South Carolina High School Math Contest, 2025

February 1, 2025

1. (early) Suppose that $x \in [0, \pi]$ is such that $\sin(2x) = 2/3$. Then find $\sin(x) + \cos(x)$.
(a) -$\frac{5}{3}$ (b) -$\sqrt{\frac{5}{3}}$ (c) 1 (d) $\frac{5}{3}$ (e) $\sqrt{\frac{5}{3}}$

Solution: (e)

We have
$$(\sin(x) + \cos(x))^2 = 1 + 2\sin(x)\cos(x) = 1 + \sin(2x) = 5/3,$$
so $\sin(x) + \cos(x) = \sqrt{\frac{5}{3}}$.

2. (early) Call a positive integer n a *prime gap* if there are prime numbers $p < q$ with $q - p = n$.
How many positive integers ≤ 10 are prime gaps?
(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Solution: (d)

All positive integers less than 10, other than 7 are prime gaps.

For the even integers we have
$$5 - 3 = 2,\ 7 - 3 = 4,\ 11 - 5 = 6,\ 13 - 5 = 8,\ 17 - 7 = 10,$$
and for the odd integers we have
$$3 - 2 = 1,\ 5 - 2 = 3,\ 7 - 2 = 5, 11 - 2 = 9.$$
For an odd integer n to be a prime gap, we must have $p = 2$, since 2 is the only even prime. Since 9 is not prime, 7 is not a prime gap.

3. (early) In ΔABC, if we have $\angle A : \angle B : \angle C = 1 : 1 : 2$, then $\overline{BC} : \overline{AC} : \overline{AB} =$
(a) $1 : 1 : 1$ (b) $1 : 1 : 2$ (c) $1 : 1 : \sqrt{2}$ (d) $1 : 1 : \sqrt{3}$ (e) $1 : 1 : 4$

Solution: (c)

ΔABC is a $45 - 45 - 90$ triangle with hypotenuse AB, so the answer is $1 : 1 : \sqrt{2}$.

4. (early) A basketball falls from above a backboard, at a height of 4096 mm. The ball bounces to 1/4 of its maximum height each time. What will the height of the ball (in mm) be after 5 bounces?
(a) 20 (b) 1 (c) $\frac{1}{2}$ (d) 2 (e) 4

Solution: (e)

Note that $4096 = 4^6$, so $4^6 * (1/4^5) = 4^1 = 4$.

5. (early) A pipe carrying oil under the sea ruptured. When the authorities arrived on the scene they noticed a big circle of contamination. By the next morning, the radius of the circle of contamination had tripled. By what factor had the area of the circle of contamination grown?
(a) 9 (b) 8 (c) 7 (d) 6 (e) 5

Solution: (a)

If the radius was originally r, then the original area was πr^2.

On the next day the radius had grown to $3r$, and so the area had grown to $\pi(3r)^2$, which is 9 times the previous area.

6. (early) Machine A can do a job in two days. It takes Machine B six days to do the same job. If the machines work together, how many days will it take them to do the job?

(a) 3/2 (b) 8 (c) 4 (d) 9/4 (e) 2

Solution: (a)

Machine A does $\frac{1}{2}$ of the job per day and Machine B does $\frac{1}{6}$ of the job per day. Let x be the number of days it takes the two machines working together to do the whole job. We solve $\frac{1}{2}x + \frac{1}{6}x = 1$ to see that $x = \frac{3}{2}$ days.

7. (early) Consider the equation $\log_4 A + \log_A 4 = 5/2$. What is the units digit of the sum of all integer solutions A to this equation with $A > 1$?

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

Solution: (e)

We have $\frac{\ln A}{2\ln 2} + \frac{2\ln 2}{\ln A} = \frac{5}{2}$, which is equivalent to

$$\ln^2 A - \frac{5}{2}(2\ln 2)\ln A + (2\ln 2)^2 = 0. \quad (1)$$

In terms of $\ln A$, the solutions are $\ln A = 4\ln 2$, $\ln 2$, and hence the solutions are $A = 16, 2$, both of which are integers > 1. Hence, the units digit of their sum is 8.

8. (early) Which of the following numbers is largest?

(a) 2025 (b) 2^{025} (c) 20^{25} (d) 202^5 (e) 20^{2^5}

Solution: (c)

We can rule out the answer choices as follows:

- $20^{2^5} = 1$.
- Since $2^{025} = 2^{25}$, we have $2^{025} < 20^{25}$.
- $2025 < 8000 = 20^3 < 20^{25}$.
- Since $202 < 20^2$, we have $202^5 < 20^{10}$.

