Industrial Mathematics Institute

2000:14

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IMI

Preprint Series

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Abstract

We prove here that if an algebraic polynomial f of degree at most n has smaller absolute values than T_n (the n-th Chebyshev polynomial of the first kind) at arbitrary $n+1$ points in $[-1,1]$, which interlace with the zeros of T_n, then the uniform norm of f' is smaller than n^2. This is an extension of a classical result obtained by Duffin and Schaeffer.

1 Introduction and statement of the result

Denote by π_n the class of algebraic polynomials of degree at most n, and by $\|\cdot\|$ the supremum norm in $[-1,1]$. The classical inequality of brothers Markov [5], [6] asserts that among all $f \in \pi_n$ satisfying

$$\|f\| \le 1 \tag{1}$$

the Chebyshev polynomial of the first kind $T_n(x) = \cos n \arccos x$ has the greatest norm of its kth derivative ($k = 1, \dots, n$). A remarkable extension of this result was found by Duffin and Schaeffer [3], who showed that this extremal property of T_n persists under a weaker assumption than (1). Namely, they showed that T_n still has the largest uniform norm of its k-th derivative in the wider class of polynomials from π_n, satisfying

$$|f(\cos(\nu\pi/n))| \le 1, \ \nu = 0, \dots, n \tag{2}$$

(actually, Duffin and Schaeffer proved a more general result, including an inequality over a strip in the complex plane, but this does not fall in the frame of the present paper). The points

$$\eta_\nu := \cos(\nu\pi/n), \ \nu = 0, \dots, n$$

are the local extremum points for T_n in $[-1,1]$, and $|T_n(\eta_\nu)| = 1$. Thus, the result of Duffin and Schaeffer may be viewed as a comparison type theorem: the inequality $|f| \le |T_n|$ at the points of local extrema for T_n induces the inequalities $\|f^{(k)}\| \le \|T_n^{(k)}\|$ for $k = 1, \dots, n$. This suggests the following

*This research was supported by ONR/ARO Research Contract DEPSCoR – DAAG55–99–1–0002

Definition. A polynomial $Q \in \pi_n$ and a mesh $\Delta = \{t_\nu\}_{\nu=0}^n$, $(1 \geq t_0 > t_1 > \ldots > t_n \geq -1)$ are said to admit Duffin and Schaeffer type inequality (DS-inequality), if for every $f \in \pi_n$ the assumption $|f(t_\nu)| \leq |Q(t_\nu)|$ for $\nu = 0, \ldots, n$ implies $\|f'\| \leq \|Q'\|$, or, more generally, $\|f^{(k)}\| \leq \|Q^{(k)}\|$ for $k = 1, \ldots, n$.

Note that in our definition the comparison points $\{t_\nu\}_{\nu=0}^n$ are not necessarily assumed to be extremum points for Q.

In 1992 A. Shadrin [13] proposed a simple proof of Markov inequality under the assumptions (2). Based on a theorem of Shadrin, Bojanov and Nikolov [2] proved a DS-inequality for $Q = P_n^{(\lambda)}$ the ultraspherical polynomials, when the mesh Δ consists of the local extremum points of $P_n^{(\lambda)}$.

Theorem A. *Let* $Q := P_n^{(\lambda)}$ $(\lambda > -1/2)$ *and* $\{t_\nu\}_{\nu=0}^n$ *be the zeros of* $(1-x^2)Q'(x)$. *If* $f \in \pi_n$ *satisfies*

$$|f(t_\nu)| \leq |Q(t_\nu)| \text{ for } \nu = 0, \ldots, n,$$

then

$$\|f^{(k)}\| \leq \|Q^{(k)}\|$$

for all $k \in \{1, \ldots, n\}$, *if* $\lambda \geq 0$, *and for* $k \geq 2$, *if* $\lambda \in (-1/2, 0)$. *Equality is possible if and only if* $f = cQ$ *with* $|c| = 1$.

The special case $\lambda = 0$ comes down to the classical inequality of Duffin and Schaeffer.

For some other DS-inequalities, we refer the reader to [4], [7], [8], [9], [10], [11]. In particular, the following result has been proved in [9]:

Theorem B. *If* $f \in \pi_n$ *satisfies* $|f(\pm 1)| \leq 1$ *and*

$$|f(x)| \leq \sqrt{1-x^2} \quad \text{at the zeros of } T_{n-1},$$

then

$$\|f^{(k)}\| \leq \|T_n^{(k)}\| \quad \text{for } k = 1, \ldots, n.$$

Moreover, equality is possible if and only $f = cT_n$ *with* $|c| = 1$.

Theorems A and B show that for $Q = T_n$ DS-inequality holds at least for two choices of "check points", namely, for those formed by the zeros of $(1-x^2)T_n'(x)$ and by the zeros of $(1-x^2)T_{n-1}(x)$. We naturally come to the question: What are the meshes Δ admitting DS-inequality with $Q = T_n$? The aim of this paper is to show that for $k = 1$ each mesh $\Delta = \{t_\nu\}_{\nu=0}^n$ whose points interlace with the zeros of T_n is admissible.

Theorem 1. *Let* $\{t_\nu\}_{\nu=0}^n$ *satisfy* $1 \geq t_0 > \xi_1 > t_1 > \ldots > \xi_n > t_n \geq -1$, *where* $\{\xi_\nu\}_{\nu=1}^n$ *are the zeros of* T_n, *i.e.*, $\xi_\nu = \cos((2\nu-1)\pi/(2n))$. *If* $f \in \pi_n$ *and*

$$|f(t_\nu)| \leq |T_n(t_\nu)| \quad \text{for} \quad \nu = 0, \ldots, n,$$

then

$$\|f'\| \leq n^2. \tag{3}$$

Moreover, equality in (3) is possible if and only if $f = cT_n$ *with* $|c| = 1$.

