RESEARCH STATEMENT

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1. Introduction

A Hecke algebra $\mathcal{H}$ is an algebra that is associated with a group G and a field k. While there is no concise definition of a Hecke algebra, there are several simple ways of picturing one. For example, a Hecke algebra is the algebra of k-valued functions that are constant on certain cosets of G. Another option is to consider $\mathcal{H}$ as a certain subalgebra of the group algebra kG.

With Hecke algebras for monomial groups, the underlying group is represented by monomial matrices (matrices with one nonzero entry in each row and each column). These Hecke algebras has been studied by Cabanes and Enguehard [1] as well as others. However, their analysis used properties of and coincidences between prime numbers, while ours uses a generic construction.

2. Generic Hecke Algebras for Monomial Groups

Let W be the symmetric group on n letters and let $S = \{ s_1, \dots, s_{n-1} \}$ be a fixed set of generators of W. Also fix a positive integer b, and consider a group H_b which can be represented by the group of $n \times n$ diagonal matrices with bth roots of unity along the diagonal. Then WH_b is a monomial group represented by $n \times n$ monomial matrices with bth roots of unity as the nonzero entries. Let a and v denote indeterminates, and define the Hecke algebra $\mathcal{H}$ to be the $\mathbb{Z}[a, v, v^{-1}]$-algebra given by the following presentation:

The generators of $\mathcal{H}$ are $\{ t_s \mid s \in S \} \cup \{ t_d \mid d \in H_b \}$ and the relations are:

$$t_d t_{d'} = t_{dd'} \quad \text{for any } d,\, d' \text{ in } H_b \tag{2.0.1}$$

$$t_d t_{s_i} = t_{s_i} t_{s_i d s_i} \quad \text{for any } d \text{ in } H_b \text{ and } 1 \le i \le n-1 \tag{2.0.2}$$

$$t_{s_i} t_{s_j} = t_{s_j} t_{s_i} \quad \text{if } |j - i| > 1 \tag{2.0.3}$$

$$t_{s_i} t_{s_{i+1}} t_{s_i} = t_{s_{i+1}} t_{s_i} t_{s_{i+1}} \quad \text{for } 1 \le i < n-1 \tag{2.0.4}$$

$$t_{s_i}^2 = v^2 t_1 + a \sum_{d \in X_{s_i}} t_d t_{s_i} \quad \text{for } 1 \le i \le n-1 \tag{2.0.5}$$

where X_{s_i} is a subset of H_b that depends on s_i.

3. Results

The first two results are properties of the basis elements t_x for $x \in WH_b$.

Theorem 1. *Suppose $w = s_{i_1} \cdots s_{i_p}$ is in W with p minimal, and suppose d is in H_b. Then we may write $t_{wd} = t_{s_{i_1}} \cdots t_{s_{i_p}} t_d$ with no ambiguity.*

Theorem 2. *The element t_1 is the identity in $\mathcal{H}$, where 1 is the identity in WH_b.*

Theorem 3. *The algebra $\mathcal{H}$ is free as an $\mathbb{Z}[a, v, v^{-1}]$-module with basis $\{ t_x \mid x \in WH_b \}$.*

Theorem 4. *Each element t_x is invertible, where x is in WH_b.*

The classic type of Hecke algebra is called the Iwahori-Hecke algebra. Its underlying group is a Coxeter group–that is, a group with generators $\{ r_1, \dots, r_p \}$ and relations $(r_i r_j)^{m_{ij}} = 1$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. In order to study Iwahori-Hecke algebras, Kazhdan and Lusztig [4] and Lusztig [5] created a theory for analyzing representations of these algebras. This in turn helped in describing the irreducible representations of the underlying Coxeter group. Having similar interests in mind, we followed the theory of Kazhdan and Lusztig. Specifically, we defined a bar involution $^{-}$ with the property that $\overline{t_x} := t_{x^{-1}}^{-1}$. Since the collection $\{ t_x \mid x \in WH_b \}$ forms a basis for $\mathcal{H}$, we could define elements $R_{x,y}$ in $\mathbb{Z}[a, v, v^{-1}]$ by

$$\overline{t_y} = \sum_{x \in WH_b} \overline{R_{x,y}}\, t_x.$$

We found several interesting properties of $R_{x,y}$. The proof of the Theorem 5 uses the results of Deodhar [2].

Theorem 5. *The coefficients $R_{x,y}$ are polynomial in a and v^{-1}.*

Theorem 6. *The polynomials $R_{x,y}$ form a partial order on WH_b.*

Theorem 6 is interesting because when $b = 1$, the group $WH_b \cong W$ and the partial order given by $R_{x,y}$ coincides with the Bruhat order, which is most commonly used on W.

