2012:02

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April 8, 2012

Abstract

We study Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases. We mostly concentrate on this study in the L_p spaces. The novelty of the paper is in obtaining better Lebesgue-type inequalities under extra assumptions on a quasi-greedy basis than known Lebesgue-type inequalities for quasi-greedy bases. We consider uniformly bounded quasi-greedy bases of L_p, $1 < p < \infty$, and prove that for such bases an extra multiplier in the Lebesgue-type inequality can be taken as $C(p)\ln(m+1)$. The known magnitude of the corresponding multiplier for general (no assumption of uniform boundedness) quasi-greedy bases is of order $m^{|\frac{1}{2}-\frac{1}{p}|}$, $p \neq 2$. For uniformly bounded orthonormal quasi-greedy bases we get further improvements replacing $\ln(m+1)$ by $(\ln(m+1))^{1/2}$.

1 Introduction

We study the efficiency of greedy algorithms for m-term nonlinear approximation with regard to bases. Our primary interest is in approximation in L_p

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Math Subject Classifications. primary: 41A65; secondary: 41A25, 41A46, 46B20.

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with respect to quasi-greedy bases. Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^\infty$ be a semi-normalized basis for X ($0 < c_0 \le \|\psi_k\| \le C_0, k \in \mathbb{N}$). All bases considered in our paper are assumed to be semi-normalized. For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f, \Psi)_X := \inf_{b_k, \Lambda} \|f - \sum_{k \in \Lambda} b_k \psi_k\|_X,$$

where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^\infty c_k(f) \psi_k.$$

We call a permutation ρ, $\rho(j) = k_j, j = 1, 2, ...,$ of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \ge |c_{k_2}(f)| \ge \dots .$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f) := G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^m c_{k_j}(f) \psi_{k_j}.$$

This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA). The best we can achieve with the algorithm G_m is

$$\|f - G_m(f)\|_X = \sigma_m(f, \Psi)_X,$$

or a little weaker

$$\|f - G_m(f)\|_X \le C \sigma_m(f, \Psi)_X$$

for all $f \in X$ with a constant C independent of f and m. The following concept of a greedy basis has been introduced in [8].

Definition 1.1. *We call a basis Ψ a greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f, \Psi, \rho)\|_X \leq C\sigma_m(f, \Psi)_X$$

with a constant C independent of f and m.

We refer the reader to a survey [23] and a book [24] for further discussion of greedy type bases. In this paper we are interested in special inequalities – Lebesgue-type inequalities – for greedy approximation.

Lebesgue [11] proved the following inequality: for any 2π-periodic continuous function f we have

$$\|f - S_n(f)\|_\infty \leq (4 + \frac{4}{\pi^2} \ln n) E_n(f)_\infty, \tag{1.1}$$

where $S_n(f)$ is the nth partial sum of the Fourier series of f and $E_n(f)_\infty$ is the error of the best approximation of f by the trigonometric polynomials of order n in the uniform norm $\|\cdot\|_\infty$. The inequality (1.1) relates the error of a particular method (S_n) of approximation by the trigonometric polynomials of order n to the best-possible error $E_n(f)_\infty$ of approximation by the trigonometric polynomials of order n. By the Lebesgue-type inequality we mean an inequality that provides an upper estimate for the error of a particular method of approximation of f by elements of a special form, say, form $\mathcal{A}$, by the best-possible approximation of f by elements of the form $\mathcal{A}$. In the case of approximation with regard to bases (or minimal systems), the Lebesgue-type inequalities are known both in linear and in nonlinear settings (see surveys [9], [22], and [23]).

By the Definition 1.1 greedy bases are those for which we have ideal (up to a multiplicative constant) Lebesgue-type inequalities for greedy approximation. In this paper we concentrate on a wider class of bases than greedy bases – quasi-greedy bases. The concept of quasi-greedy basis was introduced in [8].

Definition 1.2. *The basis Ψ is called quasi-greedy if there exists some constant C such that*

$$\sup_m \|G_m(f, \Psi)\| \leq C\|f\|.$$

Subsequently, Wojtaszczyk [27] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{n\to\infty} G_n(f) = f.$$

The main result of [26] is the following Lebesgue-type inequality for greedy approximation with respect to a quasi-greedy basis in the L_p spaces.

Theorem 1.1. *Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have*

$$\|f - G_m(f, \Psi)\|_{L_p} \leq C(p, \Psi) m^{|1/2-1/p|} \sigma_m(f, \Psi)_{L_p}. \tag{1.2}$$

Theorem 1.1 does not cover the case $p = 2$. It is mentioned in [27] that in the case $p = 2$ one has the following inequality

$$\|f - G_m(f, \Psi)\|_{L_2} \leq C(\Psi) \ln(m+1) \sigma_m(f, \Psi)_{L_2}.$$

We do not know if the above inequality is sharp in the sense that an extra factor $\log m$ cannot be replaced by a slower growing factor. The reader can find further discussion of this problem in [25].

We note that inequality (1.2) is known (see [27]) in the case of unconditional bases Ψ. It is proved in [21] that (1.2) holds for the trigonometric system $\Psi = \{e^{ikx}\}$ for all $1 \leq p \leq \infty$. It was noticed in [21] that (1.2) holds for any uniformly bounded orthonormal basis of L_2. Thus, it was known that bases satisfying very different in nature conditions – uniformly bounded orthonormal basis of L_2 or quasi-greedy basis of L_p – both guarantee that similar Lebesgue-type inequalities (1.2) hold for greedy approximation. In this paper we continue to study Lebesgue-type inequalities for greedy approximation. We try to make a bridge between the two above conditions – uniformly bounded orthonormal basis of L_2 and quasi-greedy basis of L_p. We consider uniformly bounded quasi-greedy bases of L_q and study Lebesgue-type inequalities in L_p, $q \leq p$. It turns out that even the question of existence of such bases is nontrivial. For instance, it is known (see [3]) that there is no uniformly bounded unconditional bases in L_p, $p \neq 2$. Quasi-greedy bases are close to unconditional bases. However, surprisingly, it turns out that there exist uniformly bounded quasi-greedy bases in all L_q with $1 < q < \infty$. We discuss this issue in Section 3, where we present a construction of uniformly bounded quasi-greedy bases. In particular, we prove the following theorem there.

Theorem 1.2. *There exists a uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^{\infty}$ in L_p, $1 < p < \infty$, that consists of trigonometric polynomials.*

We note that existence of uniformly bounded orthonormal quasi-greedy bases was proved in [14]. The construction in [14] is a variation on a constuction by [10]. The same type of construction was used in [27]. Our construction in Section 3 is a somewhat more general version of the known construction. We include it in the paper for completeness. It is based on the trigonometric system and this fact allows us to build bases of interest consisting of trigonometric polynomials. It is important when we consider the Hardy spaces $H_p(D)$ of analytic functions. The construction in [14] is based on the Walsh system.

It is known from [1] that the space $C[0,1]$ does not have quasi-greedy bases and the space $L_1[0,1]$ has quasi-greedy bases. In Section 4 we prove, in particular, that the space $L_1[0,1]$ does not have a uniformly bounded quasi-greedy basis.

In Section 5 we prove Lebesgue-type inequalities for greedy approximation in L_p, $2 \le p \le \infty$ under different assumptions on a basis Ψ. In that section we assume that Ψ is a uniformly bounded basis. In addition we assume that Ψ is a certain type basis (quasi-greedy basis, Riesz basis) in one of the spaces L_2, L_q, $1 < q < 2$, or L_q, $2 < q < \infty$. Here is a typical result from Section 5 (see Theorem 5.2). We will often use the notation $h(p) := |\frac{1}{2} - \frac{1}{p}|$. We also use the brief notation $\|\cdot\|_p := \|\cdot\|_{L_p}$.

Theorem 1.3. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have for $2 \le p \le \infty$

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\ln(m+1)\|f - t_m\|_2.$$

In Section 6 we continue to prove Lebesgue-type inequalities for greedy approximation in L_p under different assumptions on a basis Ψ. In that section we assume that Ψ is a semi-normalized quasi-greedy basis for a pair of spaces: L_q, $1 < q < \infty$, and L_p, $q \le p$. It turns out that this assumption results in a dramatic improvement of the corresponding Lebesgue-type inequalities. It is demonstrated by the following result (see Theorem 6.1).

Theorem 1.4. *Assume that Ψ is a semi-normalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q.$$

We now formulate some of the Lebesgue-type inequalities obtained in the paper. We already mentioned above (see Theorem 1.1) that the Lebesgue-type inequalities in L_p, $1 < p < \infty$, under assumption that Ψ is a quasi-greedy basis of L_p were obtained in [26]. First we give a definition of a democratic basis.

Definition 1.3. *We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a democratic basis for X if there exists a constant $D := D(X,\Psi)$ such that, for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$, we have*

$$\|\sum_{k\in P}\psi_k\| \le D\|\sum_{k\in Q}\psi_k\|.$$

In Section 5 we prove that if Ψ is both quasi-greedy and democratic then for any $f \in X$

$$\|f - G_m(f,\Psi)\|_X \le C\ln(m+1)\sigma_m(f,\Psi)_X. \tag{1.3}$$

We note that it is proved in [2] that bases which are simultaneously quasi-greedy and democratic are exactly almost greedy bases. As a corollary of (1.3) we obtain the Lebesgue-type inequality for a uniformly bounded quasi-greedy basis of L_p, $1 < p < \infty$ (see Corollary 5.3):

$$\|f - G_m(f,\Psi)\|_p \le C(p)\ln(m+1)\sigma_m(f,\Psi)_p. \tag{1.4}$$

Here $\sigma_m(f,\Psi)_p := \sigma_m(f,\Psi)_{L_p}$. Comparing (1.4) with (1.2) we see that an extra assumption of uniform boundedness of the basis improves the Lebesgue-type inequalities dramatically.

In Section 6, making our assumptions on the basis even stronger, we improve (1.4) to the following inequality

$$\|f - G_m(f,\Psi)\|_p \le C(p)(\ln(m+1))^{1/2}\sigma_m(f,\Psi)_p, \tag{1.5}$$

under assumption that Ψ is a uniformly bounded orthonormal quasi-greedy basis of L_p, $2 \le p < \infty$.

In Section 5 we impose assumptions on the basis in the L_q space and obtain inequalities in the L_p space:

$$\|f - G_m(f,\Psi)\|_p \le C(p,q) m^{(1-q/p)/2} \ln(m+1) \sigma_m(f,\Psi)_p \tag{1.6}$$

under assumption that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$, and $2 \le p < \infty$, $p \ge q$. We note that in the case $p = q$ inequality (1.6) turns into (1.4).

We begin a systematic presentation with Section 2, where we list some properties of quasi-greedy bases that are used in the paper.