Note that the set of all admissible meshes Δ (i.e., such that DS-inequality holds with $Q = T_n$) is not substantially larger than the one described in Theorem 1. In fact, the points of any admissible mesh must separate the zeros of T_n (see Section 4).

The proof of Theorem 1 relies on a pointwise inequality given by the next theorem, which was suggested to the author by A. Shadrin [15].

Theorem 2. *Let $Q \in \pi_n$ have n distinct zeros $\{x_\nu\}_{\nu=1}^n$, all located in $(-1,1)$. Let $\{t_j\}_{j=0}^n$ satisfy $1 \ge t_0 > x_1 > t_1 > \ldots > x_n > t_n \ge -1$. If $f \in \pi_n$ and*

$$|f(t_j)| \le |Q(t_j)| \quad \text{for } j = 0, \ldots, n,$$

then for each $k \in \{1, \ldots, n\}$ and for every $x \in [-1,1]$ there holds

$$|f^{(k)}(x)| \le \max\{|Q^{(k)}(x)|, |Q_\nu^{(k)}(x)|,\ \nu = 1, \ldots, n\},$$

where

$$Q_\nu(x) = Q(x)\frac{1 - x_\nu x}{x - x_\nu}.$$

The paper is organized as follows. In Section 2, we summarize some results from V. Markov's paper and prove Theorem 2. The proof of Theorem 1 is given in Section 3. Section 4 contains some concluding remarks and points out to a possible application of Theorem 1 to the estimation of the round-off error in the Lagrange differentiation formula.

2 Proof of Theorem 2

We start with an observation from the original work of V. Markov [6], concerning polynomial interpolation and pointwise estimates for polynomial derivatives. We formulate it in two lemmas.

Definition. Let $p \in \pi_n$ or $p \in \pi_{n+1}$, $q \in \pi_n$, and p, q have only real and simple zeros, say $\{t_j\}_{j=1}^{n(+1)}$ and $\{\tau_j\}_{j=1}^n$. The zeros of p and q are said to interlace, if

$$t_1 \le \tau_1 \le t_2 \le \ldots \le t_{n-1} \le \tau_n (\le t_{n+1}).$$

If only strict inequalities appear above, then the zeros of p and q are said to interlace strictly.

The first Markov's lemma reveals a simple (and, as a matter of fact, very useful) property of the zeros of algebraic polynomials.

Lemma 1. *Let p and q be algebraic polynomials ($p \not\equiv q$), which have only real and simple zeros. If the zeros of p and q interlace, then the zeros of p' and q' interlace strictly.*

A proof of Lemma 1 can be found in [12, Lemma 2.7.1], or in [13]. Note that for polynomials of the same degree the claim of Lemma 1 can be viewed as a monotone dependence of the zeros of the derivative with respect the zeros of the polynomial ([1, p. 39].

Given a mesh $\Delta = \{t_j\}_{j=0}^n$ $(1 \geq t_0 > t_1 > \ldots > t_n \geq -1)$, and $\epsilon := \{\epsilon_j\}_{j=0}^n$ $(\epsilon_j > 0,\ j = 0, \ldots, n)$, we define the set of polynomials

$$\Omega_n(\Delta, \epsilon) := \{f \in \pi_n : |f(t_j)| \leq \epsilon_j,\ j = 0, \ldots, n\}.$$

Clearly, $\Omega_n(\Delta, \epsilon)$ is a compact set.

Define real valued polynomials $\{P_\nu\}_{\nu=0}^n = \{P_\nu(\Delta, \epsilon; \cdot)\}_{\nu=0}^n \in \Omega_n(\Delta, \epsilon)$ by

$$|P_\nu(t_j)| = \epsilon_j \quad \text{for } j, \nu = 0, \ldots, n,$$

$$P_0(t_{j-1})P_0(t_j) < 0 \text{ for } j = 1, \ldots, n,$$

and, for each $\nu = 1, \ldots, n$,

$$P_\nu(t_{\nu-1})P_\nu(t_\nu) > 0,\ \ P_\nu(t_{j-1})P_\nu(t_j) < 0 \text{ for } j \neq \nu.$$

Evidently, the above conditions determine $\{P_\nu\}_{\nu=0}^n$ uniquely up to a multiplier -1. Theorem 2 follows easily from the next lemma.

Lemma 2. *For each* $x \in [-1, 1]$ *and for every* $k \in \{1, \ldots, n\}$,

$$\sup\{|f^{(k)}(x)| : f \in \Omega_n(\Delta, \epsilon)\} = \max\{|P_\nu^{(k)}(x)|, \nu = 0, \ldots, n\}.$$

Proof. Note first that the sup is attainable since $\Omega_n(\Delta, \epsilon)$ is a compact. Set $\omega(t) := (t - t_0) \ldots (t - t_n)$, $\omega_\nu(t) := \omega(t)/(t - t_\nu)$ $(\nu = 0, \ldots, n)$, then for $f \in \Omega_n(\Delta, \epsilon)$ and a fixed $x \in [-1, 1]$ the Lagrange interpolation formula yields

$$|f^{(k)}(x)| = |\sum_{j=0}^n \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} f(t_j)| \leq \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| \epsilon_j. \tag{4}$$

The upper bound is attained if $|f(t_j)| = \epsilon_j$ for $j = 0, \ldots, n$ and f has a suitable sign pattern at the points $\{t_j\}$. Next, we show that the polynomials $\{P_\nu\}_{\nu=0}^n$ provide a complete set of appropriate sign patterns. For any pair of indices $i, j \in \{0, \ldots, n\},\ i < j$ the zeros of ω_i and ω_j interlace (though not strictly), therefore, in view of Lemma 1, the zeros $\{\gamma_{i,\mu}\}_{\mu=1}^{n-k}$ of $\omega_i^{(k)}$ and the zeros $\{\gamma_{j,\mu}\}_{\mu=1}^{n-k}$ of $\omega_j^{(k)}$ interlace strictly. Furthermore, since the zeros of ω_i are less than or equal to the corresponding zeros of ω_j, we have the following arrangement:

$$-1 < \gamma_{0,n-k} < \ldots < \gamma_{n,n-k} < \gamma_{0,n-k-1} < \ldots < \gamma_{n,n-k-1} < \ldots < \gamma_{0,1} < \ldots < \gamma_{n,1} < 1.$$

