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You Cannot Conceive The Many Without The One
-Plato-



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Japanese Ladder Game

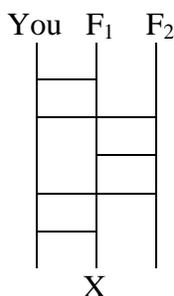
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(Joint work with Christopher Rufty)

Mathematical games and puzzles are great gateways to the beauty of math. They are fun, they are usually easy to play, and they often lead the players to a crazy mindset of wanting to study the math behind the games! This article is a brief introduction of the findings by me and one of my students, Mr. Chris Rufty, during our independent study last year. It all happened after I successfully lured him into playing a game.

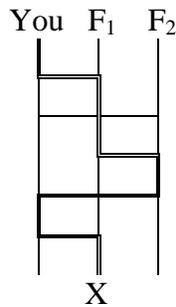
Martin Gardner introduced this game in his book *New Mathematical Diversions* [3]. One night in a bar, you and two of your friends are wondering who should pay all the tabs to end the wonderful evening. One of your friends then comes up with an idea. On a paper he draws three vertical lines, and he randomly puts one of your names at the top of each line. He then folds the top of the paper so no one can see the names. The other of your friends then draws several random horizontal lines with each one connecting two of the vertical lines. Finally, you add several random horizontal lines, and mark an X at the bottom of one of the vertical lines.

Figure 1



To determine the payer, each of you in turn traces this ladder. For example, you start to trace down from the vertical line with your name on it. Every time you meet an end of a horizontal line, continue on the horizontal line until you reach the other end of it, and then turn down the vertical line. Repeat this process until you reach the bottom of a vertical line. If you are the “lucky” one ending on the X, well, you know what to do.

Figure 2



This game is very popular in Asia. The Japanese call it *Amidakuji*, and the Chinese call it *Ghost Leg*. The name *Japanese Ladder Game* is adopted from S. T. Dougherty and J. F. Vasquez [2], and its rules represent just one of the games that can be played using such ladder structures. My goal in this article is to introduce you to the rules of the game and provide you with a simple mathematical way to solve it. I will try NOT to scare you with confounding mathematical theories, although technical terms sometimes may be used out of necessity for the sake of convenience. We must begin by clearly defining what a Japanese ladder is:

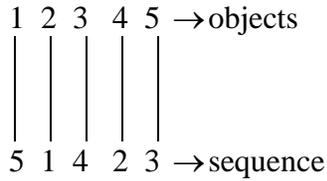
A Japanese Ladder consists of several vertical lines and several horizontal bars, or rungs, connecting two vertical lines. From the top of each vertical line a path is traced through the ladder using these three rules:

- 1. When tracing a vertical line, continue downward until an end of a rung is reached, then continue along the rung.*
- 2. When tracing a rung, continue along it until the end of the rung is reached, then continue down the vertical line.*
- 3. Repeat steps 1 and 2 until the bottom of a vertical line is reached.*

You already may have noticed that if you trace from the top of any vertical line, you will always end at the bottom of one of the vertical lines, though it may not be the same one on which you began. Then if you start from the point at which you just arrived at the bottom and change the “downward” rule to “upward,” you will be led back to the same point at which you started earlier. This simple experiment indicates an important fact in mathematics. Since this tracing technique is invertible, the ladder provides “1 to 1” and “onto” mapping. In other words, every time you trace from the top of the ladder, you will always land at a unique spot at the bottom. For convenience, in this article we will use consecutive numbers, called **Objects**, at the top of a ladder. And their re-arrangement at the bottom after tracing through the ladder will be called **Sequence**.

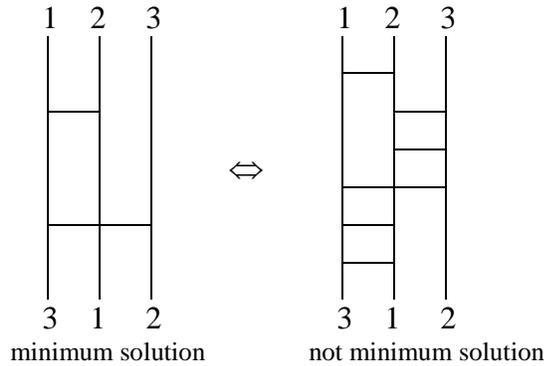
With the ladder structure, we then can introduce the **Japanese Ladder Game**. *A set of vertical lines is provided, along with the top objects and the bottom sequence. One is asked to create the rungs in the ladder necessary to match the sequence.*

Figure 3



The word “necessary” in the above request is tricky. We want to create rungs to fit the purpose, but we don’t want to create too many rungs. Hence, what necessary means in this context is finding the minimum number of rungs to match the sequence. We will call this a **Minimum Solution**. In figure 4 you will see two solutions of a game. However, the one on the right is not a minimum solution.