2 Properties of quasi-greedy bases

Quasi-greedy bases. The definition of a quasi-greedy basis is given in the Introduction (see Definition 1.2). We give here an equivalent definition (see [24], p. 34). For a set of indices Λ we define the corresponding partial sum as follows

$$S_\Lambda(f) := S_\Lambda(f,\Psi) := \sum_{k\in\Lambda} c_k(f)\psi_k.$$

Definition 2.1. *We say that a basis Ψ is quasi-greedy if there exists a constant C_Q such that, for any $f \in X$ and any finite set of indices Λ having the property*

$$\min_{k\in\Lambda} |c_k(f)| \ge \max_{k\notin\Lambda} |c_k(f)|,$$

we have

$$\|S_\Lambda(f,\Psi)\| \le C_Q\|f\|.$$

First, we present some known useful properties of quasi-greedy bases. The reader can find the following two lemmas in [24], p. 37.

Lemma 2.1. *Let Ψ be a quasi-greedy basis. Then, for any two finite sets of indices $A \subseteq B$ and coefficients $0 < t \le |c_j| \le 1$, $j \in B$, we have*

$$\|\sum_{j\in A} c_j\psi_j\| \le C(X,\Psi,t)\|\sum_{j\in B} c_j\psi_j\|.$$

It will be convenient to define the quasi-greedy constant K to be the least constant such that

$$\|G_m(f)\| \le K\|f\| \quad \text{and} \quad \|f - G_m(f)\| \le K\|f\|, \quad f \in X.$$

Lemma 2.2. *Suppose Ψ is a quasi-greedy basis with a quasi-greedy constant K. Then, for any real numbers c_j and any finite set of indices P, we have*

$$(4K^2)^{-1} \min_{j\in P} |c_j| \|\sum_{j\in P} \psi_j\| \le \|\sum_{j\in P} c_j\psi_j\| \le 2K \max_{j\in P} |c_j| \|\sum_{j\in P} \psi_j\|.$$

We present the following lemma from [1] with a proof for completeness.

Lemma 2.3. *Let Ψ be a quasi-greedy basis of X. Then for any finite set of indices Λ we have for all $f \in X$*

$$\|S_\Lambda(f,\Psi)\| \le C \ln(|\Lambda| + 1)\|f\|.$$

Proof. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

Let a sequence $k_j, j = 1, 2, ...,$ of the positive integers be such that

$$|c_{k_1}(f)| \ge |c_{k_2}(f)| \ge ... \,.$$

We will use the notation

$$a_n(f) := |c_{k_n}(f)|$$

for the decreasing rearrangement of the coefficients of f. Without loss of generality assume that f is normalized in such a way that guarantees that $|a_1(f)| \le 1$ and consider $m := |\Lambda| \ge 2$. Consider for integer $s \ge 0$

$$\tau_s := \{k : 2^{-s} \le |c_k(f)| < 2^{1-s}\}.$$

Denote

$$\Lambda_s := \Lambda \cap \tau_s, \qquad \Lambda' := \Lambda \setminus (\cup_{s\le \log_2 m}\Lambda_s).$$

The semi-normalization property of the basis Ψ implies

$$\|S_{\Lambda'}(f)\| \le \frac{2}{m}|\Lambda'|C_0 \le 2C_0.$$

For $s \le \log_2 m$ we have

$$S_{\Lambda_s}(f) = S_{\Lambda_s}(S_{\tau_s}(f)).$$

By Lemma 2.1 we obtain

$$\|S_{\Lambda_s}(f)\| \le C\|S_{\tau_s}(f)\|.$$

Our assumption that Ψ is a quasi-greedy basis implies that for all s

$$\|S_{\tau_s}(f)\| \le C\|f\|.$$

Thus, for all $s \le \log_2 m$

$$\|S_{\Lambda_s}(f)\| \le C\|f\|,$$

and, therefore,

$$\|S_{\Lambda}(f)\| \le C \ln(m+1)\|f\|.$$

□

The following Lemma 2.4 is a new result that answers Question 2 from [6]. Let

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We define the following expansional best m-term approximation of f

$$\tilde{\sigma}_m(f) := \tilde{\sigma}_m(f, \Psi) := \inf_{\Lambda, |\Lambda|=m} \|f - \sum_{k\in\Lambda} c_k(f)\psi_k\|.$$

It is clear that

$$\sigma_m(f, \Psi) \le \tilde{\sigma}_m(f, \Psi).$$

It is also clear that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f, \Psi) \le C(X, \Psi)\sigma_m(f, \Psi).$$

Lemma 2.4. *Let Ψ be a quasi-greedy basis of X. Then for all $f \in X$*

$$\tilde{\sigma}_m(f) \le C \ln(m+1)\sigma_m(f).$$

Proof. For a given $\epsilon > 0$ let p_m be an m-term polynomial

$$p_m := \sum_{k \in P} b_k \psi_k, \qquad |P| = m,$$

such that

$$\|f - p_m\| \leq \sigma_m(f) + \epsilon.$$

Then by Lemma 2.3 we obtain

$$\tilde{\sigma}_m(f) \leq \|f - S_P(f)\| = \|f - p_m - S_P(f - p_m)\| \leq C \ln(m+1)(\sigma_m(f) + \epsilon).$$

This completes the proof of Lemma 2.4.

□

We now formulate a result about quasi-greedy bases in L_p spaces. The following theorem is from [25]. We note that in the case $p = 2$ Theorem 2.1 was proved in [27].

Theorem 2.1. *Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of the L_p space, $1 < p < \infty$. Then for each $f \in X$ we have*

$$C_1(p) \sup_n n^{1/p} a_n(f) \leq \|f\|_p \leq C_2(p) \sum_{n=1}^\infty n^{-1/2} a_n(f), \quad 2 \leq p < \infty;$$

$$C_3(p) \sup_n n^{1/2} a_n(f) \leq \|f\|_p \leq C_4(p) \sum_{n=1}^\infty n^{1/p-1} a_n(f), \quad 1 < p \leq 2.$$

Remark 2.1. *Theorem 2.1 was proved in [25] under assumption that Ψ is a normalized basis. That proof works for a semi-normalized basis as well.*

Remark 2.2. *The proof of Theorem 2.1 in [25] gives the following inequalities. Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of X. If for any set of indices A of cardinality m we have $\|\sum_{k\in A} \psi_k\|_X \leq C' m^{1/2}$ then for each $f \in X$ we have*

$$\|f\|_X \leq C_1 \sum_{n=1}^\infty n^{-1/2} a_n(f). \tag{2.1}$$

If for any set of indices A of cardinality m we have $\|\sum_{k\in A} \psi_k\|_X \geq c' m^{1/2}$ then for each $f \in X$ we have

$$\|f\|_X \geq c_1 \sup_n n^{1/2} a_n(f).$$

A general version of (2.1) was obtained in [6]. We define the *fundamental function* $\varphi(m) := \varphi(m, \Psi, X)$ of a basis Ψ in X by

$$\varphi(m, \Psi, X) := \sup_{|A| \leq m} \|\sum_{k \in A} \psi_k\|.$$

Lemma 2.5. *Let Ψ be a quasi-greedy basis of X. Then for each $f \in X$ we have*

$$\|f\| \leq C \sum_{n=1}^{\infty} a_n(f) \varphi(n) \frac{1}{n}.$$

Proof. It is known (see [2], p. 581) that $\varphi(n)/n$ is monotone decreasing. Therefore, by Lemma 2.2 we obtain

$$\|f\| \leq \sum_{s=1}^{\infty} \| \sum_{n=2^{s-1}}^{2^s - 1} a_n(f) \psi_{k_n} \|$$

$$\leq C \sum_{s=1}^{\infty} a_{2^{s-1}}(f) \varphi(2^{s-1}) \leq C \sum_{n=1}^{\infty} a_n(f) \varphi(n) \frac{1}{n}.$$

□

Uniformly bounded quasi-greedy bases. It is clear that any orthonormal basis of a Hilbert space H is an unconditional basis and, therefore, a quasi-greedy basis of H. For example, the trigonometric basis is a uniformly bounded orthonormal basis of L_2. Even the question of existence of a uniformly bounded quasi-greedy basis in L_p, $p \neq 2$, is a nontrivial question. It is known (see [Gap]) that there is no uniformly bounded unconditional bases in L_p, $p \neq 2$. As we already mentioned in the Introduction, there are uniformly bounded quasi-greedy bases in L_p, $1 < p < \infty$. We build such bases in Section 3. We now present some properties of these bases. We begin with proving an analog of Lemma 2.2 from [21]. Lemma 2.2 from [21] was proved for the trigonometric system. We will prove its analog for a uniformly bounded Riesz basis of L_2.

Lemma 2.6. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then for any set Λ of indices we have for $2 \leq p \leq \infty$*

$$\|S_\Lambda(f)\|_p \leq C |\Lambda|^{h(p)} \|f\|_2.$$

Proof. Let

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

Our assumptions on Ψ imply

$$\|S_\Lambda(f)\|_2 \le C\|f\|_2$$

and

$$\|S_\Lambda(f)\|_\infty \le \sum_{k\in\Lambda} |c_k(f)|\|\psi_k\|_\infty \le C|\Lambda|^{1/2}(\sum_{k\in\Lambda} |c_k(f)|^2)^{1/2} \le C|\Lambda|^{1/2}\|f\|_2.$$

Using the inequality

$$\|g\|_p \le \|g\|_2^{2/p}\|g\|_\infty^{1-2/p}, \qquad 2 \le p \le \infty, \tag{2.2}$$

we obtain the required bound from the above inequalities. $\square$

We now prove an analog of Lemma 2.6 for uniformly bounded quasi-greedy bases.

Lemma 2.7. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for any set Λ of indices we have for $2 < p \le \infty$*

$$\|S_\Lambda(f)\|_p \le C|\Lambda|^{h(p)}\|S_\Lambda(f)\|_2. \tag{2.3}$$

We also have

$$\|S_\Lambda(f)\|_2 \le C\ln(|\Lambda| + 1)\|f\|_2. \tag{2.4}$$

Proof. First, we prove (2.4). We note that (2.4) follows from Lemma 2.3 which does not require uniform boundedness of a basis. We give another proof here that does not require uniform boundedness of a basis too. Using notation $m := |\Lambda|$ we obtain by Theorem 2.1

$$\|S_\Lambda(f)\|_2 \le C_2(2)\sum_{n=1}^{m} n^{-1/2}a_n(S_\Lambda(f)) \le C\sum_{n=1}^{m} n^{-1/2}a_n(f)$$

$$\le C\sum_{n=1}^{m} n^{-1/2}C_3(2)^{-1}\|f\|_2 n^{-1/2} \le C\ln(m+1)\|f\|_2.$$

This proves (2.4).

