n x_1 x_2,$$

where $\mathbf{x} = (x_1,x_2)\in(0,1]^2$.
Rewriting it to

$$D(\mathcal{F}_n', \mathbf{x}) = \sum_{\mathbf{p}=(p_1,p_2)\in\mathcal{F}_n}\left[\chi_{[p_1,1)\times[p_2,1)}(\mathbf{x}) + \chi_{[p_1,1)\times[1-p_2,1)}(\mathbf{x})\right] - 2b_n x_1 x_2,$$

and computing the Fourier coefficients of the $D(\mathcal{F}'_n, \mathbf{x})$ give

$$\begin{aligned}
\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \sum_{\mathbf{p}=(p_1,p_2)\in\mathcal{F}_n} \left[\widehat{\chi}_{[p_1,1)\times[p_2,1)}(\mathbf{k}) + \widehat{\chi}_{[p_1,1)\times[1-p_2,1)}(\mathbf{k})\right] - 2\widehat{b_n x_1 x_2} \\
&= \sum_{\mathbf{p}\in\mathcal{F}_n} \Bigg[\int_0^1\int_0^1 \chi_{[p_1,1)\times[p_2,1)}(x_1,x_2) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} dx_1 dx_2 \\
&\quad + \int_0^1\int_0^1 \chi_{[p_1,1)\times[1-p_2,1)}(x_1,x_2) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} dx_1 dx_2\Bigg] \\
&\quad -2b_n \int_0^1\int_0^1 x_1 x_2 e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} dx_1 dx_2 \\
&= \sum_{\mathbf{p}\in\mathcal{F}_n} \Bigg[\int_{p_1}^1 e^{-2\pi i k_1 x_1} dx_1 \int_{p_2}^1 e^{-2\pi i k_2 x_2} dx_2 \\
&\quad + \int_{p_1}^1 e^{-2\pi i k_1 x_1} dx_1 \int_{1-p_2}^1 e^{-2\pi i k_2 x_2} dx_2\Bigg] \\
&\quad -2b_n \int_0^1 x_1 e^{-2\pi i k_1 x_1} dx_1 \int_0^1 x_2 e^{-2\pi i k_2 x_2} dx_2. \qquad (3.2)
\end{aligned}$$

Note that

$$\sum_{\mu=1}^{b_n} e^{-2\pi i l\mu/b_n} = \begin{cases} b_n, & l \equiv 0 \pmod{b_n}, \\ 0, & l \not\equiv 0 \pmod{b_n}. \end{cases} \qquad (3.3)$$

Let $L(n) := \{\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2 : k_1 + b_{n-1}k_2 \equiv 0 \pmod{b_n}\}$, then

$$\sum_{\mu=1}^{b_n} e^{-2\pi i (k_1 + b_{n-1}k_2)\mu/b_n} = \begin{cases} b_n, & (k_1,k_2) \in L(n), \\ 0, & (k_1,k_2) \notin L(n). \end{cases} \qquad (3.4)$$

Now let's consider different cases:

Case 1. $k_1 = 0$, $k_2 = 0$. We have the following lemma:

Lemma 3.1. $\widehat{D}(\mathcal{F}'_n, \mathbf{0}) = -\dfrac{1}{2}$.

Proof. From (3.2) we get

$$\begin{aligned}
\widehat{D}(\mathcal{F}'_n, \mathbf{0}) &= \sum_{\mathbf{p}\in\mathcal{F}_\mathbf{n}} \left[(1-p_1)(1-p_2) + (1-p_1)p_2\right] - \frac{b_n}{2} \\
&= \sum_{\mathbf{p}\in\mathcal{F}_\mathbf{n}} \left(1-p_1\right) - \frac{b_n}{2} \\
&= \sum_{\mu=1}^{b_n} (1 - \mu/b_n) - \frac{b_n}{2} \\
&= b_n - \frac{b_n(b_n+1)}{2b_n} - \frac{b_n}{2} \\
&= -\frac{1}{2}. \qquad (3.5)
\end{aligned}$$

$\square$

Case 2. $k_1 \neq 0$, $k_2 \neq 0$.
In this case

$$\begin{aligned}\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \big[(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2 p_2}) \\ &\quad +(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2 (1-p_2)})\big] + \frac{b_n}{2\pi^2 k_1 k_2} \\ &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \big[(1-e^{-2\pi i k_1 p_1})(1-e^{-2\pi i k_2 p_2}) \\ &\quad +(1-e^{-2\pi i k_1 p_1})(1-e^{2\pi i k_2 p_2})\big] + \frac{b_n}{2\pi^2 k_1 k_2}. \end{aligned} \tag{3.6}$$

Then we have the following lemma:

Lemma 3.2. *If $k_1 \neq 0$, $k_2 \neq 0$, then*

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \frac{b_n}{2\pi^2 k_1 k_2} \tag{3.7}$$

provided that at least one of k_1 and k_2 is 0 modulo b_n.

