An Artificial Compression Based Reduced Order Model

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Introduction

- We consider the incompressible Navier-Stokes equations (NSE) on a bounded domain Ω subject to no-slip boundary conditions:

$$\begin{cases} u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f(x,t) & \forall x \in \Omega \times (0,T] \\ \nabla \cdot u = 0 & \forall x \in \Omega \times (0,T] \\ u = 0 & \forall x \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & \forall x \in \Omega. \end{cases}$$

- Defining:

$$X := H_0^1(\Omega)^d = \{H^1(\Omega)^d : v = 0 \ \text{ on } \ \partial\Omega\}$$
$$Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}.$$

- The weak formulation for the NSE can be written as: find $u : [0, T] \to X$ and $p : (0, T] \to Q$ such that, for almost all $t \in (0, T]$, satisfy

$$\begin{cases} (u_t, v) + (u \cdot \nabla u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v) & \forall v \in X \\ (\nabla \cdot u, q) = 0 & \forall q \in Q \end{cases}$$

Goals

- We seek a solution in a low dimensional ROM velocity space X_R with basis $\{\varphi_i\}_{i=1}^{R}$, and possibly pressure space Q_M with basis $\{\psi_i\}_{i=1}^{M}$.
- We want the scheme to be fast i.e. use the fewest number of basis functions possible.
- We want the scheme to be robust i.e. adding basis functions does not reduce accuracy.

Proper Orthogonal Decomposition (POD)

$$\min \sum_{n=0}^{N} \left\| u_{h,s}^{n} - \sum_{j=1}^{R} (u_{h,s}^{n}, \varphi_j)\varphi_j \right\|^2$$

$$\text{subject to } (\varphi_i, \varphi_j) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, R,$$

and

$$\min \sum_{n=0}^{N} \left\| p_{h,s}^{n} - \sum_{j=1}^{M} (p_{h,s}^{n}, \psi_j)\psi_j \right\|^2$$

$$\text{subject to } (\psi_i, \psi_j) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, M.$$

Introduction

- Often the velocity basis will be assumed to be weakly divergence-free.
- This will give rise to a velocity only ROM i.e.

$$\Big(\frac{u_R^{n+1}-u_R^n}{\Delta t},\varphi\Big)+b^*(u_R^n,u_R^{n+1},\varphi)+\nu(\nabla u_R^{n+1},\nabla\varphi)=(f^{n+1},\varphi),\quad \forall\varphi\in X_R.$$

- Issue: In engineering applications we need the pressure to calculate quantities involving the stresses such as lift and drag.

Pressure Poisson

- One approach for recovering the pressure with only the velocity is solving the Pressure Poisson Equation **(Noack et al. 2005),(Caiazzo, Iliescu et al. JCP, 2014)** ,:

$$\Delta p_M = -\nabla \cdot ((u_R \cdot \nabla) u_R) \quad \text{in } \Omega \, .$$

- Issues with this approach:
  - Correct boundary conditions unclear.
  - There are multiple consistent Pressure Poisson Equations.

Alternate Approach - No Pressure Poisson Equation

Use a ROM pressure basis $\{\psi_i\}_{i=1}^{M}$ to solve a coupled system, i.e.

$$\left(\frac{u_R^{n+1} - u_R^n}{\Delta t}, \varphi\right) + b^*(u_R^n, u_R^{n+1}, \varphi) + \nu(\nabla u_R^{n+1}, \nabla\varphi) + (p_M^{n+1}, \nabla \cdot \varphi) = (f^{n+1}, \varphi)$$
$$(\nabla \cdot u_R^{n+1}, \psi) = 0.$$

- Issue: The pressure and velocity basis will not necessarily satisfy the inf-sup/LBB_h condition

$$\inf_{q_M \in Q_M} \sup_{v_R \in X_R} \frac{(\nabla \cdot v_R, q_M)}{\|\nabla v_R\| \, \|q_M\|} \geq \beta_{is} > 0\,.$$

Supremizer Stabilization

- One way to deal with lack of LBB_h stability is the supremizer approach **(Rozza, Veroy. CMAME, 2007, Rozza et al, Numerische Mathematik, 2013. Ballarin et al IJNME, 2015)**.
- Solves a series of generalized eigenvalue problems to determine a new set of velocity basis functions $\{\xi_i\}_{i=1}^{S}$.
- Letting $X_{R_s} = \{\varphi_i\}_{i=1}^{R} \cup \{\xi_i\}_{i=1}^{S}$ ensures that LBB_h is satisfied at the online stage.

$$\inf_{q_M \in Q_M} \sup_{v_{R_s} \in X_{R_s}} \frac{(\nabla \cdot v_{R_s}, q_M)}{\|\nabla v_{R_s}\| \, \|q_M\|} \geq \beta_{is} > 0$$

- This is a very accurate approach.
- Depending on the problem may not be computationally feasible:
  - Calculating the supremizers may be very expensive.
  - Have to solve an R+M+S size system at each time step.