Figure 4



So, how shall we start the quest for a mathematical method of solving these games? Let’s check a simple case with only one rung in the ladder. In figure 5, we can see that from the top objects to the bottom sequence the two numbers switch their places. So what a rung does is create a **Permutation** of the two numbers. For this example, we use the notation (1,2) to indicate the permutation. Number 1 moves from the first place to the second place, and number 2 moves to the first place: $1 \rightarrow 2 \rightarrow 1$. Actually, the notation (2,1) indicates the same permutation: $2 \rightarrow 1 \rightarrow 2$.

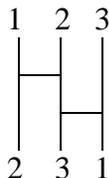
Figure 5



How about if we have two rungs? In figure 6, number 1 moves to the third place, number 3 moves to the second place, and number 2 moves to the first place. Therefore, we will use

$(1,3,2)$ to indicate this permutation: $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$. Since it forms a cycle, it doesn't matter which number we start with, as long as we have them in the correct order. More specifically, $(1,3,2) = (3,2,1) = (2,1,3)$. From now on, we will call this a 3-cycle, which means it permutes three objects. Hence, an **n-cycle** is actually a permutation of n objects.

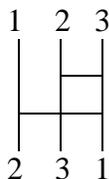
Figure 6



In the previous example (figure 5) we learned that every rung is a 2-cycle. So can we use two 2-cycles to describe this two-rung case (figure 6)? Yes, we can. But we have to start from the rung at the lower position. The rung at the lower position permutes 2 and 3, so its notation is $(2,3)$. The one at the upper position permutes 1 and 2, so we will indicate it as $(1,2)$. Now we put them together as our final answer: $(2,3)(1,2)$. We express our answer as a **Product** of two 2-cycles. The reason we start from the rung at the lower place and go upward is because in mathematics we operate every permutation notation from the right one to the left one, which is the opposite of how we write down these cycles. The 2-cycle at the very right end of this product, $(1,2)$ in this case, is actually the first one to operate. Therefore, it has to be the rung at the highest place. And now we can check this new rule to see if it really matches the sequence. Number 1 first permutes to the second place by $(1,2)$, then permutes (from the second place) to the third place by $(2,3)$: $1 \rightarrow 3$. Number 2 permutes to the first place by $(1,2)$, and does not move in $(2,3)$: $2 \rightarrow 1$. Number 3 does not move in $(1,2)$, but switches to the second place in $(2,3)$: $3 \rightarrow 2$. Organizing them together we will have $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, which is exactly the same operation as the 3-cycle $(1,3,2)$. Since these two expressions have the same result—they create the same sequence, we say they are equal: $(1,3,2) = (23)(12)$. The same principle can also apply to any n-cycle. We then have a very important result, which we have proved in our larger work-in-progress [4]: *Any n-cycle can be written as a product of (n-1) 2-cycles. Moreover, (n-1) is the least number of 2-cycles needed.*

However, this decomposition is not unique. The 3-cycle $(1,3,2)$ can also be expressed as $(1,3)(3,2)$, which creates a different ladder that matches the bottom sequence as well.

Figure 7



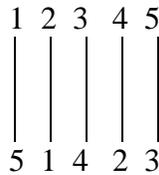
To make it easy, I will use the following way to decompose an n-cycle:

$$(a_1, a_2, \dots, a_{n-1}, a_n) = (a_1, a_2)(a_2, a_3)(a_3, a_4) \cdots (a_{n-2}, a_{n-1})(a_{n-1}, a_n).$$

For example, $(1,2,3,4) = (1,2)(2,3)(3,4)$ and $(5,4,3,2,1) = (5,4)(4,3)(3,2)(2,1)$.

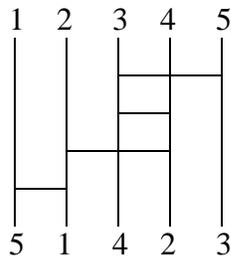
Well, from the above discussion, we know that there is a relationship between a bottom sequence and a product of 2-cycles. But to make the process easier, I will use an n-cycle as the medium. (Another reason to use n-cycles is that based on the results at which we arrived in our work-in-progress [4], I can guarantee that I only use the least number of 2-cycles, hence rungs, needed! That fits our purpose of finding minimum solutions.) Let's use the following figure (figure 8) as an example. By analyzing the sequence—number 1 to the second place: $1 \rightarrow 2$, number 2 to the fourth place: $2 \rightarrow 4$, number 3 to the fifth place: $3 \rightarrow 5$, number 4 to the third place: $4 \rightarrow 3$, and number 5 to the first place: $5 \rightarrow 1$ —we can organize it as $1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 1$, which indicates the 5-cycle $(1,2,4,3,5)$.