Therefore the answer is 20^{25}.

9. (early) If $y = 10 \cdot \dfrac{1}{\sec(\alpha)}$ and $\alpha \in (0, \frac{\pi}{2})$, then which of the following must equal β?

$$\beta - \cos(\alpha)y = \frac{2 \cdot \sin(\alpha)}{\left(\frac{1}{5} \cdot [1 - \cos^2(\alpha)]^{-1/2}\right)}$$

(a) 1 (b) α (c) $y/10$ (d) 10 (e) 0

Solution: (d)

We have

$$\left[1 - \cos^2(\alpha)\right]^{-1/2} = (\sin^2(\alpha))^{-1/2} = \csc(\alpha).$$

Here we used the assumption that $\alpha \in (0, \frac{\pi}{2})$ to conclude that $(\sin^2(\alpha))^{1/2}$ is $\sin(\alpha)$ and not $-\sin(\alpha)$.

Therefore, we have

$$\beta = \cos(\alpha) \cdot \frac{10}{\sec(\alpha)} + \frac{2 \cdot \sin(\alpha)}{\frac{1}{5}\csc(\alpha)} = 10\cos^2(\alpha) + 10\sin^2(\alpha) = 10.$$

10. (early) Suppose $2a^2 + b^2 - 4a - 6b = -11$, where a and b are integers. What is $(a^b)(b^a)$?
(a) 1 (b) 2 (c) 3 (d) -1 (e) 0

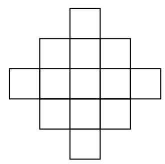
Solution: (c)

We have

$$2(a^2 - 2a) + (b^2 - 6b) = -11$$
$$2(a^2 - 2a + 1) + (b^2 - 6b + 9) = -11 + 11$$
$$2(a-1)^2 + (b-3)^2 = 0,$$

so $a = 1, b = 3$, and $(a^b)(b^a) = (1^3)(3^1) = 3$.

11. (middle) As shown in the figure, each small square is a square of the same size. How many squares of all sizes are there in this figure?



(a) 15 (b) 16 (c) 17 (d) 18 (e) 19

Solution: (d)

We count. There are:

- $1 + 3 + 5 + 3 + 1 = 13$ small 1×1 squares;
- 4 larger 2×2 squares, each including four of the smaller squares;
- 1 large 3×3 square.

Therefore the answer is $13 + 4 + 1 = 18$.

12. (middle) Let x, y, and z be integers such that $|12 - 3z| + |2y - 6| + |x + 2| = 1$. What is the minimum value of $x + y + z$?
(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

Solution: (b)

Since z is an integer, $|12 - 3z|$ must be a multiple of 3, and it must be at most 1. Hence $12 - 3z = 0$ and $x = 4$.

Similarly we have $|2y - 6| = 0$ and hence $y = 3$.

Finally, we have $|x + 2| = 1$, so that $x = -3$ or $x = -1$. The value of $x + y + z$ is minimized by taking $x = -3$, and we have $x + y + z = -3 + 3 + 4 = 4$.

13. (middle) Four fair coins are flipped. Then, each coin showing tails is flipped a second time.

On average, how many heads do you expect to flip total?
(a) 2 (b) $\frac{5}{2}$ (c) $\frac{8}{3}$ (d) 3 (e) 4

Solution: (d)

For each of the coins, there is a $\frac{1}{2}$ chance it shows heads originally, and a $\frac{1}{2} \cdot \frac{1}{2}$ chance it shows tails originally and then heads on the second flip.

Hence, each coin has a $\frac{3}{4}$ probability of showing heads at the end, so the answer is $4 \cdot \frac{3}{4} = 3$.

14. (middle) If w, x, y, z are positive integers such that $2022w + 2023x + 2024y + 2025z = 12140$, what is $w + x + y + z$?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

Solution: (e)

We have

$$w + x + y + z = \frac{2022(w + x + y + z)}{2022} < \frac{2022w + 2023x + 2024y + 2025z}{2022} = \frac{12140}{2022},$$

and

$$w + x + y + z = \frac{2025(w + x + y + z)}{2025} > \frac{2022w + 2023x + 2024y + 2025z}{2025} = \frac{12140}{2025},$$

so that

$$\frac{12140}{2025} < w + x + y + z < \frac{12140}{2022}.$$

Long division shows that both fractions are extremely close to 6, so the answer is 6.