Since $\omega_{j-1}(t_{j-1})\omega_j(t_j) < 0$ for $j = 1, \ldots, n$, the above inequalities show that for $x \in [-1,1] \setminus \{\gamma_{\nu,j}\}_{\nu=0,j=1}^{n\ \ n-k}$, the quantities $\{\omega_j^{(k)}(x)/\omega_j(t_j)\}_{j=0}^n$ either change their signs alternatively, if

$$x \in I_{n,k}^0, \quad I_{n,k}^0 = I_{n,k}^0(\Delta) := [-1, \gamma_{0,n-k}) \cup_{j=n-k}^1 (\gamma_{n,j}, \gamma_{0,j-1}) \cup (\gamma_{n,1}, 1],$$

or change signs alternatively with only one exception $\dfrac{\omega_{\nu-1}^{(k)}(x)}{\omega_{\nu-1}(t_{\nu-1})} \dfrac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)} > 0$ for some $\nu \in \{1, \ldots, n\}$. The latter situation occurs when $x \in I_{n,k}^\nu$, where

$$I_{n,k}^\nu = I_{n,k}^\nu(\Delta) := \cup_{j=1}^{n-k} (\gamma_{\nu-1,j}, \gamma_{\nu,j}).$$

Correspondingly, if $x \in I_{n,k}^\nu$ for some $\nu \in \{0, \ldots, n\}$, then (4) holds with equality sign for $f = P_\nu$. If $x = \gamma_{\nu,j}$, then $\omega_\nu^{(k)}(x) = 0$, and equality in (4) holds for $f = P_\nu$ as well as for any $f \in \pi_n$ which coincides with P_ν at the points $\{t_j : j \neq \nu\}$.

Thus, in (4) equality holds for $f = P_\nu$, if $x \in \overline{I_{n,k}^\nu}$ $(\nu = 0, \ldots, n)$, and since $\cup_{\nu=0}^n \overline{I_{n,k}^\nu} = [-1,1]$, the proof of Lemma 2 is completed. ∎

Remark 1. It follows from the proof of Lemma 2 that if for some $f \in \Omega_n(\Delta, \epsilon)$ we have equality in (4) for some $x \in I_{n,k}^\nu$ $(\nu \in \{0, \ldots, n\})$, then necessarily $f = cP_\nu$, where c is a constant with $|c| = 1$. Thus, for $x \in [-1,1] \setminus \{\gamma_{\nu,j}\}_{\nu=0,j=1}^{n\ \ n-k}$, any extremal polynomial in Lemma 2 is of the form $f = cP_\nu$, where $\nu \in \{0, \ldots, n\}$ and $|c| = 1$.

Proof of Theorem 2. Set $\epsilon_j := |Q(t_j)|$, $j = 0, \ldots, n$, and define polynomials $\{P_\nu\}_{\nu=0}^n$ as above. Based on the interlacing assumption, we conclude that $P_0 = Q$ or $P_0 = -Q$, while for $\nu = 1, \ldots, n$ the sign patterns of P_ν and Q_ν coincide. Moreover, we have

$$|Q_\nu(t_j)| = \epsilon_j \frac{1 - x_\nu t_j}{|t_j - x_\nu|} \geq \epsilon_j \text{ for } j = 0, \ldots, n \text{ and } \nu = 1, \ldots, n.$$

In the proof of Lemma 2, we deduced that for any $f \in \Omega_n(\Delta, \epsilon)$

$$|f^{(k)}(x)| \leq |P_\nu^{(k)}(x)| \text{ if } x \in \overline{I_{n,k}^\nu},\ \nu = 0, \ldots, n. \tag{5}$$

For $\nu = 0$ (5) reads as $|f^{(k)}(x)| \leq |Q^{(k)}(x)|$, while for $x \in \overline{I_{n,k}^\nu}$ $(\nu \in \{1, \ldots, n\})$ we have

$$|P_\nu^{(k)}(x)| = \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| \epsilon_j \leq \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| |Q(t_j)| = |Q_\nu^{(k)}(x)|$$

(for the last equality we used that P_ν and Q_ν have the same sign pattern). The claim of Theorem 2 now follows from Lemma 2. ∎

As an immediate consequence of Theorem 2 we get

Corollary 1. *If, in addition to the assumptions of Theorem 2, for an $k \in \{1, \ldots, n\}$*

$$\max_{1 \leq \nu \leq n} \|Q_\nu^{(k)}\| \leq \|Q^{(k)}\|,$$

then

$$\|f^{(k)}\| \leq \|Q^{(k)}\|.$$

3 Proof of Theorem 1

The proof of Theorem 1 follows from Corollary 1, applied to $Q = T_n$ and $k = 1$. The application of Corollary 1 is possible because of the following lemma:

Lemma 3. *Let the polynomials $\{P_\nu\}_{\nu=1}^n$ be defined by*

$$P_\nu(x) := T_n(x)\frac{1-\xi_\nu x}{x-\xi_\nu}.$$

Then, for $n \geq 2$,

$$\|P_\nu'\| < n^2 \quad (\nu = 1, \ldots, n). \tag{6}$$

For $n = 2, 3$ the validity of (6) is verified directly, therefore we assume in what follows $n \geq 4$. The proof of Lemma 3 goes through a number of lemmas.