Figure 8



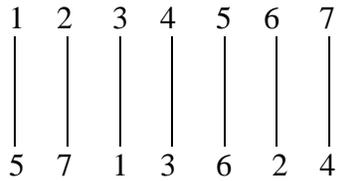
We already know that we can decompose this cycle: $(1,2,4,3,5) = (1,2)(2,4)(4,3)(3,5)$. Our next step is to draw the corresponding rungs in the ladder. If you want to start from the very left 2-cycle in the product, you have to draw the rungs from a lower place and go upward. (Remember that the left 2-cycle is the last permutation to operate!) However, I would like to start from the very right 2-cycle and then draw the rungs downward. So first I will draw a rung connecting the third and the fifth vertical lines $(3,5)$. The second 2-cycle, $(4,3)$, corresponds to a rung connecting the fourth and the third vertical lines—and don't forget, the $(4,3)$ rung has to be lower than the $(3,5)$ rung to guarantee that it operates after the $(3,5)$ rung. Following this pattern to create the $(2,4)$ rung and the $(1,2)$ rung, we successfully create a minimum solution of the Japanese ladder game. Feel free to check!

Figure 9



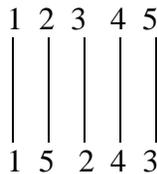
So, can you use what you just learned to find a minimum solution of the following Japanese ladder game? The answer is provided at the end of this article.

Figure 10



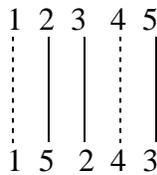
In some cases, the sequence in the game may look very complicated. But if we look closer, we will realize that we can simplify it. Check the next figure.

Figure 11



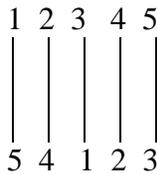
In this sequence, numbers 1 and 4 remain at the same places. So the only cycle we will get is (2,3,5). Actually, since nothing happens to numbers 1 and 4, the game is equivalent to a simplified game shown below. We don't have to consider the first and the fourth vertical lines at all.

Figure 12



Do we now have enough knowledge to solve a random Japanese ladder game? Not yet! Let's look at the following example. By analyzing the sequence of this game—number 1 to the third place: $1 \rightarrow 3$, number 3 to the fifth place: $3 \rightarrow 5$, and number 5 to the first place: $5 \rightarrow 1$ —we complete a 3-cycle, (1,3,5).

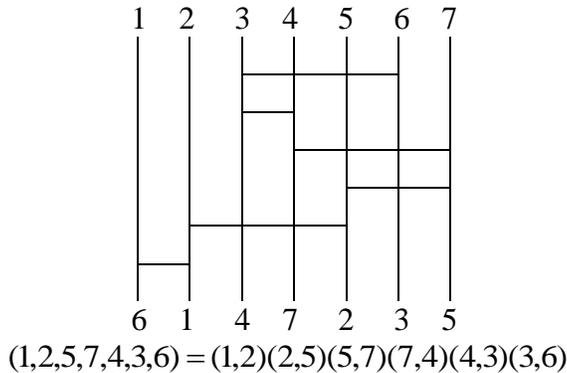
Figure 13



How about numbers 2 and 4? They switch their places and form a 2-cycle, $(2,4)$. Since these two cycles (the 3-cycle and the 2-cycle in the game) do not have an element in common, their operations do not affect each other. That means any number from 1 to 5 will only be permuted once by only one of the cycles. Also, no matter which cycle goes first, they all create the same sequence, $(1,3,5)(2,4) = (2,4)(1,3,5)$. Cases like this are considered to be the same products. Two cycles with no elements in common are called **Disjoint**. And the products of disjoint cycles, regardless of the order, are considered the same. In these cases we say the product is **Unique**. For example, $(1,2)(3,4)(5,6) = (3,4)(1,2)(5,6) = (5,6)(3,4)(1,2)$.

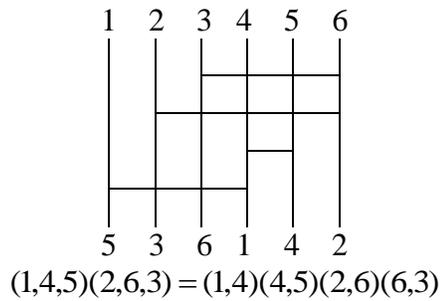
There is another important theorem we need to introduce before we start to solve a random Japanese ladder game. *Any sequence can be uniquely written as an n -cycle, or a product of disjoint cycles* [4]. This theorem guarantees us that every Japanese ladder game is solvable. And we can always find a minimum number of rungs to match the sequence provided. How? Here is a brief explanation. According to the theorem, if a sequence can be expressed as an n -cycle, we already know that we can rewrite it as a product of minimum 2-cycles. And for each 2-cycle, we can always find a corresponding rung. That means we just need to find all the rungs, and then we are done. Figures 8 and 9 clearly demonstrate an example of this. Another example is shown below.