Second, we prove (2.3). We have

$$\|S_\Lambda(f)\|_\infty \le \sum_{k\in\Lambda} |c_k(f)|\|\psi_k\|_\infty \le C\sum_{n=1}^m a_n(S_\Lambda(f))$$

$$\le C\sum_{n=1}^m n^{-1/2}\|S_\Lambda(f)\|_2 \le Cm^{1/2}\|S_\Lambda(f)\|_2.$$

The above inequality combined with (2.2) gives (2.3). □

In the following lemma we replace a quasi-greedy assumption in L_2 by the corresponding assumption in L_q, $1 < q < \infty$.

Lemma 2.8. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$. Then for any set Λ of indices we have for $q < p \le \infty$*

$$\|S_\Lambda(f)\|_p \le C|\Lambda|^{(1-q/p)/2}\|S_\Lambda(f)\|_q. \tag{2.5}$$

We also have

$$\|S_\Lambda(f)\|_q \le C\ln(|\Lambda|+1)\|f\|_q. \tag{2.6}$$

Proof. Inequality (2.6) follows from Lemma 2.3. We prove (2.5). We have

$$\|S_\Lambda(f)\|_\infty \le \sum_{k\in\Lambda} |c_k(f)|\|\psi_k\|_\infty \le C\sum_{n=1}^m a_n(S_\Lambda(f)).$$

By Proposition 2.2 (see below) we continue

$$\le C\sum_{n=1}^m n^{-1/2}\|S_\Lambda(f)\|_q \le Cm^{1/2}\|S_\Lambda(f)\|_q.$$

The above inequality combined with

$$\|g\|_p \le \|g\|_q^{q/p}\|g\|_\infty^{1-q/p}, \qquad q \le p \le \infty, \tag{2.7}$$

gives (2.5).

We note that in the case $1 < q \le 2$ we could use Theorem 2.1 instead of Proposition 2.2. □

Uniformly bounded orthonormal quasi-greedy bases. We prove in Section 3 that there exist uniformly bounded orthonormal quasi-greedy bases in L_p, $1<p<\infty$. We also prove in Section 3 that if Ψ is a uniformly bounded orthonormal quasi-greedy basis in L_p, $2\le p<\infty$ then Ψ is a quasi-greedy basis of $L_{p'}$ $(p^{-1}+p'^{-1}=1)$. Thus there are uniformly bounded bases which are quasi-greedy bases of two spaces L_p and $L_{p'}$, $2<p<\infty$. We now present some results in this direction. We prove an analog of Lemma 2.7.

Lemma 2.9. *Assume that Ψ is a semi-normalized quasi-greedy basis for both L_q and L_p with $1<q\le 2\le p<\infty$. Then for any set Λ of indices we have*

$$\|S_\Lambda(f)\|_p \le C\ln(|\Lambda|+1)\|f\|_q. \tag{2.8}$$

Proof. Using notation $m:=|\Lambda|$ we obtain by Theorem 2.1

$$\|S_\Lambda(f)\|_p \le C_2(p)\sum_{n=1}^m n^{-1/2}a_n(S_\Lambda(f)) \le C(p)\sum_{n=1}^m n^{-1/2}a_n(f)$$

$$\le C(p)\sum_{n=1}^m n^{-1/2}C_3(q)^{-1}\|f\|_q n^{-1/2} \le C(p,q)\ln(m+1)\|f\|_q.$$

This proves (2.8). □

Lemma 2.10. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis of L_p, $2<p<\infty$. Then for any set Λ of indices we have*

$$\|S_\Lambda(f)\|_2 \le C(\ln(|\Lambda|+1))^{1/2}\|f\|_{p'} \tag{2.9}$$

and

$$\|S_\Lambda(f)\|_p \le C(\ln(|\Lambda|+1))^{1/2}\|f\|_2. \tag{2.10}$$

Proof. Let $|\Lambda|=m$. By Theorem 2.2 (see below) and Theorem 2.1 we have in the case of (2.9)

$$\|S_\Lambda(f)\|_2 \le (\sum_{n=1}^m a_n(f)^2)^{1/2} \le C(\sum_{n=1}^m n^{-1}\|f\|_{p'}^2)^{1/2} \le C(\ln(m+1))^{1/2}\|f\|_{p'}.$$

In the case of (2.10) we obtain by Theorem 2.1

$$\|S_\Lambda(f)\|_p \le C\sum_{n=1}^m n^{-1/2}a_n(f)$$

$$\leq C(\ln(m+1))^{1/2}(\sum_{n=1}^{m} a_n(f)^2)^{1/2} \leq C(\ln(m+1))^{1/2}\|f\|_2.$$

□

Let us discuss in more details uniformly bounded orthonormal quasi-greedy bases. Existence of such bases is guaranteed by Theorem 3.3. We first recall the definition of bases which are called unconditional for constant coefficients, cf. [27].

Definition 2.2. *A basis Ψ is called unconditional for constant coefficients (UCC) if there exist constants C_1 and C_2 such that for each finite subset $A \subset \mathbb{N}$ and for each choice of signs $\varepsilon_i = \pm 1$ we have*

$$C_1\|\sum_{i\in A}\psi_i\| \leq \|\sum_{i\in A}\varepsilon_i\psi_i\| \leq C_2\|\sum_{i\in A}\psi_i\|.$$

It is known ([27]) that quasi-greedy bases are UCC bases. To formulate our results we need some of the basic concepts of the Banach space theory from [12]. First, let us recall the definition of type and cotype. Let $\{\varepsilon_i\}$ be a sequence of independent Rademacher variables. We say that a Banach space X has type p if there exists a universal constant C_3 such that for $f_k \in X$

$$\left(\mathrm{Ave}_{\varepsilon_k=\pm 1}\|\sum_{k=1}^{n}\varepsilon_k f_k\|^p\right)^{1/p} \leq C_3\left(\sum_{k=1}^{n}\|f_k\|^p\right)^{1/p},$$

and X is of cotype q if there exists a universal constant C_4 such that for $f_k \in X$

$$\left(\mathrm{Ave}_{\varepsilon_k=\pm 1}\|\sum_{k=1}^{n}\varepsilon_k f_k\|^q\right)^{1/q} \geq C_4\left(\sum_{k=1}^{n}\|f_k\|^q\right)^{1/q}.$$

It is known that L_p, $2 \leq p < \infty$ has type 2. Consider uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^{\infty}$ in L_p, $2 < p < \infty$. Then we obtain from its orthonormality and property UCC that for any set A of indices of cardinality m we have

$$m^{1/2} = \|\sum_{k\in A}\psi_k\|_2 \leq \|\sum_{k\in A}\psi_k\|_p \asymp (\mathrm{Ave}_{\varepsilon_k=\pm 1}\|\sum_{k\in A}\varepsilon_k\psi_k\|_p^p)^{1/p}$$

$$\asymp \|(\sum_{k\in A}|\psi_k|^2)^{1/2}\|_p \leq C(p)(\sum_{k\in A}\|\psi_k\|_p^2)^{1/2} \asymp m^{1/2}. \tag{2.11}$$

Equation (2.11) shows that for uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_p, $2 < p < \infty$, we have $\varphi(m, \Psi, L_p) \asymp m^{1/2}$. In particular, this implies that Ψ is democratic. We consider along with the basis Ψ in L_p its dual basis Ψ^* in $L_{p'}$. By orthonormality of Ψ we get that $\Psi^* = \Psi$. Properties of dual bases to quasi-greedy and almost greedy bases are discussed in detail in [2]. In particular, by Proposition 4.4 and Theorem 5.4 from [2] relation $\varphi(m, \Psi, L_p) \asymp m^{1/2}$ implies that Ψ is also a quasi-greedy basis of $L_{p'}$. We formulate this conclusion as a theorem.

Theorem 2.2. *Let Ψ be a uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_p, $2 < p < \infty$. Then Ψ is a quasi-greedy basis of $L_{p'}$.*

The definition of democratic basis is given in the Introduction (see Definition 1.3).

Proposition 2.1. *Let Ψ be a uniformly bounded quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_q, $1 < q < \infty$. Then Ψ is democratic with fundamental function $\varphi(m, \Psi, L_q) \asymp m^{1/2}$.*

Proof. The proofs in both cases $1 < q \le 2$ and $2 \le q < \infty$ are similar. We give here only a proof for $1 < q \le 2$. Using the UCC property of quasi-greedy bases and using the fact that L_q, $1 < q \le 2$, is of cotype 2 we obtain as in (2.11)

$$\|\sum_{k\in A} \psi_k\|_q \asymp (\mathrm{Ave}_{\varepsilon_k=\pm1}\|\sum_{k\in A} \varepsilon_k\psi_k\|_q^2)^{1/2} \ge Cm^{1/2}.$$

Also

$$\|\sum_{k\in A} \psi_k\|_q \asymp (\mathrm{Ave}_{\varepsilon_k=\pm1}\|\sum_{k\in A} \varepsilon_k\psi_k\|_q^2)^{1/2} \le (\mathrm{Ave}_{\varepsilon_k=\pm1}\|\sum_{k\in A} \varepsilon_k\psi_k\|_2^2)^{1/2} \le Cm^{1/2}.$$

$\square$

Combination of Proposition 2.1 and Remark 2.2 gives the following inequalities which we will often use.

Proposition 2.2. *Let Ψ be a uniformly bounded quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_q, $1 < q < \infty$. Then we have for $f \in L_q$*

$$c_1(q) \sup_n n^{1/2} a_n(f) \le \|f\|_q \le C_1(q) \sum_{n=1}^\infty n^{-1/2} a_n(f). \tag{2.12}$$

This proposition implies the following analog of Lemma 2.9.

Lemma 2.11. *Assume that Ψ is a uniformly bounded quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_q and L_p, $1 < q, p < \infty$. Then for any set Λ of indices we have*

$$\|S_\Lambda(f)\|_p \le C \ln(|\Lambda| + 1)\|f\|_q. \tag{2.13}$$

Proof. Let $|\Lambda| = m$. By Proposition 2.2 we obtain

$$\|S_\Lambda(f)\|_p \le C_1(p) \sum_{n=1}^{m} n^{-1/2} a_n(f)$$

$$\le c_1(q)^{-1} C_1(p) \sum_{n=1}^{m} n^{-1}\|f\|_q \le C \ln(|\Lambda| + 1)\|f\|_q.$$

□

3 Construction of quasi-greedy bases

In this section we describe a general scheme of construction of a quasi-greedy basis out of a given basis with special properties. This scheme is similar to the one used by Wojtaszczyk in [27]. Both schemes are based on the orthogonal Haar-type matrices, used firstly by Olevskii to construct orthogonal systems (see [4], p. 120, [15], [16]).