Other Approaches

- Even more options
  - Residual-based stabilization for POD-Galerkin **(Caiazzo, Iliescu et al. JCP, 2014)**.
  - Petrov-Galerkin **(Dahmen, Carlberg, Parish, Abdulle, Budac)**.
  - Others I am sure I missed.
- All of these approaches have merit.

Artificial Compression

- The approach we consider that circumvents some of the previously mentioned issues is the artifical compression scheme:

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f$$
$$\varepsilon p_t + \nabla \cdot u = 0.$$

- Originally proposed by Chorin and Temam, and further developed by Shen, Guermond, Layton and others.
- Does not requre LBB_h to be satisfied.
- Basis functions are constructed from *data that does not have to be weakly-divergence free*.
- Can use not discretely divergence free data.
- Do not need to worry about boundary conditions for the pressure.

Artifical Compression ROM

- The fully discrete algorithm for the Artifical Compression ROM (AC-ROM) scheme we consider is:

$$
\begin{aligned}
&\left(\frac{u_R^{n+1} - u_R^n}{\Delta t}, \varphi\right) + b^*(u_R^n, u_R^{n+1}, \varphi) + \nu(\nabla u_R^{n+1}, \nabla\varphi) \\
&- (p_M^{n+1}, \nabla \cdot \varphi) = (f^{n+1}, \varphi) \qquad\qquad \forall \varphi \in X_R \\
&\varepsilon \left(\frac{p_M^{n+1} - p_M^n}{\Delta t}, \psi\right) + (\nabla \cdot u_R^{n+1}, \psi) = 0 \qquad \forall \psi \in Q_M.
\end{aligned}
$$

- The velocity and pressure basis are constructed using POD.
- We can decouple the velocity and pressure system so only separate $M \times M$ and $R \times R$ systems need to be solved.

AC-ROM Analysis

- In the FEM setting if the basis does not satisfy LBB_h we expect to see convergence order degradation of Δt^{-1}.
- In the POD setting this may be pessimistic depending on the basis quality.
- To see this let P_R and χ_M be L^2 projections into the reduced basis velocity and pressure space respectively.

$$e_u^{n+1} = u^{n+1} - u_R^{n+1} = (u^{n+1} - P_R(u^{n+1})) + (P_R(u^{n+1}) - u_R^{n+1}) = \eta^{n+1} - \xi_R^{n+1}$$
$$e_p^{n+1} = p^{n+1} - p_M^{n+1} = (p^{n+1} - \chi_M(p^{n+1})) + (\chi_M(p^{n+1}) - p_M^{n+1}) = \kappa^{n+1} - \pi_M^{n+1}.$$

AC-ROM Analysis

- The problem term in the analysis is $(\nabla \cdot \eta^{n+1}, \pi_M^{n+1})$.
- We need a better bound than standard Cauchy-Schwarz.

Lemma (Strengthened CBS inequality)

Given a Hilbert space V and two finite dimensional subspaces $V_1 \subset V$ and $V_2 \subset V$ with trivial intersection:

$$V_1 \cap V_2 = \{0\},$$

then there exists $0 \leq \alpha < 1$ such that

$$|(v_1, v_2)| \leq \alpha \|v_1\| \|v_2\| \quad \forall v_1 \in V_1, v_2 \in V_2.$$

Principal Angle

- We want to determine α from the strengthened CBS inquality.
- Let $X_R^{div} := \text{span}\{\nabla \cdot \varphi_i\}_{i=1}^R$
- We need to calculate the principal angle between X_R^{div} and Q_M.

$$\theta_1 := \min \left\{ \arccos \left(\frac{|(v, \psi)|}{\|v\| \|\psi\|} \right) \middle| v \in X_R^{div}, \psi \in Q_M \right\},$$

with $0 \leq \theta_1 \leq \frac{\pi}{2}$.