Lemma 4. *For every $x \in [-1, 1]$ and for $\nu = 1, \ldots, n$*

$$|P_\nu'(x)| \leq R_\nu(x),$$

where

$$R_\nu(x) = \left[\frac{(1-\xi_\nu^2)^2}{(x-\xi_\nu)^4} + \frac{n^2(1-\xi_\nu x)^2}{(1-x^2)(x-\xi_\nu)^2}\right]^{1/2}$$

Proof. The result is immediate from

$$P_\nu'(x) = T_n'(x)\frac{1-\xi_\nu x}{x-\xi_\nu} - T_n(x)\frac{1-\xi_\nu^2}{(x-\xi_\nu)^2}, \tag{7}$$

the identity $[T_n(x)]^2 + (1-x^2)[T_n'(x)]^2/n^2 = 1$, and Cauchy's inequality. ∎

Lemma 5. *$R_\nu(x)$ is a strictly convex function on each of the intervals $(-1, \xi_\nu)$ and $(\xi_\nu, 1)$.*

Proof. We suppress the index ν, writing

$$R(x) = \left[\frac{(1-\xi^2)^2}{(x-\xi)^4} + \frac{n^2(1-\xi x)^2}{(1-x^2)(x-\xi)^2}\right]^{1/2} =: (g_1^2(x) + g_2^2(x))^{1/2},$$

where

$$g_1(x) := \frac{1-\xi^2}{(x-\xi)^2}, \ g_2(x) := \frac{n(1-\xi x)}{(1-x^2)^{1/2}(x-\xi)}.$$

Since

$$R'' = \frac{(g_1 g_2' - g_1' g_2)^2 + R^2(g_1 g_1'' + g_2 g_2'')}{R^3},$$

the lemma will be proved if we show that $g_1(x)g_1''(x)$ and $g_2(x)g_2''(x)$ are positive in $(-1,\xi)$ and in $(\xi,1)$. This is easily seen for the first term, while for the second term a short calculation yields

$$\begin{aligned}&\frac{(x-\xi)^4(1-x^2)^3}{n^2}g_2(x)g_2''(x)\\&\quad=2(1-\xi^2)(1-x^2)^2-2x(x-\xi)(1-\xi^2)(1-x^2)+(1-\xi x)(x-\xi)^2(2x^2+1).\end{aligned}$$

The positivity of the right hand side is easily verified with the help of the inequality

$$2(1-\xi^2)(1-x^2)^2+(1-\xi x)(x-\xi)^2(2x^2+1)\geq 2(1-x^2)|x-\xi|[2(1-\xi^2)(1-\xi x)(2x^2+1)]^{1/2}.$$

∎

We now examine the polynomials $\{P_\nu\}_{\nu=1}^n$. Due to symmetry, we may (and shall) consider only half of them, say, those with indices $1\leq\nu\leq[(n+1)/2]$. Recall that the zeros of P_ν coincide with the zeros $\{\xi_j\}_{j=1}^n$ of T_n with the exception of ξ_ν which is replaced by $1/\xi_\nu$ (in the case n odd and $\nu=(n+1)/2$, $1/\xi_\nu$ is interpreted as a zero at ∞). With this last convention, we observe that for $1\leq\nu\leq[(n+1)/2]$ the zeros of P_ν are located to the right with respect to the zeros $\{\xi_i\}$ of T_n, and interlace with them. In view of Lemma 1, the same relation holds between the zeros of the derivatives of P_ν and T_n. We are interested in the behavior of $P_\nu'(x)$, in particular, its critical points. To this end, we shall exploit (7) and the explicit form of P_ν'',

$$P_\nu''(x)=T_n''(x)\frac{1-\xi_\nu x}{x-\xi_\nu}-2T_n'(x)\frac{1-\xi_\nu^2}{(x-\xi_\nu)^2}+2T_n(x)\frac{1-\xi_\nu^2}{(x-\xi_\nu)^3}. \tag{8}$$

In the proof of the next lemmas we shall use the differential equation

$$(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0, \tag{9}$$

as well as the following simple facts:

$$\{n\sin(\alpha\pi/n)\}_{n=1}^\infty\nearrow\alpha\pi, \tag{10}$$

$$\cot\alpha\leq\frac{1}{\alpha}, \tag{11}$$

where $0<\alpha\leq\pi/2$.

Lemma 6. *The polynomials P_ν' $(\nu=1,\ldots,[(n+1)/2])$ satisfy the following:*
(i) If $2\leq\nu<\frac{n+1}{2}$, then P_ν' has exactly one local extremum to the right of 1;
(ii) P_ν' has exactly one local extremum in $(\xi_{\nu+1},\eta_\nu)$;
(iii) P_ν' is strictly monotone in $[\eta_\nu,\eta_{\nu-1}]$;
(iv) P_ν' is strictly monotone in $[-1,\eta_{n-1}]$ and in $[\eta_1,1]$.

Proof. The first claim in (iv) follows trivially, since, as was already mentioned, the zeros of P_ν are located to the right with respect to $\{\xi_j\}_{j=1}^n$. In view of Lemma 1, the same is true for the zeros of P''_ν and T''_n. Since the leftmost zero of T''_n is located to the right of η_{n-1}, so is the smallest zero of P''_ν.

Substituting $x = 1$ in (8) we get

$$P''_\nu(1) = \frac{n^2(n^2-1)}{3} - 2n^2\cot^2\frac{(2\nu-1)\pi}{4n} + \frac{\cot^2\frac{(2\nu-1)\pi}{4n}}{\sin^2\frac{(2\nu-1)\pi}{4n}}.$$

With the help of (10) and (for $\nu = 2$) (11), it is easy to see that $P''_\nu(1) > 0$ for $2 \le \nu \le [(n+1)/2]$. Since P'_ν has a negative leading coefficient and at most one critical points to the right of $x = 1$, this proves part (i) of the lemma.