Assumptions. Let X be a separable Banach space and $\Phi = \{\varphi_j\}_{j=1}^\infty$ be a semi-normalized basis of X, $0 < c_0 \le \|\varphi_j\| \le C_0$. We assume that Φ is a Besselian basis of X: for any

$$f = \sum_{j=1}^{\infty} c_j(f)\varphi_j \tag{3.1}$$

we have

$$(\sum_{j=1}^{\infty} |c_j(f)|^2)^{1/2} \le C_1\|f\|. \tag{3.2}$$

Assume that Φ can be split into two systems $F = \{f_s\}_{s=1}^\infty$, $f_s = \varphi_{m(s)}$ and $E = \{e_j\}_{j=1}^\infty$, $e_j = \varphi_{n(j)}$ with increasing sequences $\{m(s)\}$ and $\{n(j)\}$ in such a way that E has the following special property. For any sequence $\{c_j\}$ we have

$$\|\sum_{j=1}^{\infty} c_j e_j\| \le C_2(\sum_{j=1}^{\infty} |c_j|^2)^{1/2}. \tag{3.3}$$

In our construction of quasi-greedy bases we will use special matrices. Let a collection of matrices $\mathcal{A} = \{A(n)\}_{n=1}^{\infty}$, $A(n)$ is of size $n \times n$, satisfy the following properties.

M1. Singular numbers of matrices $A(n)$ and their inverse $A(n)^{-1}$ are uniformly bounded:

$$s_j(A(n)) \le C_3; \qquad s_j(A(n)^{-1}) \le C_3. \tag{3.4}$$

M2. For the elements of the first column of matrix $A(n) = [a_{ij}(n)]$ we have

$$|a_{i1}(n)| \le C_4 n^{-1/2}. \tag{3.5}$$

Construction. Let $\{n_k\}_{k=0}^{\infty}$, $n_0 = 0$, be an increasing sequence of integers such that

$$n_{k+1} \ge n_k^2 \quad \text{and} \quad \sum_{k=1}^{\infty} n_k^{-1} < \infty\,. \tag{3.6}$$

For a fixed natural number k we pick the basis elements

$$g_1^k := f_k, \quad g_i^k := e_{S_{k-1}+i-1}, \quad i = 2, \dots, n_k, \tag{3.7}$$

where $\{S_j\}$ is defined recursively as

$$S_j = S_{j-1} + n_j - 1, \quad j = 1, 2, \dots, \quad S_0 = 0.$$

We build a new system of elements $\{\psi_i^k\}_{i=1}^{n_k}$ using a matrix $A(n_k)$ in the following way:

$$(\psi_1^k, \dots, \psi_{n_k}^k)^T = A(n_k)(g_1^k, \dots, g_{n_k}^k)^T. \tag{3.8}$$

In other words, for $i \in [1, n_k]$ we have

$$\psi_i^k = \sum_{j=1}^{n_k} a_{ij}(n_k) g_j^k.$$

We define and study the new system $\Psi = \{\psi_i^k\}_{i=1,k=1}^{n_k,\infty} = \{\psi_{j(k,i)}\}$ ordered in the lexicographical way: $j(k', i') > j(k, i)$ if either $k' > k$ or $k' = k$ and $i' > i$.

Properties of Ψ. We begin with a property of an auxiliary system $G := \{g_i^k\}_{i=1,k=1}^{n_k,\infty} = \{g_{j(k,i)}\}$ ordered in the lexicographical way: $j(k',i') > j(k,i)$ if either $k' > k$ or $k' = k$ and $i' > i$.

Proposition 3.1. *The system G is a Besselian basis of X.*

Proof. It follows from the definition of G that an expansion of f with respect to G will be a rearrangement of the expansion of f with respect to Φ. Therefore, we only need to prove that G is a basis. Then the Besselian property of G follows from the Besselian property of Φ.

Let f have the expansion (3.1) with respect to Φ. Consider the series

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n_k} c_i^k g_i^k,$$

where $c_i^k = c_j(f)$ if $g_i^k = \varphi_j$. A partial sum of this series has the form

$$\sum_{k=1}^{N} c_1^k g_1^k + \sum_{k=1}^{N}\sum_{i\in P_k} c_i^k g_i^k, \quad P_k \subseteq [2, n_k]. \tag{3.9}$$

We note that in the above representation $P_k = [2, n_k]$ for all k except maybe $k = N$. By our choice of g_i^k we have that $g_i^k \in E$ for all k and $i > 1$. Therefore, for the second sum in (3.9) we use (3.3) and obtain the bound

$$\|\sum_{k=1}^{N}\sum_{i\in P_k} c_i^k g_i^k\| \le C_2(\sum_{k=1}^{N}\sum_{i\in P_k} |c_i^k|^2)^{1/2}. \tag{3.10}$$

Using the Besselian property of basis Φ (3.2) we get

$$(\sum_{k=1}^{N}\sum_{i\in P_k} |c_i^k|^2)^{1/2} \le C_1\|f\|.$$

Let $g_1^N = f_N = \varphi_{m(N)}$. Then for the first sum in (3.9) we obtain

$$\sum_{k=1}^{N} c_1^k g_1^k = \sum_{j=1}^{m(N)} c_j(f)\varphi_j - \sum_{k=1}^{K}\sum_{i\in Q_k} c_i^k g_i^k, \quad Q_k \subseteq [2, n_k].$$

The assumption that Φ is a basis implies

$$\|\sum_{j=1}^{m(N)} c_j(f)\varphi_j\| \le C\|f\|.$$

In a way similar to the above estimation of the second sum in (3.9) we get

$$\|\sum_{k=1}^{K}\sum_{i\in Q_k} c_i^k g_i^k\| \le C\|f\|.$$

This completes the proof of Proposition 3.1.

□

Proposition 3.2. *The system Ψ is a Besselian basis of X.*

Proof. Denote

$$X_k := \operatorname{span}(\psi_1^k,\dots,\psi_{n_k}^k) = \operatorname{span}(g_1^k,\dots,g_{n_k}^k).$$

Let $g \in X_k$. Then

$$g = \sum_{i=1}^{n_k} v_i g_i^k, \quad g = \sum_{i=1}^{n_k} u_i \psi_i^k.$$

Using definition of ψ_i^k in terms of g_j^k we obtain

$$\sum_{i=1}^{n_k} u_i\psi_i^k = \sum_{i=1}^{n_k} u_i(\sum_{j=1}^{n_k} a_{ij}(n_k)g_j^k).$$

Therefore,

$$v_j = \sum_{i=1}^{n_k} a_{ij}(n_k)u_i,$$

or $v = A(n_k)^T u$, where $u = (u_1,\dots,u_{n_k})^T$, $v = (v_1,\dots,v_{n_k})^T$. The property **M1** of matrix $A(n_k)$ implies that

$$\|u\|_2 \le C_3\|v\|_2.$$

This and Proposition 3.1 imply that Ψ is a Besselian system. It remains to prove that Ψ is a basis of X. It is clear that the use of Proposition 3.1 allows us to limit our prove to only one subspace X_k. In this case by (3.3) and **M2** we have for $n \in [1, n_k]$

$$\|\sum_{i=1}^{n} u_i \psi_i^k\| \le (\sum_{i=1}^{n} |u_i|) C_4 n_k^{-1/2} \|g_1^k\| + \|\sum_{i=1}^{n} u_i (\sum_{j=2}^{n_k} a_{ij}(n_k) g_j^k)\|$$

$$\le C\|u\|_2 + C(\sum_{j=2}^{n_k} |\sum_{i=1}^{n} a_{ij}(n_k) u_i|^2)^{1/2}.$$

Using our assumption **M1** we obtain

$$(\sum_{j=2}^{n_k} |\sum_{i=1}^{n} a_{ij}(n_k) u_i|^2)^{1/2} \le C_3 \|u\|_2.$$

Therefore, applying Proposition 3.1 we get

$$\|\sum_{i=1}^{n} u_i \psi_i^k\| \le C\|u\|_2 \le C\|v\|_2 \le \|f\|.$$

This completes the proof of Proposition 3.2. $\square$

Theorem 3.1. *The basis Ψ is a quasi-greedy basis of X.*

Proof. Let $f \in X$ have a representation

$$f = \sum_{k=1}^{\infty} \sum_{i=1}^{n_k} b_i^k \psi_i^k$$

with respect to Ψ. Suppose that the mth greedy approximant is given by

$$G_m(f, \Psi) = \sum_{k \in J} \sum_{i \in I_k} b_i^k \psi_i^k, \quad I_k \subseteq [1, n_k]. \tag{3.11}$$

We will prove that

$$\|G_m(f, \Psi)\| \le C\|f\|. \tag{3.12}$$

It is clear that it suffices to prove (3.12) for normalized f, $\|f\| = 1$.

At the first step we consider the following modification of the sum from (3.11).

$$\Sigma_1 := \sum_{k \in J} \sum_{i \in I_k} b_i^k (\psi_i^k - a_{i1}(n_k) f_k).$$

It follows from the definition of ψ_i^k that

$$\Sigma_1 = \sum_{k\in J}\sum_{i\in I_k} b_i^k(\sum_{j=2}^{n_k} a_{ij}(n_k)g_j^k) = \sum_{k\in J}\sum_{j=2}^{n_k}(\sum_{i\in I_k} b_i^k a_{ij}(n_k))g_j^k.$$

By (3.3) we get

$$\|\Sigma_1\| \le C(\sum_{k\in J}\sum_{j=2}^{n_k}|\sum_{i\in I_k} b_i^k a_{ij}(n_k))|^2)^{1/2}. \tag{3.13}$$

Using property **M1** and Proposition 3.2 we obtain from (3.13)

$$\|\Sigma_1\| \le C(\sum_{k\in J}\sum_{i\in I_k}|b_i^k|^2)^{1/2} \le C.$$

At the second step we consider

$$\Sigma_2 := G_m(f,\Psi) - \Sigma_1 = \sum_{k\in J}\sum_{i\in I_k} b_i^k a_{i1}(n_k)f_k.$$

We split each of I_k into three disjoint subsets:

$$I_k^1 := \{i \in I_k : |b_i^k| \le n_k^{-1}\};$$

$$I_k^2 := \{i \in I_k : |b_i^k| \ge n_k^{-1/2}\};$$

$$I_k^3 := \{i \in I_k : n_k^{-1} < |b_i^k| < n_k^{-1/2}\}.$$

Denote

$$\Sigma_2^s := \sum_{k\in J}\sum_{i\in I_k^s} b_i^k a_{i1}(n_k)f_k, \quad s = 1,2,3.$$

For Σ_2^1 we have

$$\Sigma_2^1 := \sum_{k\in J}\sum_{i\in I_k^1} b_i^k a_{i1}(n_k)f_k = \sum_{k\in J} f_k \sum_{i\in I_k^1} b_i^k a_{i1}(n_k).$$

It follows from the definition of I_k^1 and from property **M2** that

$$|\sum_{i\in I_k^1} b_i^k a_{i1}(n_k)| \le C_4 n_k^{-1/2}.$$

Therefore,

$$\|\Sigma_2^1\| \leq C \sum_{k\in J} n_k^{-1/2} \leq C.$$

We proceed to estimate Σ_2^2. We have

$$|I_k^2| n_k^{-1} \leq \sum_{i=1}^{n_k} |b_i^k|^2,$$

and

$$\|\Sigma_2^2\| \leq C_4 C_0 \sum_{k\in J} n_k^{-1/2} |I_k^2|^{1/2} (\sum_{i=1}^{n_k} |b_i^k|^2)^{1/2} \leq C_4 C_0 \sum_{k\in J} \sum_{i=1}^{n_k} |b_i^k|^2 \leq C.$$