4. Example

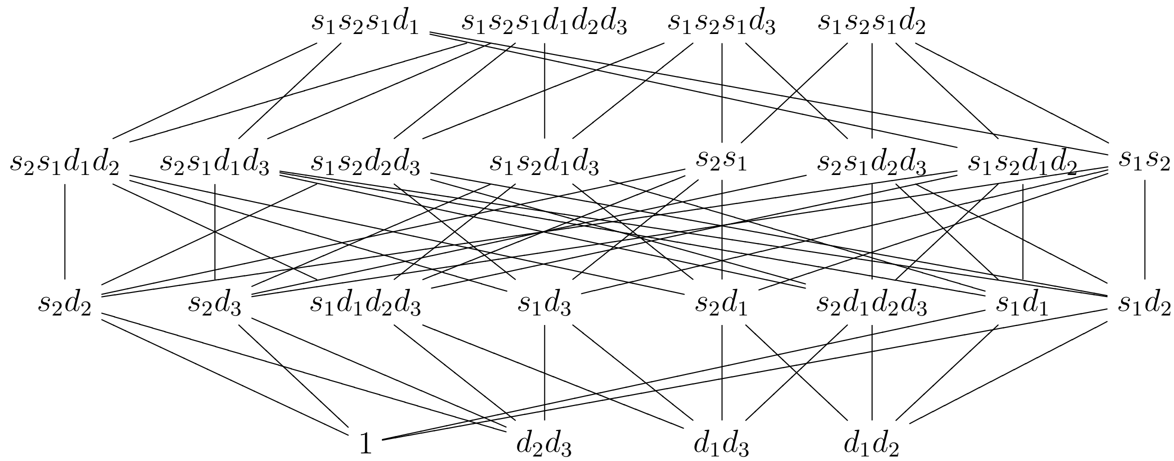
A particularly illustrative example of a Hecke algebra for monomial groups is found by specializing b to 2 and n to 3. Then S consists of the transpositions $s_1 = (1, 2)$ and $s_2 = (2, 3)$, and W is the symmetric group on 3 letters. The generators s_1 and s_2 can be represented by the matrices $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, respectively.

The group H_2 can be represented by the group of 3×3 diagonal matrices with ± 1 along the diagonal. This group is generated by matrices $\{ d_i \mid i = 1, 2, 3 \}$ that have a -1 in the ith diagonal position and 1 elsewhere on the diagonal.

The set X_{s_1} consists of the matrices d_1 and d_2, the set X_{s_2} consists of the matrices d_2 and d_3, and we have for instance

$$s_1^2 = v^2 t_1 + a t_{d_1} t_{s_1} + a t_{d_2} t_{s_1}.$$

The partial order on WH_b defined by $R_{x,y}$ can be described by a Hasse diagram: the elements in higher rows are greater than the elements in the lower rows to which they are connected. Elements that are not connected are not comparable. The Hasse diagram for this example actually has two connected components. Below is the component that contains the identity element. The other component can be obtained by multiplying on the left by d_1.



5. Future Research

Problem. *To determine if $\mathcal{H}$ has a bar-stable basis.*

Recall the bar involution that takes t_x to $t_{x^{-1}}^{-1}$. Kazhdan and Lusztig [4] defined a basis $\{ C_w \mid w \in W \}$ for the Iwahori-Hecke algebra, with the property that $\overline{C_w} = C_w$ for each w in W. I would like to determine if Hecke algebras for monomial groups have a bar-stable basis.

- If the answer is positive, this basis is likely to lead to finding a cellular decomposition of $\mathcal{H}$ in the sense of Graham and Lehrer [3], and to help in studying the representation theory of $\mathcal{H}$.
- If the answer is negative, then I'll have shown that the existence of a well-defined bar involution does not guarantee the existence of a bar-stable basis. This is a question that has not been resolved yet.

Problem. *To analyze the representation theory of $\mathcal{H}$.*

Even without the benefit of a bar-stable basis, it is possible to study the representation theory of $\mathcal{H}$. Following ideas given in the sixth chapter of [6], we can study irreducible $\mathcal{H}$ modules by studying the irreducible modules of a specific family of subalgebras.

6. Other Areas of Research

Although my primary field of study is in Hecke algebras for monomial groups, I am also interested in math education. I currently teach at an open-admission 2-year university. We have been working to motivate students by using a problem-solving approach to mathematics. Rather than asking students to memorize formulas and repetitively work problems, they are encouraged to discuss and derive methods for dealing with real-life situations. This strategy is a new one for our campus and it will take some time to collect enough data to determine if it is effective.

I am also interested in the history of mathematics, specifically from the Middle East. I am currently reading background material to determine what has already been discovered. I am also doing a preliminary analysis of some manuscripts written in 1486 CE.

References

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