- It then follows that $\alpha = \cos(\theta_1)$.
- This can be done using the SVD. Since R and M will be small the cost will be negligible.

Principal Angle Calculation

- Calculating the principal angle is not that expensive in the ROM setting.
- Let $\{\nabla \cdot \varphi_i^{orth}\}_{i=1}^{R}$ denote the orthonormalized basis of X_R^{div} . We consider the matrices

$$\mathbb{Q} = [\psi_1, \psi_2, \dots \psi_M] \text{ and } \mathbb{X} = [\nabla \cdot \varphi_2^{orth}, \nabla \cdot \varphi_2^{orth}, \dots \nabla \cdot \varphi_R^{orth}].$$

  Multiplying these two matrices and taking the SVD gives

$$\mathbb{X}^\top \mathbb{Q} = U \Sigma V.$$

- The first principal angle will then be given in terms of the first nonzero entry of Σ, by $\theta_1 = \text{arccos}(\sigma_1)$.

AC-ROM Analysis

Theorem (Error analysis of AC-ROM)

Under appropriate regularity assumptions we have the following error bound:

$$\|e_u^{N+1}\|^2 + \epsilon\|e_p^{N+1}\|^2 + \frac{\nu\Delta t}{2}\|\nabla e_u^{N+1}\|^2 + \Delta t\sum_{n=1}^{N+1}\frac{\nu}{2}\|\nabla e_u^n\|^2$$
$$\leq C\left(\Delta t + (1+\alpha^2\Delta t^{-1})\|\nabla\eta\|^2 + \|\kappa\|^2\right).$$

- The term $\alpha^2\Delta t^{-1}$ arises due to the lack of LBB_h stability.
- If α^2 is sufficiently small we do not expect to see order reduction with respect to Δt.

AC-ROM analysis

- We can show that α is actually an upper bound for the inf-sup constant.

Lemma

Suppose the POD basis is inf-sup stable for some constant β_{is} then it holds that $\alpha \geq \beta_{is}$.

Proof.

$$\alpha = \sup_{q_M \in Q_M} \sup_{v_R \in X_R} \frac{(\nabla \cdot v_R, q_M)}{\|\nabla \cdot v_R\| \, \|q_M\|} \geq \inf_{q_M \in Q_M} \sup_{v_R \in X_R} \frac{(\nabla \cdot v_R, q_M)}{\|\nabla v_R\| \, \|q_M\|} \geq \beta_{is} .$$

□

- Small α is good for AC-ROM convergence, bad for saddle point problem.

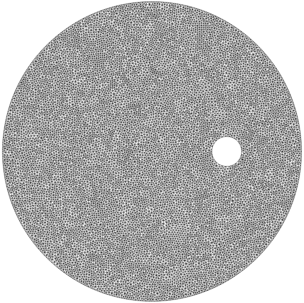
Numerical Experiments

We examine the two-dimensional flow between two offset circles. The domain is given by

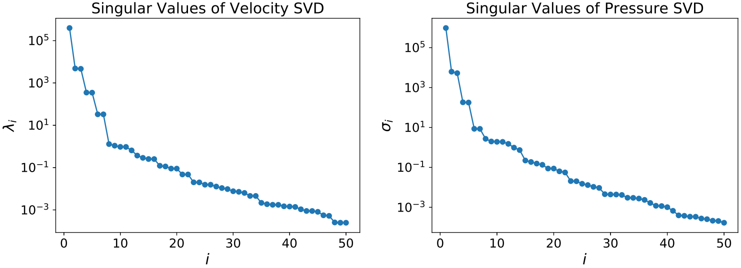
$$\Omega = \left\{ (x, y) : x^2 + y^2 \leq 1 \text{ and } \left(x - \frac{1}{2} \right)^2 + y^2 \geq \frac{1}{100} \right\},$$

and is driven by the body force

$$f(x, y) = \big(-4y(1 - x^2 - y^2)\,,\, 4x(1 - x^2 - y^2) \big).$$



- All computations done using FEniCS.
- Discretize in space via the P^2-P^1 Taylor-Hood element pair with 114,224 velocity and 14,421 pressure degrees of freedom.
- Let $\nu = \frac{1}{100}$, $u_0 = (0,0)$, $p_0 = 0$ and $u = (0,0)$ on $\partial\Omega$.
- Using a BE-AC scheme velocity/pressure snapshots are taken every $\Delta t = 2.5e-4$ seconds from $T = 12$ to $T = 16$.