We proceed to Σ_2^3. We note that the bound on Σ_1 combined with Proposition 3.2 imply that for any N

$$\| \sum_{k=1}^{N} \sum_{i=1}^{n_k} b_i^k f_k a_{i1}(n_k) \| \leq C. \tag{3.14}$$

Denote

$$K := \max\{k \in J : I_k^3 \neq \emptyset\}.$$

This means that there is a b_i^K, $i \in I_K^3$, such that $|b_i^K| < n_K^{-1/2}$. The fact that $K \in J$, our assumption that $n_{k+1} \geq n_k^2$ and the definition of greedy approximant imply that for all $k \in [1, K]$ we have that either I_k^3 is empty or that $k \in J$. Thus

$$\Sigma_2^3 = \sum_{i\in I_K^3} b_i^K f_K a_{i1}(n_K) + \sum_{k=1}^{K-1} \sum_{i=1}^{n_k} b_i^k f_k a_{i1}(n_k) - \sigma_1 - \sigma_2, \tag{3.15}$$

where σ_1 has form of Σ_2^1 and σ_2 has form of Σ_2^2. Therefore, it is sufficient to bound only the first term in the right side of (3.15). We have

$$\| \sum_{i\in I_K^3} b_i^K f_K a_{i1}(n_K) \| \leq C n_K^{-1/2} n_K^{1/2} (\sum_{i=1}^{n_k} |b_i^K|^2)^{1/2} \leq C.$$

This completes the proof of Theorem 3.1.

$\square$

Extra assumptions. First of all we note that if $\Phi \subset H$ is from a Hilbert space H and forms an orthonormal basis there then G also forms an orthonormal basis in H. Second, if matrices $A(n)$ are orthogonal matrices then Ψ is an orthonormal basis of H.

Next, assume that Y is a subspace of X with a stronger norm: $\|f\|_X \le \|f\|_Y$. Assume that the basis Φ is from Y and $\|\varphi_j\|_Y \le B$, $j = 1, \dots$. We impose an extra assumption on matrices too.

M3. Assume that for all n

$$\sum_{j=1}^{n} |a_{ij}(n)| \le C_5. \tag{3.16}$$

Under condition **M3** we easily derive from the definition of Ψ that

$$\|\psi_i^k\|_Y \le C_5 B.$$

Examples. Let $X = L_p(0, 2\pi)$, $2 < p < \infty$, $Y = L_\infty(0, 2\pi)$. Consider $\Phi = \mathcal{T}$ to be the trigonometric system $\{e^{ikx}\}$. Define $E := \{e^{i2^j x}\}_{j=1}^\infty$. It is well known that (3.3) holds for this system. By Riesz theorem $\mathcal{T}$ is a basis of L_p, $1 < p < \infty$. Trivially, $\mathcal{T}$ has Besselian property in L_p, $2 < p < \infty$. Thus applying the above construction we obtain the following theorem.

Theorem 3.2. *There exists a uniformly bounded quasi-greedy basis* $\Psi = \{\psi_j\}_{j=1}^\infty$ *in* L_p, $2 < p < \infty$, *that consists of trigonometric polynomials.*

Moreover, as it is pointed out above if matrices $A(n)$ are orthogonal matrices then Ψ is an orthonormal basis of H. Thus, we have the following variant of Theorem 3.2.

Theorem 3.3. *There exists a uniformly bounded orthonormal quasi-greedy basis* $\Psi = \{\psi_j\}_{j=1}^\infty$ *in* L_p, $2 < p < \infty$.

4 Uniformly bounded quasi-greedy systems

Combining Theorem 2.2 and Theorem 3.3 with the preceding discussion gives the following result.

Theorem 4.1. *There exists a uniformly bounded orthonormal system* $\Psi = \{\psi_j\}_{j=1}^\infty$ *consisting of trigonometric polynomials which is a quasi-greedy basis for* $L_p[0, 1]$ *for all* $1 < p < \infty$.

Remark 4.1. *The orthonormal system constructed in Theorem 4.1 will also be a quasi-greedy basis for other function spaces, including the Lorentz spaces* $L_{p,q}[0,1]$ *for* $1 < p < \infty$, $1 \le q < \infty$.

The main result of this section is that there is no analogue of Theorem 4.1 for $L_1[0,1]$. It is known that $L_1[0,1]$ has a quasi-greedy basis [1, Theorem 7.1] and, by a theorem of Szarek [20], that $L_1[0,1]$ does not admit any uniformly integrable *Schauder* basis (see also [7]). On the other hand, the trigonometric system is a uniformly bounded *Markushevich basis*. Therefore, it is natural to ask whether $L_1[0,1]$ admits a uniformly bounded (or uniformly integrable) quasi-greedy Markushevich basis Ψ. We answer this question negatively.

First, we recall the relevant definitions. Let X be a separable Banach space. Let $\Psi = \{\psi_j\}_{j=1}^{\infty} \subset X$ be a fundamental and semi-normalized system, i.e. there exist positive constants a and b such that

$$a \le \|\psi_j\| \le b \qquad (j \ge 1), \tag{4.1}$$

with biorthogonal sequence $\{\psi_j^*\}_{j=1}^{\infty} \subset X^*$. Ψ is said to be a Markushevich basis if the mapping $f \mapsto \{\psi_j^*(f)\}_{j=1}^{\infty}$ $(f \in X)$ is one-one. In other words, each $f \in X$ is uniquely determined by its coefficient sequence $\{\psi_j^*(f)\}_{j=1}^{\infty}$. We say that Ψ is quasi-greedy if there exists a constant C such that

$$\|G_m(f,\Psi)\| \le C\|f\| \qquad (m \ge 1, f \in X). \tag{4.2}$$

Wojtaszczyk [27] proved that (4.2) is equivalent to the norm convergence of $\{G_m(f)\}$ to f for all $f \in X$.

It follows easily from (4.1) and (4.2) that $\{\psi_j^*\}_{j=1}^{\infty}$ is semi-normalized in X^*. Indeed, for $f \in X$, we have

$$|\psi_j^*(f)| \le |a_1(f)| \le (1/a)\|G_1(f)\| \le (C/a)\|f\|,$$

and hence $\|\psi_j^*\| \le C/a$. On the other hand, since $\psi_j^*(\psi_j) = 1$, we also have $\|\psi_j^*\| \ge 1/\|\psi_j\| \ge 1/b$.

The following result was proved for quasi-greedy bases (actually for the larger class of *thresholding-bounded* bases) in [1, Lemma 8.2]. The proof easily carries over to quasi-greedy Markushevich bases (cf. also the proof of Lemma 2.3 above).

Proposition 4.1. *Suppose that Ψ is a semi-normalized quasi-greedy Markushevich basis for X. There exists a constant C such that for all finite sets $\Lambda \subset \mathbb{N}$ with $|\Lambda| = N \geq 2$, we have*

$$\max_{\pm} \| \sum_{n \in \Lambda} \pm \psi_n^*(f) \psi_n \| \leq C \ln N \|f\| \qquad (f \in X).$$

In particular,

$$\|S_\Lambda(f)\| \leq C \ln N \|f\| \qquad (f \in X).$$

Recall that a bounded operator $T \colon X \to Y$, where X and Y are Banach spaces, is *absolutely summing* if there exists a constant C such that, for all $n \geq 1$ and for all finite sequences $\{f_j\}_{j=1}^n \subset X$, we have

$$\sum_{j=1}^n \|T(f_j)\| \leq C \max_{\pm} \| \sum_{j=1}^n \pm f_j \|.$$

The smallest such constant is denoted $\pi_1(T)$. A Banach space X is called a *GT space* [18] if every bounded operator $T \colon X \to \ell_2$ is absolutely summing. X is a GT space if and only if there exists a constant B such that $\pi_1(T) \leq B\|T\|$ for all bounded $T \colon X \to \ell_2$. Grothendieck [5] proved that $L_1(\mu)$ spaces are GT space.

The proof of the following result is based on the methods used in [1, Section 8].

Theorem 4.2. *Suppose that X is a GT space. Let Ψ be a semi-normalized quasi-greedy Markushevich basis for X. Then Ψ is democratic and its fundamental function satisfies $\varphi(n) \asymp n$.*

Proof. For $1 \leq p \leq \infty$, recall that a Markushevich basis Ψ is said to be p-Besselian if there exists a constant C_p such that

$$(\sum_{n=1}^{\infty} |\psi_n^*(f)|^p)^{1/p} \leq C_p \|f\| \qquad (f \in X)$$

(with the obvious modification for $p = \infty$). Since ψ is quasi-greedy, $C_\infty = \sup_{n \geq 1} \|\psi_n^*\| < \infty$, so Ψ is ∞-Besselian.

We will derive Theorem 4.2 from the following Theorem 4.3.

Theorem 4.3. *Suppose that Ψ is a semi-normalized quasi-greedy Markushevich basis for a GT space X. Then Ψ is r-Besselian for all $r > 1$.*

We need the following key lemma.