Now we find the sign of P''_ν at the points $\xi_{\nu+1},\ \eta_\nu$, and $\eta_{\nu-1}$. First, we shall show that

$$\operatorname{sign}\{P''_\nu(\xi_{\nu+1})\} = (-1)^{\nu+1}. \tag{12}$$

Putting $x = \xi_{\nu+1}$ in (8) and using that $T''_n(\xi_{\nu+1}) = \xi_{\nu+1}T'_n(\xi_{\nu+1})/(1-\xi_{\nu+1}^2)$ and $\operatorname{sign}\{T'_n(\xi_{\nu+1})\} = (-1)^\nu$, we get

$$\operatorname{sign}\{P''_\nu(\xi_{\nu+1})\} = (-1)^{\nu+1}\operatorname{sign}\{2(1-\xi_\nu^2)(1-\xi_{\nu+1}^2) + \xi_{\nu+1}(\xi_\nu - \xi_{\nu+1})(1-\xi_\nu\xi_{\nu+1})\}.$$

Now (12) is obvious if $\xi_{\nu+1} \ge 0$. The only possibility where $\xi_{\nu+1} < 0$ is $\nu = m$ and $n = 2m$ or $n = 2m-1$. An easy calculation shows that for $n \ge 4$ (12) is true in this case, too.

Next, we prove both (ii) and (iii) by showing that

$$\operatorname{sign}\{P''_\nu(\eta_\mu)\} = (-1)^\nu \ \text{ for } \mu = \nu, \nu-1,\ \mu \ne 0. \tag{13}$$

Using (8) and (9), we obtain

$$P''_\nu(\eta_\mu) = \frac{T_n(\eta_\mu)}{(\xi_\nu - \eta_\mu)^3(1-\eta_\mu^2)}[n^2(1-\xi_\nu\eta_\mu)(\xi_\nu-\eta_\mu)^2 - 2(1-\xi_\nu^2)(1-\eta_\mu^2)]. \tag{14}$$

Since $\operatorname{sign}\{T_n(\eta_\mu)\} = (-1)^\mu$, it suffices to prove that the term in the square brackets is positive. Using the inequality $(1-\xi_\nu^2)(1-\eta_\mu^2) < (1-\xi_\nu\eta_\mu)^2$ we obtain

$$n^2(1-\xi_\nu\eta_\mu)(\xi_\nu-\eta_\mu)^2 - 2(1-\xi_\nu^2)(1-\eta_\mu^2) > (1-\xi_\nu\eta_\mu)[n^2(\xi_\nu-\eta_\mu)^2 - 2(1-\xi_\nu\eta_\mu)].$$

After simple manipulations, using the trigonometric representation of ξ_ν and η_μ we find that the inequality $n^2(\xi_\nu-\eta_\mu)^2 - 2(1-\xi_\nu\eta_\mu) \ge 0$ is equivalent to

$$\frac{1}{n^2\sin^2\frac{\pi}{4n}} + \frac{1}{n^2\sin^2\frac{(2\nu+2\mu-1)\pi}{4n}} \le 2.$$

This last inequality will hold for all $\nu \in \{1,\dots,[(n+1)/2]\}$ and $\mu = \nu, \nu-1, (\mu \ne 0)$, if it is true for $\nu = \mu = 1$, i.e., if

$$\frac{1}{n^2\sin^2\frac{\pi}{4n}} + \frac{1}{n^2\sin^2\frac{3\pi}{4n}} \le 2.$$

Since the left hand side is a decreasing function of n (see (10)), and for $n = 3$ it is $(\sin^{-2}(\pi/12) + 2)/9 = (4\sqrt{3} + 10)/9 < 2$, (13) is proved. Now we conclude from (12) and (13) (with $\mu = \nu$) that P_ν'' has a zero in $(\xi_{\nu+1}, \eta_\nu)$ for $\nu = 1, \ldots, [(n+1)/2]$. In addition, (13) implies that this zero is unique, and no zeros of P_ν'' exist in $[\eta_\nu, \eta_{\nu-1}]$ $(\nu \geq 2)$, otherwise there would be at least three zeros in $(\xi_{\nu+1}, \xi_{\nu-1})$, a contradiction. For the same reason, P_1'' has a simple zero in (ξ_2, η_1), and no zeros of P_1'' exist in $[\eta_1, 1]$. This is exactly the claim of (iii) for $\nu = 1$ and of the second part of (iv) for $\nu = 1$.

To prove the second part of (iv) for $2 \leq \nu < (n+1)/2$, we shall show that

$$P_\nu''(\eta_1) > 0. \tag{15}$$

Having established (15), the second part of (iv) will follow immediately. Indeed, we found in the beginning of this proof that $P_\nu''(1) > 0$ for $2 \leq \nu < (n+1)/2$, and if P_ν' was not monotone in $[\eta_1, 1]$, then P_ν'' would have at least three zeros (two zeros, if $\nu = (n+1)/2$) to the right of η_1, which is impossible. The proof of (15) goes along the lines of the proof of (13). Equation (14) with $\mu = 1$ shows that (15) follows if

$$n^2(1 - \xi_\nu\eta_1)(\xi_\nu - \eta_1)^2 - 2(1 - \xi_\nu^2)(1 - \eta_1^2) > 0,$$

or, in view of $(1 - \xi_\nu^2)(1 - \eta_1^2) \leq (1 - \xi_\nu\eta_1)^2$, if

$$n^2(\xi_\nu - \eta_1)^2 - 2(1 - \xi_\nu\eta_1) > 0.$$

The latter inequality is equivalent to the inequality

$$\frac{1}{n^2 \sin^2 \frac{(2\nu-3)\pi}{4n}} + \frac{1}{n^2 \sin^2 \frac{(2\nu+1)\pi}{4n}} \leq 2,$$

whose validity is easily verified with the help of (10). Lemma 6 is proved. ∎

Lemma 7. *The following estimates for $\|P_\nu'\|$ hold true:*

(i) *For $\nu = 1, 2$,*

$$\|P_\nu'\| \leq \max\{|P_\nu'(-1)|, |P_\nu'(1)|, R_\nu(\eta_{n-1}), R_\nu(\eta_\nu)\};$$