15. (middle) Fifty USC football players have an assigned seat on the bus for a game. A single, rookie player takes a random seat not knowing where people sit. After that, each player takes the assigned seat if it is unoccupied, and one of the unoccupied seats at random otherwise. What is the probability that the last football player to board gets to sit in their assigned seat?

(a) 1/100 (b) 1/20 (c) 1/200 (d) 1/2 (e) 1/50

Solution: (d)

(Haonan) Name the players 1,2,...,50 in the order they take the seats. According to the rule, each seat assignment corresponds to a loop: $a_1 = 1 < a_2 < ... < a_k \leq 50$ meaning a_1 takes the seat of a_2... and a_k takes the seat of a_1. Whenever the last person 50 is in the loop, i.e. $a_k = 50$, remove it to get another loop. This gives a bijection (if $a_k < 50$ add $a_{k+1} = 50$ in the loop). So the probability that the last person gets his/her seat is the same as probability that he/she doesn't get it.

16. (middle) A bag contains six green marbles, seven blue marbles, and eight red marbles.

If you remove three marbles at random from the bag, what is the probability that they are all of different colors?

(a) $\frac{1}{6}$ (b) $\frac{2}{9}$ (c) $\frac{24}{95}$ (d) $\frac{26}{95}$ (e) $\frac{8}{21}$

Solution: (c)

The probability of drawing a green marble, a blue marble, and a red marble in that order is

$$\frac{6}{21} \cdot \frac{7}{20} \cdot \frac{8}{19} = \frac{4}{95}.$$

The probability of drawing green, blue, and red in any other particular order is the same. Since there are six possible orders, the total probability is $6 \cdot \frac{4}{95}$.

17. (middle) Find the sum

$$\sum_{k=1}^{9999} \log\left(1 - \frac{1}{k+1}\right)$$

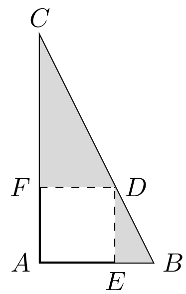
(a) -6 (b) -4 (c) -2 (d) 0 (e) 2

Solution: (b)

We have that

$$\sum_{k=1}^{9999} \log\left(1 - \frac{1}{k+1}\right) = \sum_{k=1}^{9999} \log\left(\frac{k}{k+1}\right) = \sum_{k=1}^{9999} (\log k - \log(k+1)) = \log(1) - \log(9999+1) = -\log(10^4) = -4.$$

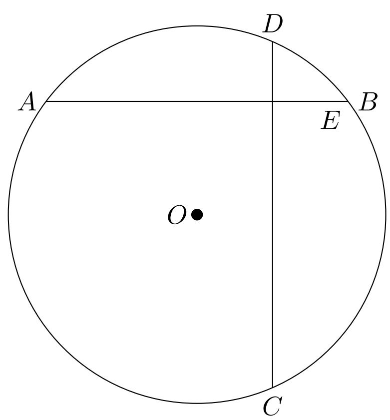
18. (middle) Inside the right triangle ΔABC, $AEDF$ forms a square. If $\overline{BD} = 2$ and $\overline{CD} = 4$, then what is the area of the shaded part, *i.e.*, the sum of the areas of the two triangles ΔCFD and ΔDEB?
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5



Solution: (d)

Rotate the triangle ΔDEB 90° clockwise around D. By assumption, the triangles ΔCFD and ΔDEB now form a right triangle whose right angle sides have lengths 2 and 4, respectively. So the answer is $\frac{1}{2} \times 2 \times 4 = 4$.

19. (middle) Suppose that A, B, C, D are four points on the circle that is centered at O. The line segments AB and CD are perpendicular to each other and divide the circle into four parts. The distances from O to AB and CD are 2 and 1, respectively. Consider the areas of the four parts. Find the difference between the sum of maximum and minimum, and the sum of the remaining two.
(a) 2 (b) 4 (c) 6 (d) 8 (e) π



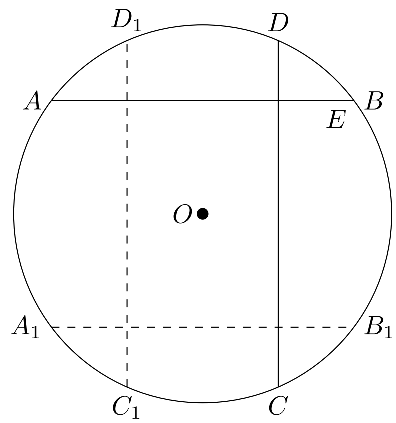
Solution: (d)