Lemma 4.1. *Suppose that Ψ is a semi-normalized quasi-greedy Markushevich basis for a GT space X. If Ψ is p-Besselian for some $2 \le p \le \infty$, then Ψ is r-Besselian for all r satisfying $1/r < 1/p + 1/2$.*

Proof. We shall give the proof for the case $2 < p < \infty$ as the case $p = \infty$ requires only minor changes. Let $1/s = 1/p + 1/2$. Suppose that $\Lambda \subset \mathbb{N}$, with $|\Lambda| = N$, and that $(\eta_n)_{n\in\Lambda}$ is any fixed choice of signs. Choose $f \in X$, with $\|f\| = 1$, such that

$$\sum_{n\in\Lambda} \eta_n \psi_n^*(f) \ge \frac{1}{2} \|\sum_{n\in\Lambda} \eta_n \psi_n^*\|.$$

Next consider $T\colon X \to \ell_2(\Lambda)$ defined as follows:

$$T(g) = (\psi_n^*(g)|\psi_n^*(f)|^{s-1})_{n\in\Lambda} \qquad (g \in X).$$

Then, applying Hölder's inequality and using the fact that Ψ is p-Besselian, we get

$$\begin{aligned}\|T(g)\| &= (\sum_{n\in\Lambda} |\psi_n^*(f)|^{2s-2}|\psi_n^*(g)|^2)^{1/2}\\ &\le (\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1-1/s}(\sum_{n\in\Lambda} |\psi_n^*(g)|^p)^{1/p}\\ &\le C_p(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1-1/s}\|g\|.\end{aligned}$$

Hence $\|T\| \le C_p(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1-1/s}$. Since X is a GT space, we have

$$\begin{aligned}\sum_{n\in\Lambda} |\psi_n^*(f)|^s &= \sum_{n\in\Lambda} |\psi_n^*(f)|\|T(\psi_n)\|\\ &\le B\|T\| \sup_{\varepsilon_n=\pm1} \|\sum_{n\in\Lambda} \varepsilon_n \psi_n^*(f)\psi_n\|\\ &\le BC_p(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1-1/s} \sup_{\varepsilon_n=\pm1} \|\sum_{n\in\Lambda} \varepsilon_n \psi_n^*(f)\psi_n\|.\end{aligned}$$

Thus,

$$(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1/s} \le BC_p \sup_{\varepsilon_n=\pm1} \|\sum_{n\in\Lambda} \varepsilon_n \psi_n^*(f)\psi_n\|.$$

Since $|\Lambda| = N$, Proposition 4.1 yields

$$\sup_{\varepsilon_n=\pm 1} \|\sum_{n\in\Lambda} \varepsilon_n \psi_n^*(f)\psi_n\| \leq C'(\ln N),$$

where C' is independent of N. Hence

$$(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1/s} \leq BC'C_p(\ln N).$$

Thus,

$$\begin{aligned}
\|\sum_{n\in\Lambda} \eta_n \psi_n^*\| &\leq 2\sum_{n\in\Lambda} \eta_n \psi_n^*(f) \\
&\leq 2(\sum_{n\in\Lambda} |\psi_n^*(f)|^s)^{1/s} N^{1-1/s} \\
&\leq BC'C_p(\ln N)N^{1-1/s}.
\end{aligned}$$

Now suppose that $g \in X$ with $\|g\| = 1$. For $a > 0$, let

$$\Lambda(a) = \{n \in \mathbb{N} \colon |\psi_n^*(g)| \geq a\} \quad \text{and} \quad N(a) = |\Lambda(a)|.$$

Then, for some choice of signs (η_n), we have

$$\begin{aligned}
aN(a) &\leq \sum_{n\in\Lambda(a)} \eta_n \psi_n^*(g) \\
&\leq \|\sum_{n\in\Lambda(a)} \eta_n \psi_n^*\| \\
&\leq BC'C_p(\ln N(a))N(a)^{1-1/s}.
\end{aligned}$$

Thus, for some constant C'', we have $N(a) \leq C''a^{-t}$ provided t satisfies

$$\frac{1}{r} < \frac{1}{t} < \frac{1}{s}.$$

Note that

$$\sup_{n\geq 1} |\psi_n^*(g)| \leq \sup_{n\geq 1} \|\psi_n^*\|_\infty = C_\infty.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} |\psi_n^*(g)|^r &\leq \sum_{n=0}^{\infty} N(2^{-n}C_\infty)(2^{1-n}C_\infty)^r \\
&\leq 2^r C'' \sum_{n=0}^{\infty} (2^{-n}C_\infty)^{r-t} < \infty.
\end{aligned}$$

Hence Ψ is r-Besselian. $\square$

Applying the lemma twice, starting with $p = \infty$, it follows that Ψ is r-Besselian for all $r > 1$. This proves Theorem 4.3.

In particular, Ψ is 2-Besselian with constant $C_2 < \infty$. Hence, for every finite $\Lambda \subset \mathbb{N}$, the mapping $T\colon X \to \ell_2(\Lambda)$ given by $f \mapsto (\psi_n^*(f))_{n\in\Lambda}$ satisfies $\|T\| \le C_2$. Since X is a GT space the absolutely summing norm of T satisfies $\pi_1(T) \le BC$. Thus,

$$|\Lambda| = \sum_{n\in\Lambda} \|T(\psi_n)\|_2 \le BC \max_{\pm} \|\sum_{n\in\Lambda} \pm\psi_n\|.$$

Since Ψ is quasi-greedy, and hence unconditional for constant coefficients, it follows that $\varphi(n) \asymp n$. $\square$

The following Proposition 4.2 is a stronger version of Proposition 4.1 under an extra assumption that X is a GT space.

Proposition 4.2. *Suppose that Ψ is a semi-normalized quasi-greedy Markushevich basis for a GT space X. There exists a constant C such that for all finite sets $\Lambda \subset \mathbb{N}$ with $\Lambda| = N \ge 2$, we have*

$$\sum_{n\in\Lambda} |\psi_n^*(g)| \le C \ln N \|g\|, \quad (g \in X).$$

Proof. Let $\xi_n = \pm$ be such that

$$\sum_{n\in\Lambda} |\psi_n^*(g)| = \sum_{n\in\Lambda} \xi_n \psi_n^*(g).$$

Then

$$\sum_{n\in\Lambda} |\psi_n^*(g)| \le \|\sum_{n\in\Lambda} \xi_n \psi_n^*\| \|g\|. \tag{4.3}$$

We now estimate $\|\sum_{n\in\Lambda} \xi_n \psi_n^*\|$. Let as above f be such that $\|f\| = 1$ and

$$\sum_{n\in\Lambda} \xi_n \psi_n^*(f) \ge \frac{1}{2} \|\sum_{n\in\Lambda} \xi_n \psi_n^*\|. \tag{4.4}$$

Consider operator $T : X \to \ell_2(\Lambda)$

$$T(\varphi) := (\psi_n^*(\varphi))_{n\in\Lambda}.$$

By Theorem 4.3 Ψ is 2-Besselian and therefore

$$\|T(\varphi)\| = (\sum_{n\in\Lambda} |\psi_n^*(\varphi)|^2)^{1/2} \le C_2\|\varphi\|.$$

Using the assumption that X is a GT space we obtain

$$\sum_{n\in\Lambda} |\psi_n^*(f)| = \sum_{n\in\Lambda} |\psi_n^*(f)|\|T(\psi_n)\| \le BC_2 \sup_{\epsilon=\pm} \|\sum_{n\in\Lambda} \epsilon_n\psi_n^*(f)\psi_n\| \le C\ln N. \tag{4.5}$$

We used Proposition 4.1 at the last inequality. Combining (4.3) – (4.5) we complete the proof. □

We note that the following Proposition 4.3, which is stronger than Proposition 4.2, follows from Lemma 3.2 from [2] and Theorem 4.2.

Proposition 4.3. *Suppose that Ψ is a semi-normalized quasi-greedy Markushevitch basis for a GT space X. There exists a constant C such that for all $g \in X$ we have*

$$a_n(g) \le Cn^{-1}\|g\|.$$

Recall that a system $\{f_j\} \subset L_1[0,1]$ is *uniformly integrable* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\lambda(A) < \delta$, where λ denotes Lebesgue measure, then $\int_A |f_j|\, d\lambda < \varepsilon$ for all $j \ge 1$. Clearly, uniformly bounded systems are uniformly integrable.

Theorem 4.4. *Let Ψ be a semi-normalized quasi-greedy Markushevich basis for $L_1[0,1]$. Then no subsequence of Ψ is uniformly integrable. Hence every subsequence of Ψ contains a further subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. Let $\{f_j\} \subset L_1[0,1]$ be any uniformly integrable system. Given $\varepsilon > 0$, choose $M > 0$ such that $\|f_j\chi_{\{|f_j|>M\}}\|_1 < \varepsilon$ for all j. Then

$$\mathrm{Ave}_\pm \|\sum_{j=1}^n \pm f_j\|_1 \le n\varepsilon + \mathrm{Ave}_\pm \|\sum_{j=1}^n \pm f_j\chi_{\{|f_j|\le M\}}\|_2 \le n\varepsilon + M\sqrt{n}.$$

Hence $\mathrm{Ave}_\pm \|\sum_{j=1}^n \pm f_j\|_1 = o(n)$. Since $L_1[0,1]$ is a GT space, Theorem 4.2 implies that $\{f_j\}$ is not a subsequence of any quasi-greedy Markushevich basis. Finally, it is well-known that semi-normalized sequences in $L_1[0,1]$ are either uniformly integrable or contain a subsequence equivalent to the unit vector basis of ℓ_1. □

Remark 4.2. *Complemented subspaces of L_1 spaces are GT spaces. Hence the previous theorem extends to quasi-greedy Markushevich bases of complemented (infinite-dimensional) subspaces of $L_1[0,1]$. A related result of Popov [19] asserts that complemented subspaces of $L_1[0,1]$ do not admit a uniformly integrable Schauder basis.*

Next we consider the Hardy spaces $H_p(D)$ $(1 \leq p < \infty)$ of analytic functions on the disk $D := \{z \in \mathbb{C} \colon |z| < 1\}$ equipped with the norm

$$\|f\|_p = \sup_{0<r<1} (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}|^p \, d\theta)^{1/p}.$$

Using the system $\{z^n\}_{n=0}^{\infty}$ instead of $\mathcal{T}$ in the proof of Theorem 3.2 yields the following result.

Theorem 4.5. *There exists an orthonormal system of uniformly bounded analytic polynomials which is a quasi-greedy basis for $H_p(D)$ for $1 < p < \infty$.*

Using some deep results from Banach space theory we can extend the latter result also to the case $p = 1$.

Theorem 4.6. *$H_1(D)$ admits a semi-normalized uniformly bounded quasi-greedy basis of analytic polynomials.*

Proof. By Paley's inequality [17]

$$(\sum_{n=1}^{\infty} |\hat{f}(2^n)|^2)^{1/2} \asymp \| \sum_{n=1}^{\infty} \hat{f}(2^n) z^{2^n} \|_1 \leq 2\|f\|_1$$

Hence $P(f) = \sum_{n=1}^{\infty} \hat{f}(2^n) z^{2^n}$ is a bounded projection on $H_1(D)$. Let $X := \ker P$ and let $H := [e_j]_{j=1}^{\infty}$, where $e_j := z^{2^j}$. Then $H_1(D)$ is linearly isomorphic to $X \oplus H$ (equipped with the sum norm), which in turn is isomorphic to $X \oplus \ell_2$. Since X contains a complemented subspace isomorphic to ℓ_2 (e.g., the subspace spanned by $(z^{3^j})_{j=1}^{\infty}$), it follows that X is also isomorphic to $X \oplus \ell_2$ and thus also isomorphic to $H_1(D)$. Hence, by a theorem of Maurey [13], X has a normalized unconditional basis $(f_j)_{j=1}^{\infty}$. The intersection of X with the linear space of analytic polynomials is dense in X. Hence we may assume that each f_j is an analytic polynomial. Since $H_1(D)$ has cotype 2 (see Definition 2.2 above), it follows that $\{f_j\}_{j=1}^{\infty}$ is Besselian. Then $\Phi = \{f_j\}_{j=1}^{\infty} \cup \{e_j\}_{j=1}^{\infty}$ is a basis for $H_1(D)$ satisfying (3.2) and (3.3). Assume the matrices $\{A(n)\}$ satisfy **M1-M3** and, in addition, that $n_k \geq \|f_k\|_\infty^2$. By the construction, the system Ψ is a semi-normalized quasi-greedy basis. Moreover, by **M2** and **M3**,

$$\sup_{j\geq 1} \|\psi_j\|_\infty \leq \sup_{k\geq 1} \frac{\|f_k\|_\infty}{\sqrt{n_k}} + C_5 \sup_{k\geq 1} \|e_k\| \leq 1 + C_5.$$

Thus, Ψ is uniformly bounded. $\square$

Finally, let us mention that Theorem 3.2 may be generalized to certain closed subspaces of $L_p[0,1]$, for $p > 2$, including those spanned by any subsequence of the trigonometric system. Recall that $\{\psi_j\} \subset L_2[0,1]$ is a *Riesz sequence* if $\|\sum c_j\psi_j\|_2 \asymp (\sum |c_j|^2)^{1/2}$ for all scalars $\{c_j\}$.