(ii) *For $\nu = 3, \ldots, [(n+1)/2]$,*

$$\|P_\nu'\| \leq \max\{|P_\nu'(-1)|, |P_\nu'(1)|, R_\nu(\eta_{n-1}), R_\nu(\eta_\nu), R_\nu(\eta_{\nu-1}), R_\nu(\eta_1)\}.$$

Proof. According to Lemma 6, P_1' is monotone in $[-1, \eta_{n-1}]$ and $[\eta_1, 1]$, therefore on these intervals

$$|P_1'(x)| \leq \max\{|P_1'(-1)|, |P_1'(\eta_{n-1})|, |P_1'(\eta_1)|, |P_1'(1)|\}.$$

On the complementary interval $[\eta_{n-1}, \eta_1]$, we have $|P_1'(x)| \le R_1(x)$ (Lemma 4), and since R_1 is convex there (Lemma 5), it follows that $R_1(x) \le \max\{R_1(\eta_{n-1}), R_1(\eta_1)\}$ for $x \in [\eta_{n-1}, \eta_1]$. This proves (i) for $\nu = 1$.

The proof of (i) for $\nu = 2$ relies on the observation that, by Lemma 6, P_2' is monotone in $[-1, \eta_{n-1}]$ and $[\eta_2, 1]$, while $|P_2'(x)| \le \max\{R_2(\eta_{n-1}), R_2(\eta_2)\}$ in $[\eta_{n-1}, \eta_2]$, by virtue of Lemmas 4 and 5.

Part (ii) can be proved in the same fashion, exploiting the monotonicity of P_ν' on the intervals $[-1, \eta_{n-1}]$, $[\eta_\nu, \eta_{\nu-1}]$ and $[\eta_1, 1]$, and the convexity of R_ν on $[\eta_{n-1}, \eta_\nu]$ and $[\eta_{\nu-1}, \eta_1]$. We leave the details to the reader. ∎

Our last lemma estimates the quantities appearing in Lemma 7.

Lemma 8. *The following inequalities hold true:*

(i) $|P_\nu'(\pm 1)| < n^2$ $(\nu = 1, \dots, [(n+1)/2])$;

(ii) $R_\nu(\eta_1) < n^2$ $(\nu = 1, 3, 4, \dots, [(n+1)/2])$;

(iii) $R_\nu(\eta_\nu) < n^2$ $(\nu = 1, \dots, [(n+1)/2])$;

(iv) $R_\nu(\eta_{\nu-1}) < n^2$ $(\nu = 3, \dots, [(n+1)/2])$;

(v) $R_\nu(\eta_{n-1}) < n^2$ $(\nu = 1, \dots, [(n+1)/2])$.

Proof. Substituting $x = \pm 1$ in (7) we get

$$P_\nu'(1) = n^2 - \cot^2 \frac{(2\nu-1)\pi}{4n}, \quad |P_\nu'(-1)| = n^2 - \tan^2 \frac{(2\nu-1)\pi}{4n}.$$

Then (10) and $0 < (2\nu - 1)\pi/(2n) \le \pi/4$ show the validity of a slightly sharper inequalities than (i), namely

$$n^2 - 1 \le |P_\nu'(-1)| < n^2 - \frac{\pi}{4n}$$

and

$$(1 - 16/\pi^2)n^2 < P_\nu'(1) < n^2 - 1.$$

Now, we prove (ii). A short calculation yields

$$R_\nu(\eta_1) = \left[\frac{(1-\xi_\nu^2)^2}{(\eta_1 - \xi_\nu)^4} + \frac{n^2(1-\xi_\nu\eta_1)^2}{(1-\eta_1^2)(\eta_1-\xi_\nu)^2}\right]^{1/2} =: \{[A(\nu)]^2 + [B(\nu)]^4\}^{1/2}, \qquad (16)$$

where

$$A(\nu) = \frac{n}{2}\left|2\cot\frac{\pi}{n} + \cot\frac{(2\nu-3)\pi}{4n} - \cot\frac{(2\nu+1)\pi}{4n}\right|,$$

$$B(\nu) = \frac{1}{2}\left|\cot\frac{(2\nu-3)\pi}{4n} + \cot\frac{(2\nu+1)\pi}{4n}\right|.$$

Assume first that $3 \leq \nu \leq [(n+1)/2]$, then it is easy to see that $A(\nu) \leq A(3)$ and $B(\nu) \leq B(3)$. We use (11) to obtain

$$B(3) = \frac{1}{2}[\cot \frac{3\pi}{4n} + \cot \frac{7\pi}{4n}] < \frac{20n}{21\pi},$$

$$\begin{aligned} A(3) &= \frac{n}{2}[\cot \frac{3\pi}{4n} + 2\cot \frac{\pi}{n} - \cot \frac{7\pi}{4n}] \\ &< \frac{n}{2}[\cot \frac{3\pi}{4n} + 2\cot \frac{\pi}{n}] \\ &< \frac{5n^2}{3\pi}. \end{aligned}$$

Therefore, for $3 \leq \nu \leq [(n+1)/2]$,

$$R_\nu(\eta_1) < \left[\left(\frac{5n^2}{3\pi} \right)^2 + \left(\frac{20n}{21\pi} \right)^4 \right]^{1/2} < 0.54n^2 < n^2.$$

Similarly, for $\nu = 1$, we find

$$A(1) = \frac{n}{2}[\cot \frac{\pi}{4n} - 2\cot \frac{\pi}{n} + \cot \frac{3\pi}{4n}] < \frac{n}{2}[\cot \frac{\pi}{4n} + \cot \frac{3\pi}{4n}] < \frac{8n^2}{3\pi},$$

$$B(1) = \frac{1}{2}[\cot \frac{\pi}{4n} - \cot \frac{3\pi}{4n}] < \frac{1}{2}\cot \frac{\pi}{4n} < \frac{2n}{\pi}.$$

Hence,

$$R_1(\eta_1) < \left[\left(\frac{8n^2}{3\pi} \right)^2 + \left(\frac{2n}{\pi} \right)^4 \right]^{1/2} < 0.95n^2 < n^2.$$

Thus, (ii) is proved.