Figure 14



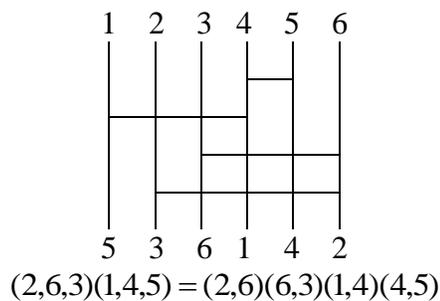
If the sequence can be uniquely expressed as a product of disjoint cycles, applying the established technique we can decompose each cycle as a product of 2-cycles. For example, $(1,4,5)(2,6,3)$ is a product of two disjoint 3-cycles. We can decompose these two 3-cycles: $(1,4,5) = (1,4)(4,5)$ and $(2,6,3) = (2,6)(6,3)$. We then can rewrite the product of two 3-cycles as a product of 2-cycles: $(1,4,5)(2,6,3) = (1,4)(4,5)(2,6)(6,3)$. All we then need to do is draw all the rungs according to the established rules and then we are done. The following figure shows us the case of $(1,4,5)(2,6,3)$.

Figure 15



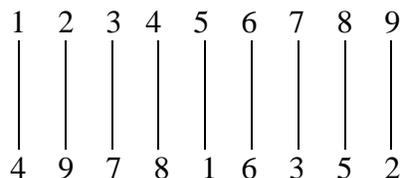
I should remind you that since $(1,4,5)$ and $(2,6,3)$ are disjoint, $(1,4,5)(2,6,3) = (2,6,3)(1,4,5)$. So for the same sequence, we may create another minimum solution like this: $(2,6,3)(1,4,5) = (2,6)(6,3)(1,4)(4,5)$. Check the next figure.

Figure 16



Now you have learned all the necessary techniques. You should be able to solve any random Japanese ladder game now. So I will end this article with a small quiz: another Japanese ladder game. Try it yourself! Oh, and next time if you and your friends end up in the bar again, you know how to avoid the X mark.

Figure 17



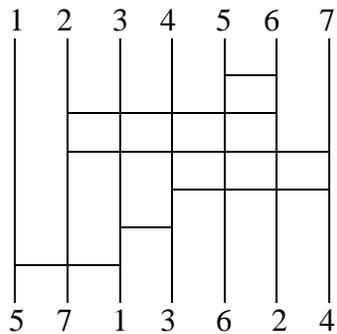
Note

A Japanese ladder game can be played with an additional rule. Dougherty and Vasquez [1], D. Senft [5], and R. Tucker [6] all mention this rule in their papers: rungs can only connect two adjacent vertical lines. That means rungs are not allowed to cross over vertical lines. With this rule the game becomes more difficult and requires more mathematical theories. Interested readers are welcome to check these theories in [1] and [5].

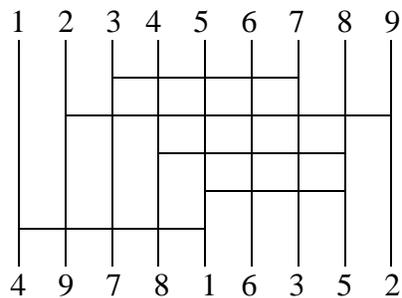
Reference

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- [3] Martin Gardner, *New Mathematical Diversions*, Mathematical Association of America, 1995, 23-33.
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<http://www2.edc.org/makingmath/studentWork/amidaKuji/AmidaKujiByDavidSenft.pdf>
- [6] R. Tucker, Amida, *Mathematical Gazette*, Vol.61, October, 1977, 213-215.

Solution of Figure 10 (not unique)



Solution of Figure 17 (not unique)



Dr. Wei-Kai Lai is an assistant professor at USC Salkehatchie. He received his Ph.D. in mathematics from the University of Mississippi. His research focuses on geometric properties of positive tensor products, Banach lattices, and positive operators. Over the past two years Dr. Lai has mentored three students for their undergraduate research on topics including advanced geometry, complex analysis and integration theory, and Japanese ladder games. Related findings have been presented at Carolina Mathematics Seminars and in *The College Mathematics Journal*.



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