Proposition 4.4. *Let X be a closed subspace of $L_p[0,1]$ for $2 \leq p < \infty$. Suppose that X has a uniformly bounded Schauder basis $\{\psi_j\}$ which is a Riesz sequence in $L_2[0,1]$. Then X admits a uniformly bounded quasi-greedy basis.*

Proof. Since $L_p[0,1]$ has an unconditional basis and $\{\psi_j\}$ is weakly null, it follows by a standard "gliding hump" argument that some subsequence $\{\psi_j\}_{j\in A}$ is unconditional. Since $L_p[0,1]$ has type 2 (for the upper estimate) and $\{\psi_j\}$ is a Riesz sequence in $L_2[0,1]$ (for the lower estimate) it follows that $\|\sum_{j\in A} c_j\psi_j\|_p \asymp (\sum_{j\in A} |c_j|^2)^{1/2}$ for all scalars $\{c_j\}$, i.e., $\{\psi_j\}_{j\in A}$ is a sequence in X that is equivalent to the unit vector basis of ℓ_2. Since $\{\psi_j\}$ is a Riesz sequence in $L_2[0,1]$, we have, for all $f \in X$,

$$\|\sum c_j(f)\psi_j\|_p \geq \|\sum c_j(f)\psi_j\|_2 \geq k_1(\sum_{j\in A} |c_j(f)|^2)^{1/2} \geq k_2\|\sum_{j\in A} c_j(f)\psi_j\|_p,$$

where k_1 and k_2 are constants. Hence the projection $Pf = \sum_{j\in A} c_j(f)\psi_j$ is bounded on X, which implies that X is linearly isomorphic to $[\psi_j]_{j\notin A} \oplus [\psi_j]_{j\in A}$. The fact that $\{\psi_j\}$ is a Riesz sequence in $L_2[0,1]$ implies that $\{\psi_j\}$ is a (uniformly bounded) Besselian basis for X. The proof is completed as in the discussion preceding Theorem 3.2.

$\square$

5 Lebesgue-type inequalities I

Our main interest in this section is to prove Lebesgue-type inequalities for greedy approximation in L_p, $2 \leq p \leq \infty$ under different assumptions on a basis Ψ. In this section we assume that Ψ is a uniformly bounded basis. In addition we assume that Ψ is a certain type basis (quasi-greedy basis, Riesz basis) in one of the spaces L_2, L_q, $1 < q < 2$, or L_q, $2 < q < \infty$. We will often use the following lemma.

Lemma 5.1. *Suppose that $X \subset Y$ are two Banach spaces such that $\|\cdot\|_Y \leq \|\cdot\|_X$. Assume that a basis of X Ψ satisfies the following property: For any set of indices Λ*

$$\|S_\Lambda(f)\|_X \leq w(|\Lambda|)\|f\|_Y.$$

Then for each $f \in X$ and any m-term polynomial

$$p_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have

$$\|f - S_P(f)\|_X \leq \|f - p_m\|_X + w(m)\|f - p_m\|_Y.$$

Proof. It is a simple one line proof. We have

$$\|f-S_P(f)\|_X = \|f-p_m(f)-S_P(f-p_m(f)\|_X \leq \|f-p_m\|_X+w(m)\|f-p_m\|_Y.$$

□

We now proceed to a systematic presentation of new results.

Theorem 5.1. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have for $2 \leq p \leq \infty$

$$\|f - G_m(f,\Psi)\|_p \leq \|f - t_m\|_p + Cm^{h(p)}\|f - t_m\|_2.$$

Corollary 5.1. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then we have for $2 \leq p \leq \infty$*

$$\|f - G_m(f,\Psi)\|_p \leq Cm^{h(p)}\sigma_m(f,\Psi)_p.$$

Proof. Denote by Q the set of indices picked by the greedy algorithm after m iterations

$$G_m(f) := G_m(f, \Psi) = \sum_{k \in Q} c_k(f)\psi_k.$$

We use the representation

$$f - G_m(f) = f - S_Q(f) = f - S_P(f) + S_P(f) - S_Q(f). \tag{5.1}$$

First, we bound $\|f - S_P(f)\|_p$. By Lemma 5.1 and Lemma 2.6 we get

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\|f - t_m\|_2. \tag{5.2}$$

Second, we write

$$\begin{aligned}\|S_P(f) - S_Q(f)\|_p &= \|S_{P\setminus Q}(f) - S_{Q\setminus P}(f)\|_p \\ &\le \|S_{P\setminus Q}(f)\|_p + \|S_{Q\setminus P}(f)\|_p.\end{aligned} \tag{5.3}$$

Using Lemma 2.6 we obtain

$$\|S_P(f) - S_Q(f)\|_p \le Cm^{h(p)}(\|S_{P\setminus Q}(f)\|_2 + \|S_{Q\setminus P}(f)\|_2). \tag{5.4}$$

The definition of Q implies

$$\|S_{P\setminus Q}(f)\|_2 \le C(\sum_{k \in P\setminus Q} |c_k(f)|^2)^{1/2}$$

$$\le C(\sum_{k \in Q\setminus P} |c_k(f)|^2)^{1/2} \le C\|S_{Q\setminus P}(f)\|_2. \tag{5.5}$$

Next,

$$\|S_{Q\setminus P}(f)\|_2 = \|S_{Q\setminus P}(f - t_m)\|_2 \le C\|f - t_m\|_2. \tag{5.6}$$

Combining (5.1) – (5.6) we complete the proof of Theorem 5.1. $\square$

We now impose a little weaker assumption on a basis Ψ than the one in Theorem 5.1.

Theorem 5.2. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for any m-term polynomial*

$$t_m = \sum_{k \in P} b_k\psi_k, \quad |P| = m,$$

we have for $2 \le p \le \infty$

$$\|f - G_m(f, \Psi)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\ln(m+1)\|f - t_m\|_2.$$

Corollary 5.2. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for $2 \le p \le \infty$*

$$\|f - G_m(f,\Psi)\|_p \le Cm^{h(p)} \ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof. This proof goes along the lines of proof of Theorem 5.1. However, the details are different because we need to use properties of quasi-greedy bases instead of properties of Riesz bases. We use notations from the proof of Theorem 5.1 and the representation (5.1). By Lemma 5.1 and Lemma 2.7 we get for $\|f - S_P(f)\|_p$

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + Cm^{h(p)} \ln(m+1)\|f - t_m\|_2. \tag{5.7}$$

Using Lemma 2.7 we obtain from (5.3)

$$\|S_P(f) - S_Q(f)\|_p \le Cm^{h(p)}(\|S_{P\setminus Q}(f)\|_2 + \|S_{Q\setminus P}(f)\|_2). \tag{5.8}$$

Next, we have by Theorem 2.1

$$\|S_{Q\setminus P}(f)\|_2 = \|S_{Q\setminus P}(f - t_m)\|_2 \le C_2(2)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f - t_m))$$

$$\le C\sum_{n=1}^{m} n^{-1/2}a_n(f - t_m) = C\sum_{n=1}^{m} n^{-1}(n^{1/2}a_n(f - t_m))$$

$$\le C\ln(m+1)\sup_n n^{1/2}a_n(f - t_m) \le C\ln(m+1)\|f - t_m\|_2. \tag{5.9}$$

For the $S_{P\setminus Q}(f)$ we have

$$\|S_{P\setminus Q}(f)\|_2 \le C_2(2)\sum_{n=1}^{m} n^{-1/2}a_n(S_{P\setminus Q}(f))$$

$$\le C_2(2)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f)) = C_2(2)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f - t_m))$$

which has been estimated in (5.9)

$$\le C\ln(m+1)\|f - t_m\|_2. \tag{5.10}$$

Combining (5.7) – (5.10) we complete the proof of Theorem 5.2.

□

Theorem 5.3. *Assume that Ψ is a democratic quasi-greedy basis of X. Then for any $f \in X$*

$$\|f - G_m(f,\Psi)\|_X \le C \ln(m+1) \sigma_m(f,\Psi)_X.$$

Corollary 5.3. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_p, $1 < p < \infty$. Then*

$$\|f - G_m(f,\Psi)\|_p \le C(p) \ln(m+1) \sigma_m(f,\Psi)_p.$$

Proof. It is known (see [2]) that democratic and quasi-greedy basis is an almost greedy bases. Therefore, the following inequality

$$\|f - G_m(f,\Psi)\|_X \le C \tilde{\sigma}_m(f,\Psi)_X$$

holds for any $f \in X$. It remains to apply Lemma 2.4 to complete the proof of Theorem 5.3. Corollary 5.3 follows from Theorem 5.3 and Proposition 2.1. $\square$

Theorem 5.4. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k \in P} b_k \psi_k, \quad |P| = m,$$

we have for $q \le p \le \infty$

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q) m^{(1-q/p)/2} \ln(m+1) \|f - t_m\|_q.$$

Corollary 5.4. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$. Then for $q \le p \le \infty$*

$$\|f - G_m(f,\Psi)\|_p \le C(p,q) m^{(1-q/p)/2} \ln(m+1) \sigma_m(f,\Psi)_p.$$

Proof. This proof goes along the lines of proof of Theorem 5.2. We use notations from the proof of Theorem 5.2 and the representation (5.1). By Lemma 5.1 and Lemma 2.8 we get for $\|f - S_P(f)\|_p$

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p,q) m^{(1-q/p)/2} \ln(m+1) \|f - t_m\|_q. \quad (5.11)$$

Using Lemma 2.8 we obtain from (5.3)

$$\|S_P(f) - S_Q(f)\|_p \le C m^{(1-q/p)/2} (\|S_{P \setminus Q}(f)\|_q + \|S_{Q \setminus P}(f)\|_q). \quad (5.12)$$

By Lemma 2.3 we estimate

$$\|S_{Q\setminus P}(f)\|_q = \|S_{Q\setminus P}(f-t_m)\|_q \le C\ln(m+1)\|f-t_m\|_q.$$

We give another proof of this bound because it will be used in estimating $\|S_{P\setminus Q}(f)\|_q$. We have by Proposition 2.2

$$\|S_{Q\setminus P}(f)\|_q = \|S_{Q\setminus P}(f-t_m)\|_q \le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f-t_m))$$

$$\le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(f-t_m) = C(q)\sum_{n=1}^{m} n^{-1}(n^{1/2}a_n(f-t_m))$$

$$\le C(q)\ln(m+1)\sup_n n^{1/2}a_n(f-t_m) \le C(q)\ln(m+1)\|f-t_m\|_q. \tag{5.13}$$

For the $S_{P\setminus Q}(f)$ we have

$$\|S_{P\setminus Q}(f)\|_q \le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{P\setminus Q}(f))$$

$$\le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f)) = C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f-t_m))$$

which has been estimated in (5.13)

$$\le C(q)\ln(m+1)\|f-t_m\|_q. \tag{5.14}$$

Combining (5.11) – (5.14) we complete the proof of Theorem 5.4.