Next, we prove (iii). For $1 \leq \nu \leq [(n+1)/2]$, we have

$$R_\nu(\eta_\nu) = \left[\frac{(1-\xi_\nu^2)^2}{(\xi_\nu - \eta_\nu)^4} + \frac{n^2(1-\xi_\nu\eta_\nu)^2}{(1-\eta_\nu^2)(\xi_\nu - \eta_\nu)^2} \right]^{1/2} =: \{[C(\nu)]^2 + [D(\nu)]^4\}^{1/2},$$

where

$$C(\nu) = \frac{n}{2}[\cot \frac{\pi}{4n} + \cot \frac{(4\nu-1)\pi}{4n} - 2\cot \frac{\nu\pi}{n}],$$

$$D(\nu) := \frac{1}{2}[\cot \frac{\pi}{4n} - \cot \frac{(4\nu-1)\pi}{4n}].$$

Unlike the situation with $A(\nu)$ and $B(\nu)$, we observe that $C(\nu)$ and $D(\nu)$ increase with ν, and for $n \geq 3$

$$D(\nu) \leq D((n+1)/2) = \frac{n}{n \sin \frac{\pi}{2n}} \leq \frac{2n}{3},$$

$$\begin{aligned}
C(\nu) \leq C((n+1)/2) &= \frac{n}{2}[\cot\frac{\pi}{4n} + 2\tan\frac{\pi}{2n} - \tan\frac{\pi}{4n}] \\
&= \frac{n}{\sin\frac{\pi}{2n}} + n[\tan\frac{\pi}{2n} - \tan\frac{\pi}{4n}] \\
&< \frac{n^2}{n\sin\frac{\pi}{2n}} + \frac{\pi}{4}\frac{1}{\cos^2\frac{\pi}{2n}} \\
&\leq \frac{1}{3}(2n^2+\pi).
\end{aligned}$$

With this (iii) is proved, since

$$R_\nu(\eta_\nu) < n^2\left[\left(\frac{2}{3}+\frac{\pi}{3n^2}\right)^2 + \left(\frac{2}{3}\right)^4\right]^{1/2} < 0.91n^2 < n^2.$$

The same arguments as above lead to the proof of (iv): $R_\nu(\eta_{\nu-1}) = [(\tilde{C}(\nu))^2 + (\tilde{D}(\nu))^4]^{1/2}$, where

$$\tilde{C}(\nu) = \frac{n}{2}[\cot\frac{\pi}{4n} + 2\cot\frac{(\nu-1)\pi}{n} - \cot\frac{(4\nu-3)\pi}{4n}],$$

$$\tilde{D}(\nu) = \frac{1}{2}[\cot\frac{\pi}{4n} + \cot\frac{(4\nu-3)\pi}{4n}].$$

Observing that $\tilde{C}(\nu)$ and $\tilde{D}(\nu)$ decrease with ν, for $3 \leq \nu \leq [(n+1)/2]$ we find the estimates

$$\tilde{D}(\nu) \leq \tilde{D}(3) = \frac{1}{2}[\cot\frac{\pi}{4n} + \cot\frac{9\pi}{4n}] < \frac{20n}{9\pi},$$

$$\begin{aligned}
\tilde{C}(\nu) \leq \tilde{C}(3) &= \frac{n}{2}[\cot\frac{\pi}{4n} + 2\cot\frac{2\pi}{n} - \cot\frac{9\pi}{4n}] \\
&< \frac{n}{2}[\cot\frac{\pi}{4n} + \cot\frac{7\pi}{4n}] \\
&< \frac{16n^2}{7\pi},
\end{aligned}$$

and hence

$$R_\nu(\eta_{\nu-1}) < \left[\left(\frac{16n^2}{7\pi}\right)^2 + \left(\frac{20n}{9\pi}\right)^4 + \right]^{1/2} < 0.89n^2 < n^2.$$

Finally, (v) can be proved in the same way as (i)–(iv). Alternatively, one can use the inequality

$$\frac{1-\xi\eta}{|\xi-\eta|} \geq \frac{1+\xi\eta}{\xi+\eta} \quad (0 \leq \xi, \eta < 1,\ \xi \neq \eta)$$

to compare pairwise $A(\nu)$ and $B(\nu)$ with the corresponding terms appearing in $R_\nu(\eta_{n-1}) = R_\nu(-\eta_1)$. The result is $R_\nu(\eta_{n-1}) \leq R_\nu(\eta_1) < n^2$. We omit the details. ∎

Proof of Lemma 3. The inequality follows from Lemmas 7 and 8. ∎

Proof of Theorem 1. Inequality (3) follows immediately from Corollary 1 and Lemma 3. It remains to clarify in which cases a equality is possible. Let $\Delta = \{t_j\}_{j=0}^n$ be a fixed mesh satisfying the assumptions of Theorem 1. Let $\epsilon = (\epsilon_0, \dots, \epsilon_n) =: (|T_n(t_0)|, \dots, |T_n(t_n)|)$, and the polynomials $P_0 = T_n,\ P_\nu\ (\nu = 1, \dots, n)$ be defined as in Section 2. Suppose that $f \in \Omega(\Delta, \epsilon)$ is an extremal polynomial, i.e., $\|f'\| = n^2$. According to Remark 1 and Lemma 3, for $x \in \cup_{\nu=1}^n \overline{I}_{n,1}^\nu$ there holds

$$|f'(x)| \le \max_{1 \le \nu \le n} \|P_\nu'\| < n^2,$$

therefore $\|f'\|$ is attained for $x \in I_{n,1}^0$. However, when $x \in I_{n,1}^0$ we have

$$|f'(x)| \le |P_0'(x)| = |T_n'(x)| \le T_n'(1) = n^2,$$

and equality holds only for $x = \pm 1$ and $f = cT_n$ with $|c| = 1$. Theorem 1 is proved. ∎