Proof. Without loss of generality assume $k_1 \equiv 0 \pmod{b_n}$, then

$$e^{-2\pi i k_1 p_1} = e^{\frac{-2\pi i k_1 \mu}{b_n}} = 1.$$

So from (3.6) we get

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \frac{b_n}{2\pi^2 k_1 k_2}. \tag{3.8}$$

$\square$

Lemma 3.3. *Assume $k_1 \not\equiv 0 \pmod{b_n}$ and $k_2 \not\equiv 0 \pmod{b_n}$, then*

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \begin{cases} \dfrac{-b_n}{2\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \equiv 0 \pmod{b_n}, k_1 - k_2 b_{n-1} \equiv 0 \pmod{b_n}, \\ \dfrac{-b_n}{4\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \equiv 0 \pmod{b_n}, k_1 - k_2 b_{n-1} \not\equiv 0 \pmod{b_n}, \\ \dfrac{-b_n}{4\pi^2 k_1 k_2}, & k_1 + k_2 b_{n-1} \not\equiv 0 \pmod{b_n}, k_1 - k_2 b_{n-1} \equiv 0 \pmod{b_n}, \\ 0, & k_1 + k_2 b_{n-1} \not\equiv 0 \pmod{b_n}, k_1 - k_2 b_{n-1} \not\equiv 0 \pmod{b_n}. \end{cases} \tag{3.9}$$

Proof. We first rewrite (3.6) as

$$\begin{aligned}\widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{4\pi^2 k_1 k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \big[(1-e^{-2\pi i k_1 p_1} - e^{-2\pi i k_2 p_2} + e^{-2\pi i (k_1 p_1 + k_2 p_2)}) \\ &\quad +(1-e^{-2\pi i k_1 p_1} - e^{2\pi i k_2 p_2} + e^{-2\pi i (k_1 p_1 - k_2 p_2)})\big] + \frac{b_n}{2\pi^2 k_1 k_2}. \end{aligned} \tag{3.10}$$

By (3.3)

$$\sum_{\mathbf{p}\in\mathcal{F}_n} e^{\pm 2\pi x i k_j p_j} = 0, \quad \text{for} \quad j = 1, 2.$$

Thus

$$\begin{aligned}\widehat{D}(\mathcal{F}'_n,\mathbf{k}) &= \frac{-1}{4\pi^2 k_1 k_2}\sum_{\mathbf{p}\in\mathcal{F}_n}\left[2+e^{-2\pi i(k_1p_1+k_2p_2)}+e^{-2\pi i(k_1p_1-k_2p_2)}\right]+\frac{b_n}{2\pi^2 k_1k_2}\\ &= \frac{-1}{4\pi^2 k_1 k_2}\sum_{\mathbf{p}\in\mathcal{F}_n}\left[e^{-2\pi i(k_1p_1+k_2p_2)}+e^{-2\pi i(k_1p_1-k_2p_2)}\right]\\ &= \frac{-1}{4\pi^2 k_1 k_2}\sum_{\mu=1}^{b_n}\left[e^{\frac{-2\pi i\mu(k_1+k_2b_{n-1})}{b_n}}+e^{\frac{-2\pi i\mu(k_1-k_2b_{n-1})}{b_n}}\right]. \qquad (3.11)\end{aligned}$$

If both $k_1+k_2b_{n-1}\equiv 0 \pmod{b_n}$ and $k_1-k_2b_{n-1}\equiv 0 \pmod{b_n}$ hold, i.e. $(k_1,k_2)\in L(n)$ and $(k_1,-k_2)\in L(n)$, we get

$$\widehat{D}(\mathcal{F}'_n,\mathbf{k}) = \frac{-b_n}{2\pi^2 k_1k_2}. \qquad (3.12)$$

Note that for odd b_n the congruences $k_1+k_2b_{n-1}\equiv 0 \pmod{b_n}$, $k_1-k_2b_{n-1}\equiv 0 \pmod{b_n}$ imply $k_1\equiv 0 \pmod{b_n}$ that violates the assumptions of Lemma 3.3. Thus this case is possible only for even b_n.
If only one of $k_1+k_2b_{n-1}\equiv 0 \pmod{b_n}$, $k_1-k_2b_{n-1}\equiv 0 \pmod{b_n}$ holds, or in other words only one of (k_1,k_2), $(k_1,-k_2)$ is in $L(n)$, then

$$\widehat{D}(\mathcal{F}'_n,\mathbf{k}) = \frac{-b_n}{4\pi^2 k_1k_2}. \qquad (3.13)$$

If $k_1+k_2b_{n-1}\not\equiv 0 \pmod{b_n}$ and $k_1-k_2b_{n-1}\not\equiv 0 \pmod{b_n}$, i.e. both (k_1,k_2) and $(k_1,-k_2)$ are not in $L(n)$, then we get

$$\widehat{D}(\mathcal{F}'_n,\mathbf{k}) = 0. \qquad (3.14)$$

□

Case 3. $k_1\neq 0$, $k_2=0$. We have the following lemma:

Lemma 3.4. *If $k_1\neq 0, k_2=0$,*

$$\widehat{D}(\mathcal{F}'_n,\mathbf{k}) = \begin{cases}\dfrac{b_n}{2\pi i k_1}, & k_1\equiv 0 \pmod{b_n},\\ 0, & k_1\not\equiv 0 \pmod{b_n}.\end{cases} \qquad (3.15)$$

Proof. We obtain from (3.2),

$$\begin{aligned}\widehat{D}(\mathcal{F}'_n,\mathbf{k}) &= \frac{-1}{2\pi i k_1}\sum_{\mathbf{p}\in\mathcal{F}_n}\left[(1-e^{-2\pi i k_1p_1})(1-p_2)+(1-e^{-2\pi i k_1p_1})p_2\right]+\frac{b_n}{2\pi i k_1}\\ &= \frac{-1}{2\pi i k_1}\sum_{\mathbf{p}\in\mathcal{F}_n}\left[1-e^{-2\pi i k_1p_1}\right]+\frac{b_n}{2\pi i k_1}. \qquad (3.16)\end{aligned}$$

If $k_1\equiv 0 \pmod{b_n}$, then $e^{-2\pi i k_1p_1}=1$, so we get

$$\widehat{D}(\mathcal{F}'_n,\mathbf{k}) = \frac{b_n}{2\pi i k_1}. \qquad (3.17)$$

If $k_1 \not\equiv 0 \pmod{b_n}$, then $\sum_{\mathbf{p}\in\mathcal{F}_n} e^{-2\pi i k_1 p_1} = 0$, so we get

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = 0. \tag{3.18}$$

□

Case 4. $k_1 = 0, k_2 \neq 0$. We have the following lemma:

Lemma 3.5. *If $k_1 = 0, k_2 \neq 0$,*

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \begin{cases} \dfrac{b_n}{2\pi i k_2}, & k_2 \equiv 0 \pmod{b_n}, \\ 0, & k_2 \not\equiv 0 \pmod{b_n}. \end{cases} \tag{3.19}$$

Proof. We obtain from (3.2),

$$\begin{aligned} \widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[(1-p_1)(1-e^{-2\pi i k_2 p_2}) + (1-p_1)(1-e^{2\pi i k_2 p_2})\right] + \frac{b_n}{2\pi i k_2} \\ &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[(1-p_1)(2-e^{-2\pi i k_2 p_2} - e^{2\pi i k_2 p_2})\right] + \frac{b_n}{2\pi i k_2} \\ &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[2 - e^{-2\pi i k_2 p_2} - e^{2\pi i k_1 p_2} - 2p_1 + p_1 e^{-2\pi i k_2 p_2} + p_1 e^{2\pi i k_2 p_2}\right] + \frac{b_n}{2\pi i k_2}. \end{aligned}$$

If $k_2 \equiv 0 \pmod{b_n}$, then $e^{\pm 2\pi i k_2 p_2} = 1$, and we get

$$\widehat{D}(\mathcal{F}'_n, \mathbf{k}) = \frac{b_n}{2\pi i k_2}. \tag{3.20}$$

If $k_2 \not\equiv 0 \pmod{b_n}$, then $\sum_{\mathbf{p}\in\mathcal{F}_n} e^{\pm 2\pi i k_2 p_2} = 0$, and we get

$$\begin{aligned} \widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{-1}{2\pi i k_2} \sum_{\mathbf{p}\in\mathcal{F}_n} \left[2 - 2p_1 + p_1 e^{-2\pi i k_2 p_2} + p_1 e^{2\pi i k_2 p_2}\right] + \frac{b_n}{2\pi i k_2} \\ &= \frac{-1}{2\pi i k_2} \sum_{\mu=1}^{b_n} \left[2 - 2\frac{\mu}{b_n} + \frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right] + \frac{b_n}{2\pi i k_2} \\ &= \frac{-1}{2\pi i k_2} \left[2b_n - (b_n+1) - b_n + \sum_{\mu=1}^{b_n} \left(\frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right)\right] \\ &= \frac{1}{2\pi i k_2} + \frac{-1}{2\pi i k_2} \left[\sum_{\mu=0}^{b_{n-1}} \left(\frac{\mu}{b_n} e^{-\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right) + 2\right]. \end{aligned} \tag{3.21}$$

Let

$$f(x) = \sum_{\mu=0}^{b_{n-1}} e^{\frac{2\pi i \mu x}{b_n}} = \frac{e^{2\pi i x} - 1}{e^{\frac{2\pi i x}{b_n}} - 1}.$$