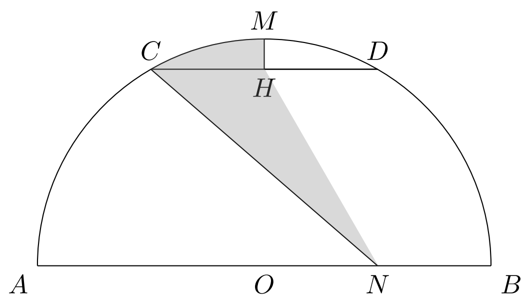
By symmetry, the desired answer is the area of the rectangle in the middle, which is $(2 \times 1) \times (2 \times 2) = 8$.

20. (middle) The diagram shows a semicircle with center O and radius $\overline{OA} = \overline{OB} = 1$. Suppose that M is in the middle of the arc AB, with A, C, D, B equally spaced on the semicircle. Suppose further that H is the midpoint of CD and N is any point on AB. Find the area of the shaded portion formed by $NHMC$.
(a) $\frac{\pi}{24}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{3}$

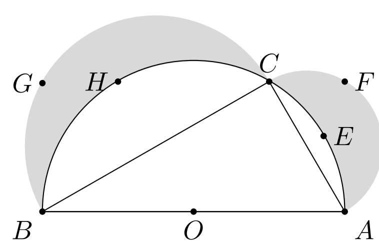
Solution: (b)

We may choose N to be O, then the area of the shaded portion is equal to 1/6 of the area of the semicircle. So the answer is $\pi \times 1^2 \times \frac{1}{2} \times \frac{1}{6} = \frac{\pi}{12}$.





21. (late) The diagram depicts a semicircle with center O. Suppose $\overline{AB} = 4$ and $\angle ABC = 30°$. Construct two semicircles using AC and BC as diameters, respectively, as the picture shows. Find the sum of the areas of the shaded portions formed by $AFCE$ and $BGCH$.
(a) $\sqrt{3}$ (b) $2\sqrt{3}$ (c) π (d) $\frac{\pi}{2}$ (e) $\frac{\pi}{2} + \sqrt{3}$



Solution: (b)

Note that $\overline{AC} = 2$ and $\overline{BC} = 2\sqrt{3}$. The desired sum of areas, denoted by S, satisfies $S = S_1 + S_2 + S_3 - S_4$, where

- S_1 is the area of the triangle ABC which is $S_1 = \frac{1}{2} \times 2 \times 2\sqrt{3} = 2\sqrt{3}$
- S_2 is the area of the semicircle formed by BGC which is $S_2 = \frac{1}{2} \times \pi \cdot (\sqrt{3})^2 = \frac{3\pi}{2}$
- S_3 is the area of the semicircle formed by AFC which is $S_3 = \frac{1}{2} \times \pi \cdot 1^2 = \frac{\pi}{2}$
- S_4 is the area of the semicircle formed by ACB which is $S_4 = \frac{1}{2} \times \pi \cdot 2^2 = 2\pi$

Therefore, the answer is $S = 2\sqrt{3}$.

22. (late) Recall the rules for leap years: The year n has 365 days, unless:

- n is divisible by 4, in which case year n has 366 days, unless:

- n is divisible by 100, in which case year n has 365 days, unless:
- n is divisible by 400, in which case year n has 366 days.

Today, February 1, 2025, is a Saturday. On what day of the week will February 1, 20025 be?
(a) Saturday (b) Monday (c) Tuesday (d) Thursday (e) Friday

Solution: (a)

We have $365 = 7 \cdot 52 + 1$. Therefore, at the end of each non-leap year, the weekday of February 1 will advance by 1 the following year; at the end of each leap year, the weekday will advance by 2.

Between 2025 and 20024, inclusive, there are:

- 18000 years; the weekday will advance by 1 at the end of each.
- 4500 years divisible by 4; the weekday will advance by an additional 1 at the end of each.
- 180 years divisible by 100; the weekday will advance by one less at the end of each (relative to the two weekdays listed above).
- 45 years divisible by 400; the weekday will advance by one more at the end of each (relative to all of the above combined).

Therefore, the weekday of February 1 will advance by

$$18000 + 4500 - 180 + 45 = 22365,$$

or more properly, the remainder when 22365 is divided by 7. It can be checked that 22365 is divisible exactly by 7; hence, February 1, 20025 will be the same day of the week as February 1, 2025.