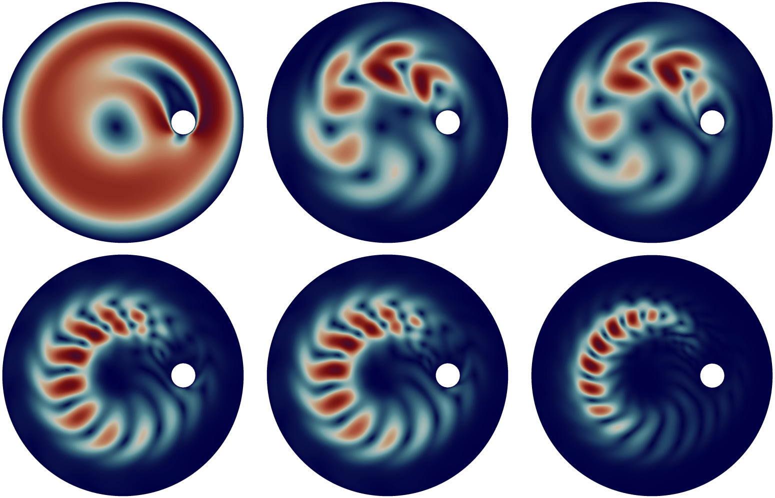


Figure: $|u_R(x)|$ with R from 1 (top left) to 6 (bottom right).

- For the online stage we compute using an equal number of pressure and velocity modes $N_v = N_p = 3, 5, 7$.
- Show Video

Numerical Experiments

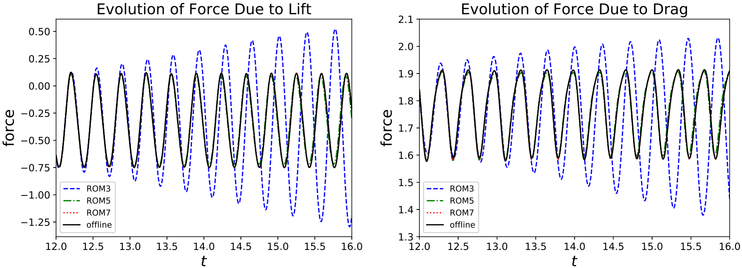


Figure: Evolution of lift (left) and drag (right) for $3, 5$ and 7 velocity/pressure basis functions compared to the benchmark.

Numerical Experiments

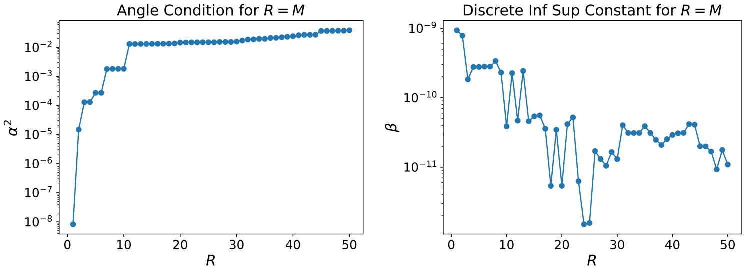


Figure: Prinicpal angle values (left) and inf-sup constant (right) for $N_v = N_p$ with varying R.

Numerical Experiments

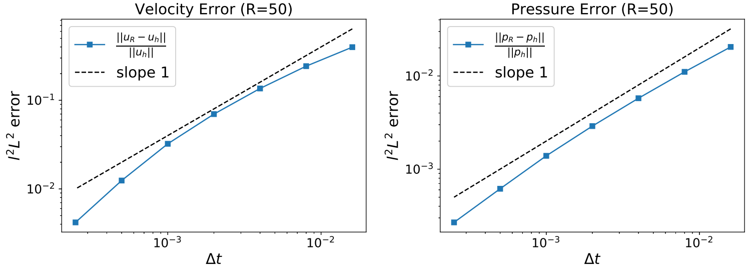


Figure: Convergence study of the pressure and velocity errors in time with $N_v = N_p = 50$ basis functions.

Conclusion

- AC-ROM scheme decouples pressure and velocity.
- Does not require the fulfillment of the inf-sup/LBB condition.
- Does not require weakly divergence-free snapshots.