On one hand,

$$f'(x) = \sum_{\mu=0}^{b_{n-1}} \frac{2\pi i \mu}{b_n} e^{\frac{2\pi i \mu x}{b_n}}, \tag{3.22}$$

and thus

$$f'(k_2 b_{n-1}) = \sum_{\mu=0}^{b_{n-1}} \frac{2\pi i \mu}{b_n} e^{\frac{2\pi i \mu k_2 b_{n-1}}{b_n}}; \tag{3.23}$$

on the other hand

$$f'(x) \quad = \quad \frac{2\pi i e^{2\pi i x}(e^{\frac{2\pi i x}{b_n}} - 1) - (e^{2\pi i x} - 1)\frac{2\pi i}{b_n} e^{\frac{2\pi i x}{b_n}}}{\left(e^{\frac{2\pi i x}{b_n}} - 1\right)^2}. \tag{3.24}$$

Note that $e^{2\pi i k_2 b_{n-1}} = 1$ and thus

$$\begin{aligned} f'(k_2 b_{n-1}) &= \frac{2\pi i (e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1)}{\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right)^2} \\ &= \frac{2\pi i}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1}. \end{aligned} \tag{3.25}$$

Comparing (3.23) and (3.25) we find

$$\sum_{\mu=0}^{b_{n-1}} \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}} = \frac{1}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1}.$$

In the same way we get

$$\sum_{\mu=0}^{b_{n-1}} \frac{\mu}{b_n} e^{\frac{-2\pi i k_2 \mu b_{n-1}}{b_n}} = \frac{1}{e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1}.$$

Therefore,

$$\begin{aligned} \sum_{\mu=0}^{b_{n-1}} \left[\frac{\mu}{b_n} e^{\frac{-2\pi i k_2 \mu b_{n-1}}{b_n}} + \frac{\mu}{b_n} e^{\frac{2\pi i k_2 \mu b_{n-1}}{b_n}}\right] &= \frac{1}{e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1} + \frac{1}{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1} \\ &= \frac{\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right) + \left(e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1\right)}{\left(e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 1\right)\left(e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} - 1\right)} \\ &= \frac{e^{\frac{2\pi i k_2 b_{n-1}}{b_n}} + e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - 2}{2 - e^{\frac{-2\pi i k_2 b_{n-1}}{b_n}} - e^{\frac{2\pi i k_2 b_{n-1}}{b_n}}} \\ &= -1. \end{aligned}$$

Hence from (3.21)

$$\begin{aligned} \widehat{D}(\mathcal{F}'_n, \mathbf{k}) &= \frac{1}{2\pi i k_2} + \frac{-1}{2\pi i k_2}(-1+2) \\ &= 0. \end{aligned} \tag{3.26}$$

$\square$

Remark 3.6. *We define the sets*

$$\begin{aligned} S_1 &= \{(k_1,k_2):\ k_1,k_2\neq 0,\, k_1\equiv 0 \pmod{b_n}\},\\ S_2 &= \{(k_1,k_2):\ k_1,k_2\neq 0,\, k_2\equiv 0 \pmod{b_n}\},\\ S_3 &= \{(k_1,0):\ k_1\equiv 0 \pmod{b_n},\quad k_1\neq 0\},\\ S_4 &= \{(0,k_2):\ k_2\equiv 0 \pmod{b_n},\quad k_2\neq 0\},\\ S_5 &= \{(k_1,k_2):\ (k_1,k_2)\in L(n)\setminus\{\mathbf{0}\},\quad k_1,k_2\not\equiv 0 \pmod{b_n}\},\\ S_6 &= \{(k_1,k_2):\ (k_1,-k_2)\in L(n)\setminus\{\mathbf{0}\},\quad k_1,k_2\not\equiv 0 \pmod{b_n}\}. \end{aligned}$$

Based on previous lemmas, we have the following observations. The results of lemmas 3.2, 3.3, 3.4, and 3.5 imply that for $\mathbf{k}\in S_1\cup\ldots\cup S_6$ *we have*

$$|\widehat{D}(\mathcal{F}_n',\mathbf{k})| \ll \frac{b_n}{\prod_{j=1}^{2}\max(|k_j|,1)}. \tag{3.27}$$

In all other cases, the corresponding Fourier coefficients are equal to zero, see (3.14), (3.18) *and* (3.26).

For $\mathbf{k}\in S_1$, we write $k_1 = lb_n$, where $l\in\mathbb{Z}\setminus\{0\}$. Then $|\widehat{D}(\mathcal{F}_n',\mathbf{k})| = \dfrac{1}{2\pi^2|k_1 l|}$. We deal with S_2, S_3, and S_4 similarly. We are now ready to proceed to the main theorem.