23. (late) Let f and g be the functions from the xy-plane to the xy-plane with the following properties: The function g reflects the xy-plane across the line $x + y = 0$, and the function f reflects the xy-plane across the line $x + y = 1$.

    Let $h = (g \circ f)^{10}$. In other words,

    $$h = (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f) \circ (g \circ f).$$

    If $h(3, 8) = (a, b)$, what is $a + b$?
    (a) -11 (b) -9 (c) 9 (d) 11 (e) 21

    Solution: (b)

    We have $g(x, y) = (-y, -x)$, $f(x, y) = (1 - y, 1 - x)$, and $(g \circ f)(x, y) = (x - 1, y - 1)$. Therefore, $h(x, y) = (x - 10, y - 10)$ and so $h(3, 8) = (-7, -2)$.

24. (late) Consider the square $ABCD$ with side length 2, and draw arcs with A, B, C, D as the centers and each with radius 2, as in the diagram. Find the area of the shaded region $x + 4y$.
    (a) $-12 + \frac{8\pi}{3} + 4\sqrt{3}$ (b) π (c) $\frac{4\pi}{3} - \sqrt{3}$ (d) $4 - \frac{2\pi}{3} - \sqrt{3}$ (e) $\sqrt{3}$

    Solution: (a)

    Computing the area of the square and of any of the four quarter circles, we have

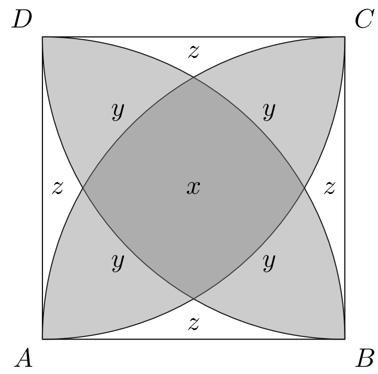
    $$x + 4y + 4z = 4$$

    $$x + 3y + 2z = \pi.$$

    Now, we compute $x + 2y + z =$ the area of the intersection of the interior of the two quarter circles with base AB. Label E for the intersection point of these circles, and observe that ΔABE is equilateral.

    The line segments AB and EB bound one sixth of a circle, with area $\frac{1}{6} \cdot 4\pi$. Similarly, the line segments BA and EA bound a sixth of a circle with the same area. The interiors of these sixth-circles intersect in ΔABE, so that

    $$x + 2y + z = 2 \cdot \frac{1}{6} \cdot 4\pi - \frac{1}{2} \cdot 2 \cdot \sqrt{3} = \frac{4\pi}{3} - \sqrt{3}.$$



We can now solve our system of three linear equations in the variables x, y, and z. We have

$$z = (x + 4y + 4z) + (x + 2y + z) - 2(x + 3y + 2z) = 4 - \frac{2}{3}\pi - \sqrt{3}$$

so that

$$x + 4y = (x + 4y + 4z) - 4 \cdot z = 4 - 4(4 - \frac{2}{3}\pi - \sqrt{3}) = -12 + \frac{8\pi}{3} + 4\sqrt{3}.$$

25. (late) Suppose that $\sin A = \cos B = \tan C$, where A, B, and C are the angles of a triangle. What is the value of $(\cos A)^3 + (\cos A)^2 - \cos A$?

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}}{2}$ (e) $\frac{1}{4}$

Solution: (a)

From $\sin A = \cos B$, we know $A = \frac{\pi}{2} \pm B$. If we had $A = \frac{\pi}{2} - B$, then we would have $C = \frac{\pi}{2}$, so that $\tan C$ would be undefined. Therefore, we must have $A = \frac{\pi}{2} + B$, so that $C = \pi - A - B = \frac{3\pi}{2} - 2A$.

The equation $\sin A = \tan C$ implies that $\sin A = \tan(\frac{3\pi}{2} - 2A) = 1/\tan(2A)$, which means $\sin A \tan(2A) = 1$. Thus, we have

$$1 = \sin A \tan(2A) = \sin A \frac{\sin 2A}{\cos 2A} = \sin A \frac{2 \sin A \cos A}{2\cos^2 A - 1} = \frac{2(1 - \cos^2 A)\cos A}{2\cos^2 A - 1},$$

so

$$2\cos^2 A - 1 = 2(1 - \cos^2 A)\cos A$$

implying $\cos^3 A + \cos^2 A - \cos A = \frac{1}{2}$.