□

6 Lebesgue-type inequalities II

In this section we continue to prove Lebesgue-type inequalities for greedy approximation in L_p under different assumptions on a basis Ψ. In this section we assume that Ψ is a quasi-greedy basis for a pair of spaces: L_q, $1<q<\infty$, and L_p, $q\le p$.

Theorem 6.1. *Assume that Ψ is a semi-normalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q.$$

Corollary 6.1. *Assume that Ψ is a semi-normalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then*

$$\|f - G_m(f,\Psi)\|_p \le C(p,q)\ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof. This proof goes along the lines of proof of Theorem 5.2. We use notations from the proof of Theorem 5.1 and the representation (5.1). By Lemma 5.1 and Lemma 2.9 we get for $\|f - S_P(f)\|_p$

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q. \tag{6.1}$$

We obtain from (5.3)

$$\|S_P(f) - S_Q(f)\|_p \le \|S_{P\setminus Q}(f)\|_p + \|S_{Q\setminus P}(f)\|_p. \tag{6.2}$$

Next, we have by Theorem 2.1

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f - t_m)\|_p \le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f - t_m))$$

$$\le C(p)\sum_{n=1}^{m} n^{-1/2}a_n(f - t_m) = C(p)\sum_{n=1}^{m} n^{-1}(n^{1/2}a_n(f - t_m))$$

$$\le C(p)\ln(m+1)\sup_n n^{1/2}a_n(f - t_m) \le C(p,q)\ln(m+1)\|f - t_m\|_q. \tag{6.3}$$

For the $S_{P\setminus Q}(f)$ we have by Theorem 2.1

$$\|S_{P\setminus Q}(f)\|_p \le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{P\setminus Q}(f))$$

$$\leq C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q \setminus P}(f)) = C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q \setminus P}(f - t_m))$$

which has been estimated in (6.3)

$$\leq C(p,q) \ln(m+1) \|f - t_m\|_q. \tag{6.4}$$

Combining (6.1) – (6.4) we complete the proof of Theorem 6.1.

$\square$

Remark 6.1. *The statement of Corollary 6.1 holds even if we drop the assumption that Ψ is quasi-greedy basis of L_q.*

Proof. Assumption that Ψ is semi-normalized for both L_q and L_p, $q \leq 2 \leq p$, implies that it is semi-normalized in L_2. Then as in Proposition 2.1 we can prove that Ψ is democratic with $\varphi(m) \asymp m^{1/2}$. It remains to apply Theorem 5.3. $\square$

Now we prove sharper results for uniformly bounded orthonormal quasi-greedy basis.

Theorem 6.2. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis for L_p, $2 \leq p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k \in P} b_k \psi_k, \quad |P| = m,$$

we have

$$\|f - G_m(f, \Psi)\|_p \leq \|f - t_m\|_p + C(p) \ln(m+1) \|f - t_m\|_{p'}, \tag{6.5}$$

$$\|f - G_m(f, \Psi)\|_p \leq \|f - t_m\|_p + C(p)(\ln(m+1))^{1/2} \|f - t_m\|_2. \tag{6.6}$$

Corollary 6.2. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis for L_p, $2 \leq p < \infty$. Then*

$$\|f - G_m(f, \Psi)\|_p \leq C(p)(\ln(m+1))^{1/2} \sigma_m(f, \Psi)_p.$$

Proof. By Theorem 2.2 Ψ is a quasi-greedy basis of $L_{p'}$. Thus, (6.5) follows from Theorem 6.1 with $q = p'$. We now prove (6.6). As in the proof of Theorem 6.1 we obtain by Lemma 5.1 and Lemma 2.10

$$\|f - S_P(f)\|_p \leq \|f - t_m\|_p + C(p)(\ln(m+1))^{1/2} \|f - t_m\|_2. \tag{6.7}$$

By Theorem 2.1 we obtain

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f-t_m)\|_p \le C(p)\sum_{n=1}^m n^{-1/2}a_n(f-t_m)$$

$$\le C(p)(\sum_{n=1}^m n^{-1})^{1/2}(\sum_{n=1}^m a_n(f-t_m)^2)^{1/2} \le C(p)(\ln(m+1))^{1/2}\|f-t_m\|_2. \quad (6.8)$$

As in the proof of Theorem 6.1 we get

$$\|S_{P\setminus Q}(f)\|_p \le C(p)\sum_{n=1}^m n^{-1/2}a_n(f-t_m)$$

and by the intermediate step in (6.8)

$$\le C(p)(\ln(m+1))^{1/2}\|f-t_m\|_2.$$

It remains to use representation (5.1) and inequality (6.2). $\square$

If Ψ is assumed to be uniformly bounded, then the Lebesgue-type inequality of Theorem 6.1 holds whenever $q \le p$.

Theorem 6.3. *Assume that Ψ is a uniformly bounded quasi-greedy basis for both L_q and L_p with $1 < q \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f-t_m\|_p + C(p,q)\ln(m+1)\|f-t_m\|_q.$$

Proof. As in the proof of Theorem 6.1 we obtain by Lemma 5.1 and Lemma 2.11

$$\|f - S_P(f)\|_p \le \|f-t_m\|_p + C(p,q)\ln(m+1)\|f-t_m\|_q. \quad (6.9)$$

By Proposition 2.2 we obtain

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f-t_m)\|_p \le C(p,q)\sum_{n=1}^m n^{-1/2}a_n(f-t_m)$$

$$\leq C(p,q)\sum_{n=1}^{m} n^{-1}\|f-t_m\|_q \leq C(p,q)\ln(m+1)\|f-t_m\|_q. \tag{6.10}$$

As in the proof of Theorem 6.1 we get

$$\|S_{P\setminus Q}(f)\|_p \leq C(p,q)\sum_{n=1}^{m} n^{-1/2}a_n(f-t_m)$$

and by the intermediate step in (6.10)

$$\leq C(p,q)\ln(m+1))\|f-t_m\|_q.$$

It remains to use representation (5.1) and inequality (6.2). $\square$

Acknowledgements. The second author wishes to thank hospitality from the Department of Mathematics of the University of South Carolina during his stay in fall 2011, and specially Prof. S.J. Dilworth and Prof. V.N. Temlyakov.

References

[1] S.J. Dilworth, N.J. Kalton, and Denka Kutzarova, On the existence of almost greedy bases in Banach spaces, Studia Math., **158** (2003), 67–101.

[2] S.J. Dilworth, N.J. Kalton, Denka Kutzarova, and V.N. Temlyakov, The Thresholding Greedy Algorithm, Greedy Bases, and Duality, Constr. Approx., **19** (2003), 575–597.

[3] V.F. Gaposhkin, On unconditional bases in L^p-spaces, Uspekhi Mat. Nauk, **13** (1958), 179–184.

[4] Ulf Grenander and Gabor Szegö, *Toeplitz forms and their applications*, Univ. of California Press, Los Angeles, CA, 1958.

[5] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1953/1956), 1–79.

[6] E. Hernández, Lebesgue-type inequalities for quasi-greedy bases, arXiv:submit/0349580, 2011.

[7] K.S. Kazarian, On the logarithmic growth of the arithmetic means of the sums of Lebesgue functions of bounded byorthogonal systems, Doklady AN Arm.SSR 69 (1979), 140-145. (Russian).

[8] S.V. Konyagin and V.N. Temlyakov, A remark on greedy approximation in Banach spaces, East. J. Approx. **5** (1999), 365-379.

[9] S.V. Konyagin and V.N. Temlyakov, Greedy Approximation with Regard to bases and General Minimal Systems, Serdica Math. J., **28** (2002), 305–328.

[10] S. Kostyukovsky and A. Olevskii, Note on decreasing rearrangement of Fourier series, J. Appl. Anal. **3** (1997), 137–142.

[11] H. Lebesgue, Sur les intégrales singuliéres, Ann. Fac. Sci. Univ. Toulouse (3), **1** (1909), 25–117.

[12] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Springer-Verlag, Berlin, 1977.

[13] Bernard Maurey, *isomorphismes entre espaces H_1*, Acta Math. **145** (1980), 79–120.

[14] Morten Nielsen, An example of an almost greedy uniformly bounded orthonormal basis for $L_p(0,1)$, J. Approx. Theory, **149** (2007), 188–192.

[15] A.M. Olevskii, *On an orthonormal system and its applications*, Mat. Sb., **71(113)** (1966), 297-336; English transl. in Amer. Math. Soc. Transl. (2) **76** (1968).

[16] A.M. Olevskii, Fourier Series with Respect to General Orthonormal Systems, Springer-Verlag, Berlin, 1975.

[17] R. E. A. C. Paley, *On the lacunary coefficients of power series*, Ann. Math. **34** (1933), 615–616.

[18] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS **60**, Amer. Math. Soc, Providence R.I., 1986.

[19] M. M. Popov, *A property of convex basic sequences in L_1*, Methods Funct. Anal. Topology **11** (2005), 409–416.

[20] S. J. Szarek, *Bases and biorthogonal systems in the spaces C and L^1*, Ark. Mat. **17** (1979), 255–271.

[21] V.N. Temlyakov, Greedy Algorithm and m-Term Trigonometric Approximation, Constr. Approx., **14** (1998), 569–587.

[22] V.N. Temlyakov, Nonlinear method of approximation, Found. Compt. Math., **3** (2003), 33-107.

[23] V.N. Temlyakov, Greedy approximation, Acta Numerica, **17** (2008), 235–409.

[24] V.N. Temlyakov, Greedy approximation, Cambridge University Press, 2011.

[25] V.N. Temlyakov, M. Yang, P. Ye, Greedy approximation with regard to non-greedy bases, Adv. Comput. Math. **34** (2011), 319–337.

[26] V.N. Temlyakov, M. Yang, P. Ye, Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases, East J. Approx. **17** (2011), 127–138.

[27] P. Wojtaszczyk, Greedy Algorithm for General Biorthogonal Systems, J. Approx. Theory **107** (2000), 293-314.