Theorem 3.7. *For the symmetrized Fibonacci set* $\mathcal{F}_n'\subset[0,1]^2$, *we have*

$$\|D(\mathcal{F}_n',\mathbf{x})\|_2 \ll \sqrt{\log b_n}, \tag{3.28}$$

Proof. By Parseval's theorem,

$$\begin{aligned} \|D(\mathcal{F}_n',\mathbf{x})\|_2^2 = \|\widehat{D}(\mathcal{F}_n',\mathbf{k})\|_2^2 &\leq |\widehat{D}(\mathcal{F}_n',\mathbf{0})|^2 + \sum_{i=1}^{6}\sum_{\mathbf{k}\in S_i}|\widehat{D}(\mathcal{F}_n',\mathbf{k})|^2\\ &\ll \sum_{\mathbf{k}\in L(n)\setminus\{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2}\max(k_j^2,1)}\\ &\quad + \sum_{(k_1,-k_2)\in L(n)\setminus\{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2}\max(k_j^2,1)}\\ &\quad +2\sum_{l\neq 0}\sum_{k\neq 0}\frac{1}{(kl)^2} + 2\sum_{l\neq 0}\frac{1}{l^2}. \end{aligned}$$

It is easy to see that the last two sums converge to some constants and the first two are completely similar to each other. We can thus estimate

$$\|D(\mathcal{F}_n',\mathbf{x})\|_2^2 \ll \sum_{\mathbf{k}\in L(n)\setminus\{\mathbf{0}\}} \frac{b_n^2}{\prod_{j=1}^{2}\max(k_j^2,1)}. \tag{3.29}$$

We now use the following lemma, see Lemma 2.1 from Chapter 4 of [20].

Lemma 3.8. *Denote*

$$\Gamma(N) := \big\{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbf{Z}^d : \prod_{j=1}^{d} \max(|k_j|, 1) \le N\big\}$$

and

$$Z_l := \big(\Gamma(2^{l+1}\gamma b_n) \setminus \Gamma(2^l \gamma b_n)\big) \cap L(n), \qquad l = 0, 1, \cdots,$$

then there exists an absolute constant $\gamma > 0$ such that for any $n > 2$

$$\Gamma(\gamma b_n) \cap \big(L(n) \setminus \mathbf{0}\big) = \emptyset,$$

and

$$|Z_l| \ll 2^l (l+1) \log b_n, \qquad l = 0, 1, \cdots . \tag{3.30}$$

Therefore, the summation in (3.29) can be estimated as

$$\|D(\mathcal{F}_n', \mathbf{x})\|_2^2 \ll \sum_{l \ge 0} \sum_{\mathbf{k} \in Z_l} \frac{1}{|2^l|^2}, \tag{3.31}$$

and using the cardinality estimate of Z_l in (3.30), we get,

$$\begin{aligned} \|D(\mathcal{F}_n', \mathbf{x})\|_2^2 &\ll \sum_{l \ge 0} \frac{2^l (l+1) \log b_n}{(2^l)^2} \\ &= \log b_n \sum_{l \ge 0} \frac{l+1}{2^l} \\ &\ll \log b_n. \end{aligned}$$

Hence

$$\|D(\mathcal{F}_n', \mathbf{x})\|_2 \ll \sqrt{\log b_n}.$$

□

Remark 3.9. *In this section we symmetrize the original Fibonacci set to obtain a $2b_n-$point set $\mathcal{F}_n' = \{(p_1, p_2) \cup \{(p_1, 1-p_2) : (p_1, p_2) \in \mathcal{F}_n\}$. Obviously, the L_∞ discrepancy of $\mathcal{F}_n'$ satisfies the same upper bound as $\mathcal{F}_n$ in the order of magnitude and thus is optimal. Theorem 3.7 verifies the sharpness of its L_2 discrepancy.*

In fact, we can also demonstrate that a $4b_n-$point set $\tilde{\mathcal{F}}_n = \{(p_1, p_2) \cup (1 - p_1, p_2) \cup \{(p_1, 1-p_2) \cup \{(1-p_1, 1-p_2) : (p_1, p_2) \in \mathcal{F}_n\}$ achieves the minimal L_2 discrepancy as well. The computation is completely analogous, and, in Case 4 (Lemma 3.5), it is much more straightforward.

We now derive a formula which provides the exact value of $\|D(\mathcal{F}_n', \mathbf{x})\|_2$. For simplicity, we shall first assume that b_n is odd, and thus $S_5 \cap S_6 = \emptyset$. We start with the contribution of $\mathbf{k} \in S_5$, using the notation introduced in Remark 3.6. In this case, $\widehat{D}(\mathcal{F}'{}_n, \mathbf{k}) = -\frac{b_n}{4\pi^2 k_1 k_2}$. We shall make use of the well-known identity, see e.g. [1]:

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+x)^2} = \frac{\pi^2}{\sin^2(\pi x)}. \tag{3.32}$$