4 Concluding remarks

1. The requirement in Theorem 1 that the points $\Delta = \{t_j\}_{j=0}^n$ interlace strictly with the zeros of T_n was only imposed in order to avoid unimportant complications in the proof. Actually, Theorem 1 is valid under the weaker assumption that $\{t_j\}_{j=0}^n$ interlace with $\{\xi_j\}_{j=1}^n$. If a comparison point t_j coincides with a zero of T_n, then the polynomials from the corresponding class $\Omega_n(\Delta, \epsilon)$ must vanish at that point. In the case when all $\{\xi_\nu\}_{\nu=1}^n$ belong to Δ Theorem 1 holds trivially, since in that case $\Omega_n(\Delta, \epsilon) = \{cT_n(x) : |c| \le 1\}$.
2. So far, we cannot extend Theorem 1 to higher order derivatives, i.e., to prove $\|f^{(k)}\| \le \|T_n^{(k)}\|$ for all $k \ge 2$. However, it should be pointed out that this inequality holds true for $k = n-1$ and for $k = n$. This is easily seen from the proof of Lemma 2: for any polynomial $f \in \Omega_n(\Delta, \epsilon)$ and for $k = n-1, n$ we have $\|f^{(k)}\| = |f^{(k)}(-1)|$ or $\|f^{(k)}\| = |f^{(k)}(1)|$, and for $x = \pm 1$ the extremal polynomials in Lemma 2 are of the form $cP_0 = \pm cT_n$, $|c| = 1$.
3. According to Lemma 2, a necessary condition for a mesh $\Delta = \{t_j\}_{j=0}^n$ to admit DS-inequality with an extremal polynomial $Q = T_n$ is, the sign pattern of $(T_n(t_0), \dots, T_n(t_n))$ to coincide (up to a factor -1) with the sign pattern of some of the polynomials $\{P_\nu\}_{\nu=0}^n$. Theorem 1 asserts DS-inequality for all meshes Δ having the sign structure of P_0. One may think that DS-inequality also holds for any other mesh $\Delta = \{t_j\}_{j=0}^n$ for which the sign pattern of $(T_n(t_0), \dots, T_n(t_n))$ coincides with the sign pattern of some P_ν, $\nu \in \{1, \dots, n\}$. However, the example below shows that this is not true, in general.

Let $t_j = \eta_{j+1}$ for $j = 0, 1, \ldots, n-2$, $t_n = \eta_n$ and $t_{n-1} = \zeta$, where $\zeta \in (-1, \xi_n)$. Define polynomial

$$q(x) = \begin{cases} T_n(x) & \text{for } x = t_j, \quad j = 0, \ldots, n-2, n, \\ -T_n(x) & \text{for } x = t_{n-1}. \end{cases}$$

Clearly, q has the same sign structure as P_{n-1}, and $|q(t_j)| = |T_n(t_j)|$ $(j = 0, \ldots, n)$. The explicit form of q is

$$q(x) = T_n(x) + a(1+x)T_n'(x), \quad \text{where } a = -2T_n(\zeta)/((1+\zeta)T_n'(\zeta)) > 0,$$

and for $k = 1, \ldots, n$ we have

$$\|q^{(k)}\| \geq q^{(k)}(1) > T_n^{(k)}(1) = \|T_n^{(k)}\|.$$

4. As was mentioned in [8, p. 174], inequalities of DS-type may be viewed as exact estimates for the roundoff error in Lagrange differentiation formulas. We describe below briefly a possible application of the result of Theorem 1.

Let $\Delta = \{t_j\}_{j=0}^n$ be a mesh whose points interlace strictly with the zeros of T_n. Suppose that inaccurate data $\{\tilde{f}(t_j)\}_{j=0}^n$ for a function $f \in C^{n+1}[-1, 1]$ is given, where

$$|f(t_j) - \tilde{f}(t_j)| \leq \delta_j \quad (j = 0, \ldots, n).$$

If $f'(x) \approx L_n'(\tilde{f}; x)$ is the Lagrange differentiation formula based on this information, then for the error $R(f; x) := f'(x) - L_n'(\tilde{f}; x)$ there holds

$$R(f; x) = R^{round}(f; x) + R^{trunc}(f; x)$$

with $R^{round}(f; x) = L_n'(\tilde{f} - f; x)$ being the error caused by inaccuracy of the data and $R^{trunc}(f; x)$ the error caused by the fact that f is not necessarily a polynomial (truncation error). We have the estimate

$$\|R(f; \cdot)\| \leq \|R^{round}(f; \cdot)\| + \|R^{trunc}(f; \cdot)\|.$$

The exact bound for the truncation error in the Lagrange differentiation formula in the general case has been obtained by Shadrin [14] (in our case $\|R^{trunc}(f; \cdot)\| \leq \|f^{(n+1)}\| \|\omega'\|/(n+1)!$). For the roundoff error, Theorem 1 provides the following exact upper bound:

$$\|R^{round}(f; \cdot)\| \leq Mn^2, \quad \text{where } M = \max_{0 \leq j \leq n} \frac{\delta_j}{|T_n(t_j)|}.$$

This upper bound is attained when $\delta_j / |T_n(t_j)| = M$ for $j = 0, \ldots, n$.

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