26. (late)

Let

$$S = \frac{3^2 + 4}{3^2 - 4} + \frac{4^2 + 4}{4^2 - 4} + \frac{5^2 + 4}{5^2 - 4} + \cdots + \frac{13^2 + 4}{13^2 - 4}.$$

What is the closest integer to S?

(a) 12 (b) 13 (c) 14 (d) 15 (e) 16

Solution: (d)

From $\frac{m^2+4}{m^2-4} = 1 + \frac{8}{m^2-4} = 1 + 2(\frac{1}{m-2} - \frac{1}{m+2})$, we have

$$\begin{aligned} S &= \sum_{m=3}^{13} \frac{m^2 + 4}{m^2 - 4} = 11 + 2\sum_{m=3}^{13}\left(\frac{1}{m-2} - \frac{1}{m+2}\right) \\ &= 11 + 2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} - \frac{1}{13} - \frac{1}{14} - \frac{1}{15}\right) \\ &= 15 - 2\left(\frac{1}{13} + \frac{1}{14} + \frac{1}{15}\right) > 15 - 2 \cdot \frac{3}{13} = 15 - \frac{6}{13} > 15 - \frac{6.5}{13} = 14.5 \end{aligned} \tag{2}$$

So we have $14.5 < S < 15$, so that the answer is 15.

27. (late) If m, a, b, c are positive integers such that $a\log_m 2 + b\log_m 3 + c\log_m 5 = 2025$, then what is the minimum possible value of $m + a + b + c$?
(a) 6102 (b) 6103 (c) 6104 (d) 6105 (e) 6106

Solution: (d)

We have $\log_m(2^a3^b5^c) = 2025$ and hence $2^a3^b5^c = m^{2025}$. This implies that 2, 3, and 5 all divide m, so that $m = 2^\alpha 3^\beta 5^\gamma$ for positive integers α, β, γ. Given that

$$2^a3^b5^c = m^{2025} = (2^\alpha 3^\beta 5^\gamma)^{2025},$$

we have $a = 2025\alpha$, $b = 2025\beta$, $c = 2025\gamma$.

The minimum values of m, a, b, and c are attained by choosing $\alpha = \beta = \gamma = 1$. We then have $m = 30$ and $m + a + b + c = 30 + 2025 \times 3 = 6105$.

28. (late) Let $f(x) = x^{2025} - 2x^{2024} - x + 3$. If $g(x) = f(f(x))$, what is the remainder when $g(x)$ is divided by $x - 2$?
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution: (b)

The answer is $g(2) = f(f(2))$. We compute that

$$f(2) = 2^{2024}(2-2) - 2 + 3 = 1,$$

and

$$f(f(2)) = f(1) = 1 - 2 - 1 + 3 = 1.$$

29. Let f be a function from $\mathbb{Z}$ to $\mathbb{Z}$ satisfying the following two properties:

$$f(4a) + 4f(b) = f(f(f(a+b))), \quad f(1) \neq 0.$$

Find the value of $|f(2025)|$:
(a) 0 (b) 150 (c) 2025 (d) 4050 (e) None of the above

Solution: (d)

We can find that

$$f(0) + 4f(b) = f(f(f(b))), \quad f(4) + 4f(b) = f(f(f(b+1))) = f(0) + 4f(b+1),$$
$$\Rightarrow f(b+1) - f(b) = \frac{f(4) - f(0)}{4} = \text{constant}.$$

Therefore, $f(x)$ must be a linear function of the form $f(x) = mx + n$. To determine m, n we write:

$$4am + n + 4mb + 4n = m(m(m(a+b) + n) + n) + n = m^3(a+b) + n + nm + nm^2.$$

Matching the coefficients in front of $(a + b)$ and the constant gives that

$$4m = m^3, \quad 5n = n + nm + nm^2.$$

So m can be $\pm 2, 0$, and n can only be 0. So the function $f(x)$ can only be $\pm 2x$. Now $|f(2025)| = 4050$.

30. For each positive integer m, let $\sigma(m)$ denote the sum of the positive integer divisors of m (including m itself).

There is a single positive integer $n < 120$ for which $\sigma(\sigma(\sigma(n))) > 2025$. What is the remainder when n is divided by 5?
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution: (d)

We have $n = 108$. I don't know of any elegant and systematic way of showing this, or proving its uniqueness. One way or another, it comes down to educated guessing and trial and error. We do have that $\sigma(n)$ is much larger than n precisely when n has many small divisors, so numbers like 108 are the first thing to try.