Denote $k_1 + k_2 b_{n-1} = l b_n$, for $l \in \mathbb{Z}$ and toward the end of the computation write $k_2 = m b_n + r$, where $m \in \mathbb{Z}$ and $r = 1, ..., b_n - 1$. We have, by Lemma 3.3

$$\begin{aligned}
\sum_{k \in S_5} \left| \widehat{D}(\mathcal{F}'{}_n, \mathbf{k}) \right|^2 &= \frac{b_n^2}{16\pi^4} \sum_{k_2 \not\equiv 0 \bmod b_n} \frac{1}{k_2^2} \sum_{l \in \mathbb{Z}} \frac{1}{b_n^2} \cdot \frac{1}{\left(l - \frac{b_{n-1} k_2}{b_n}\right)^2} \\
&= \frac{1}{16\pi^2} \sum_{k_2 \not\equiv 0 \bmod b_n} \frac{1}{k_2^2 \sin^2\left(\frac{\pi b_{n-1} k_2}{b_n}\right)} \\
&= \frac{1}{16\pi^2} \sum_{r=1}^{b_n - 1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right)} \sum_{m \in \mathbb{Z}} \frac{1}{b_n^2} \cdot \frac{1}{\left(m + \frac{r}{b_n}\right)^2} \\
&= \frac{1}{16 b_n^2} \sum_{r=1}^{b_n - 1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right) \cdot \sin^2\left(\frac{\pi r}{b_n}\right)}, \qquad (3.33)
\end{aligned}$$

where we have used identity (3.32) in the second and the last equalities above. It is obvious that the contribution of $\mathbf{k} \in S_6$ is identical. If b_n is even, a "correction term" $\frac{1}{8 b_n^2}$ arises due to the fact that $S_5 \cap S_6 \neq \emptyset$ (we leave the computation to the reader).

Using the inclusion-exclusion principle and the identity

$$\sum_{l \in \mathbb{N}} \frac{1}{l^2} = \frac{\pi^2}{6}, \qquad (3.34)$$

we obtain, by Lemma 3.2

$$\begin{aligned}
\sum_{k \in S_1 \cup S_2} \left| \widehat{D}(\mathcal{F}'{}_n, \mathbf{k}) \right|^2 &= 4 \sum_{l_1 \in \mathbb{N}, k_2 \in \mathbb{N}} \frac{b_n^2}{4\pi^4 \cdot l_1^2 b_n^2 \cdot k_2^2} + 4 \sum_{k_1 \in \mathbb{N}, l_2 \in \mathbb{N}} \frac{b_n^2}{4\pi^4 \cdot k_1^2 \cdot l_2^2 b_n^2} \\
&\quad -4 \sum_{l_1 \in \mathbb{N}, l_2 \in \mathbb{N}} \frac{b_n^2}{4\pi^4 b_n^4 l_1^2 l_2^2} \\
&= 8 \cdot \frac{1}{4\pi^4} \cdot \frac{\pi^2}{6} \cdot \frac{\pi^2}{6} - 4 \frac{1}{144 b_n^2} = \frac{1}{36} \left(2 - \frac{1}{b_n^2}\right). \qquad (3.35)
\end{aligned}$$

(The multiplication by 4 above accounts for all possible choices of signs).

Finally, Lemmas 3.4 and 3.5 yield

$$\sum_{k \in S_3 \cup S_4} \left| \widehat{D}(\mathcal{F}'{}_n, \mathbf{k}) \right|^2 = 2 \cdot \frac{b_n^2}{4\pi^2} \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{1}{b_n^2 l^2} = \frac{1}{6}. \qquad (3.36)$$

Putting together equations (3.33), (3.35), and (3.36), and the relation $\widehat{D}(\mathcal{F}'{}_n, \mathbf{0}) = \frac{1}{2}$ (Lemma 3.1) we obtain

Theorem 3.10. *For $n \geq 2$ we have*

(3.37)

$$\|D(\mathcal{F}'_n, \mathbf{x})\|_2^2 = \frac{1}{8 b_n^2} \sum_{r=1}^{b_n - 1} \frac{1}{\sin^2\left(\frac{\pi b_{n-1} r}{b_n}\right) \cdot \sin^2\left(\frac{\pi r}{b_n}\right)} + \frac{17}{36} - \frac{1}{36 b_n^2} \quad \text{when } b_n \text{ is odd },$$

(3.38)
$$\|D(\mathcal{F}_n',\mathbf{x})\|_2^2 = \frac{1}{8b_n^2}\sum_{r=1}^{b_n-1}\frac{1}{\sin^2\left(\frac{\pi b_{n-1}r}{b_n}\right)\cdot\sin^2\left(\frac{\pi r}{b_n}\right)} + \frac{17}{36} + \frac{7}{72b_n^2} \quad \text{when } b_n \text{ is even.}$$

It can be shown directly that the main therm in the equations above is of the order $\log b_n \asymp n$. Besides, numerical experiments indicate that

$$\frac{1}{b_n^2}\sum_{r=1}^{b_n-1}\frac{1}{\sin^2\left(\frac{\pi b_{n-1}r}{b_n}\right)\cdot\sin^2\left(\frac{\pi r}{b_n}\right)} \approx 0.1193n. \tag{3.39}$$

4. Centered L_p discrepancy

Consider the following modification of the classical L_p discrepancy function. For a parameter $a\in[0,1/2]$ define the following univariate characteristic function for $t\in[0,1)$.

$$S(a,t) := \chi_{[1/2-a,1/2+a]}(t),$$

and for the multivariate case $\mathbf{x}\in[0,1/2]^d$, $\mathbf{y}\in[0,1]^d$

$$S(\mathbf{x},\mathbf{y}) := \prod_{j=1}^d S(x_j,y_j).$$

Define for a set $\xi := \{\xi^\mu\}_{\mu=1}^N \subset [0,1]^d$ the centered L_p discrepancy as follows

$$D^c(\xi,N,d)_p := \left\|\sum_{\mu=1}^N S(\mathbf{x},\xi^\mu) - N\int_{[0,1]^d} S(\mathbf{x},\mathbf{y})d\mathbf{y}\right\|_{L_p([0,1/2]^d,\mathbf{x})}.$$

In this section we estimate $D^c(\xi,N,d)_p$ from above in the case $d=2$, $p<\infty$, $N=b_n$ and

$$\xi^\mu = (\mu/b_n, \{\mu b_{n-1}/b_n\}), \qquad \mathcal{F}_n := \{\xi^\mu\}_{\mu=1}^{b_n}.$$

We apply here the technique that is based on the Fourier representation of $S(\mathbf{x},\mathbf{y})$ as a function on $\mathbf{y}$. First, we find the Fourier representation of the univariate function

$$\hat{S}(a,k) = \int_0^1 S(a,t)e^{-2\pi ikt}dt = (-1)^k(2\pi ik)^{-1}(e^{2\pi ika} - e^{-2\pi ika}).$$

It is clear that $\hat{S}(a,0)=2a$. Second, it follows directly from the definition of $S(\mathbf{x},\mathbf{y})$ and the above formulas that

$$|\hat{S}(\mathbf{x},\mathbf{k})| = \prod_{j=1}^d |\hat{S}(x_j,k_j)| \le \prod_{j=1}^d \max(|k_j|,1)^{-1}. \tag{4.1}$$

Denote

$$\Phi(\mathbf{k}) = \sum_{\mu=1}^{b_n} e^{2\pi i(\mathbf{k},\xi^\mu)}.$$

Then for a trigonometric polynomial f one has

$$\Phi_n(f) := \sum_{\mu=1}^{b_n} f\big(\mu/b_n, \{\mu b_{n-1}/b_n\}\big) = \sum_{\mathbf{k}} \hat{f}(\mathbf{k})\Phi(\mathbf{k}). \tag{4.2}$$

It is known and easy to see that the following relation holds

$$\Phi(\mathbf{k}) = \begin{cases} b_n, & \mathbf{k} \in L(n), \\ 0, & \mathbf{k} \notin L(n). \end{cases} \tag{4.3}$$

Therefore, in the case $p = 2$, that we discuss first

$$D^c(\mathcal{F}_n, b_n, 2)_2 \le \left\| \sum_{\mathbf{k} \neq (0,0)} \Phi(\mathbf{k}) \hat{S}(\mathbf{x}, \mathbf{k}) \right\|_2 . \tag{4.4}$$

Using the fact that functions $\hat{S}(\mathbf{x}, \mathbf{k})$ and $\hat{S}(\mathbf{x}, \mathbf{k}')$ are orthogonal on $[0,1]^2$ if $(|k_1|, |k_2|) \neq (|k_1'|, |k_2'|)$ and using the bound (4.1) and using the (3.30) again we get from (4.4)

$$D^c(\mathcal{F}_n, b_n, 2)_2^2 \ll \sum_{l=0}^{\infty} b_n^2 (2^l b_n)^{-2} |Z_l| \ll \log b_n \sum_{l=0}^{\infty} \frac{2^l (l+1)}{2^{2l}} \ll \log b_n.$$

Thus,

$$D^c(\mathcal{F}_n, b_n, 2)_2 \ll \sqrt{\log b_n}.$$

We now proceed to the case $p \in [2, \infty)$. Let

$$\psi_l(\mathbf{x}) := \sum_{k \in Z_l} \hat{S}(\mathbf{x}, \mathbf{k}).$$

Then

$$D^c(\mathcal{F}_n, b_n, 2)_p \le b_n \sum_{l=0}^{\infty} \|\psi_l\|_p. \tag{4.5}$$

By the corollary of the Littlewood-Paley theorem we have for $\|\psi_l\|_p$

$$\|\psi_l\|_p \ll \left(\sum_{\mathbf{s}} \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_p^2 \right)^{1/2},$$

where for $\mathbf{s} = (s_1, s_2)$, s_j are nonnegative integers

$$\delta_{\mathbf{s}}(f, \mathbf{x}) := \sum_{\substack{[2^{s_j - 1}] \le |k_j| < 2^{s_j}, \\ j = 1,2}} \hat{f}(\mathbf{k}) e^{i(\mathbf{k}, \mathbf{x})}.$$

It is not difficult to see that for ψ_l only those $\delta_{\mathbf{s}}(\psi_l)$ can be nonzero for which

$$\big| \, \|\mathbf{s}\|_1 - \log_2(2^l \gamma b_n) \big| \le C.$$

In addition by lemma 3.8 the number of terms of $\delta_{\mathbf{s}}(\psi_l)$ is not greater than $C 2^l$. Therefore,

$$\big\| \delta_{\mathbf{s}}(\psi_l) \big\|_p \le \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_2^{2/p} \big\| \delta_{\mathbf{s}}(\psi_l) \big\|_\infty^{1-2/p} \ll 2^{-l/p} b_n^{-1} \tag{4.6}$$

and

$$\|\psi_l\|_p \ll (l + \log b_n)^{1/2} 2^{-l/p} b_n^{-1}. \tag{4.7}$$

The bounds (4.5) and (4.6) imply

$$D^c(\mathcal{F}_n, b_n, 2)_p \le C(p) \sqrt{\log b_n}.$$

We discuss the relations between centered L_p discrepancy and standard L_p discrepancy. We have

$$S(a,t) = \chi_{[0,\frac{1}{2}+a]}(t) - \chi_{[0,\frac{1}{2}-a]}(t).$$

This allows us to obtain the following inequality

$$D^c(\xi, N, d)_p \le 2^d \|D(\xi, \mathbf{x})\|_p.$$

The centered L_p discrepancy can be bounded from below by the L_p discrepancy of a symmetrized set $\bar{\xi}$, that we define momentarily. We describe it in the case $d = 2$. Let R_1 and R_2 be reflection operators that act as follows: for $\mathbf{u} = (u_1, u_2) \in [0,1]^2$

$$R_1(\mathbf{u}) := (1-u_1, u_2), \qquad R_2(\mathbf{u}) := (u_1, 1-u_2).$$

For a set $\xi = \{\xi^j\}_{j=1}^N \subset [0,1]^2$, define the symmetrized set

$$\bar{\xi} := \xi \cup R_1(\xi) \cup R_2(\xi) \cup R_2(R_1(\xi)).$$

This set contains $4N$ points, counting multiplicity. The sets

$$G_1(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2}+x_1\right) \times \left[\frac{1}{2}, \frac{1}{2}+x_2\right), \qquad G_2(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2}-x_1\right) \times \left[\frac{1}{2}, \frac{1}{2}+x_2\right),$$

$$G_3(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2}-x_1\right) \times \left[\frac{1}{2}, \frac{1}{2}-x_2\right), \qquad G_4(\mathbf{x}) := \left[\frac{1}{2}, \frac{1}{2}+x_1\right) \times \left[\frac{1}{2}, \frac{1}{2}-x_2\right),$$

counting the same number of points of $\bar{\xi}$ since we split the same number of the points in set $\bar{\xi}$ on the boundary evenly.
Then for the centered L_p discrepancy of $\bar{\xi}$ we have

$$\begin{aligned} D^c(\bar{\xi}, 4N, 2)_p^p &= 4\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left| \sum_{\mathbf{u}\in\bar{\xi}\cap[\frac{1}{2},1]\times[\frac{1}{2},1]} \chi_{G_1(\mathbf{x})}(\mathbf{u}) - 4N\cdot x_1x_2 \right|^p dx_1dx_2 \\ &= \int_0^1\int_0^1 \left| \sum_{\mathbf{v}\in\eta} \chi_{G_1(\mathbf{z})}(\mathbf{v}) - N\cdot z_1z_2 \right|^p dz_1dz_2, \end{aligned}$$

where

$$\mathbf{z} = 2\mathbf{x}, \mathbf{v} = 2\left(\mathbf{u} - \frac{\mathbf{1}}{2}\right) + \frac{\mathbf{1}}{2} = 2\mathbf{u} - \frac{\mathbf{1}}{2};$$

$$\eta := \left\{ \mathbf{v} = 2\mathbf{u} - \frac{\mathbf{1}}{2}, \quad \mathbf{u} \in \bar{\xi} \cap \left([\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \right) \right\}.$$

Thus,

$$D^c(\bar{\xi}, 4N, 2)_p \ge \|D(\eta - \frac{\mathbf{1}}{2}, \mathbf{x})\|_p.$$

Clearly,

$$D^c(\bar{\xi}, 4N, 2)_p \le 4D^c(\xi, N, 2)_p.$$

Therefore,

$$D^c(\xi, N, 2)_p \ge \frac{1}{4}\|D(\eta - \frac{1}{2}, \mathbf{x})\|_p.$$

It is known that for all $p > 1$ and any set $\mathcal{P}_N$ of N points one has

$$\|D(\mathcal{P}_N, \mathbf{x})\|_p \ge C\sqrt{\log N}, \tag{4.8}$$

where C is some positive absolute constant.

This implies that the Fibonacci sets $\mathcal{F}_n$ have optimal centered L_p discrepancy for $p \in (1, \infty)$ in the sense of order.

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