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Symmetric framelets

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Abstract

We study *tight wavelet frames* associated with symmetric compactly supported refinable functions, which are obtained with *the unitary extension principle.* We give a criterion for existence of two symmetric or antisymmetric compactly supported framelets.

All refinable masks of length up to 6, satisfying this criterion, are found.

AMS subject classification: 42C15
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1 Introduction

The main goal of our paper is to present a criterion for existance of two symmetric or antisymmetric *framelets* generated by a symmetric refinable function. We consider only functions of one variable in the space $\mathbb{L}^2(\mathbb{R})$ with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

As usual we denote by $\hat{f}(\omega)$ Fourier transform of the function $f(x) \in \mathbb{L}^2(\mathbb{R})$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-ix\omega}\, dx.$$

Suppose a real-valued function $\varphi \in \mathbb{L}^2(\mathbb{R})$ satisfies the following conditions:
(a) $\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega)$, where m_0 is an essentially bounded 2π-periodic function;
(b) $\lim_{\omega\to 0} \hat{\varphi}(\omega) = (2\pi)^{-1/2}$;
then the function φ is called *refinable* or *scaling*, m_0 is called *a symbol* of φ, and the relation in item (a) is called *a refinement equation.*

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Every refinable function generates *multiresolution analysis* (MRA) of the space $\mathbb{L}^2(\mathbb{R})$, i.e., a nested sequence

$$\cdots \subset V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^j \subset \dots$$

of closed linear subspaces of $\mathbb{L}^2(\mathbb{R})$ such that
(a) $\cap_{j\in\mathbb{Z}} V^j = \varnothing$;
(b) $\overline{\cup_{j\in\mathbb{Z}} V^j} = \mathbb{L}^2(\mathbb{R})$;
(c) $f(x) \in V^j \Leftrightarrow f(2x) \in V^{j+1}$.
To obtain the MRA we just have to take as above V^j the closure of the linear span of the functions $\{\varphi(2^j x - n)\}_{n\in\mathbb{Z}}$. Fulfillment of item (a) and (b) for the obtained spaces V^j was proved in [1]. Property (c) is evident.

The most popular approach to the design of orthogonal and bi-orthogonal wavelets is based on construction of MRA of the space $\mathbb{L}^2(\mathbb{R})$, generated with a given refinable function. S.Mallat [6] showed that if the system $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ constitutes a Riesz basis of the space V^0, then there exists a refinable function $\phi \in V^0$ with a symbol m_ϕ such that the functions $\{\phi(x-n)\}_{n\in\mathbb{Z}}$ form an orthonormal basis of V^0. If we denote by W^j the orthogonal complement of the space V^j in the space V^{j+1}, then the function ψ (which is called *a wavelet*), defined by the relation

$$\hat{\psi}(2\omega) := m_\psi(\omega)\hat{\phi}(\omega),$$

where $m_\psi(\omega) = \overline{e^{i\omega} m_\phi(\omega+\pi)}$, generates orthonormal basis $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ of the space W^0. Thus, the system

$$\left\{2^{k/2}\psi(2^k x - n)\right\}_{n,k\in\mathbb{Z}} \tag{1}$$

constitutes an orthonormal basis of the space $\mathbb{L}^2(\mathbb{R})$.

We see that if we have a refinable function, generating a Riesz basis, then we have explicit formulae for the wavelets, associated with this functions. It gives a simple method for constructing wavelets. Generally speaking, any orthonormal basis of $\mathbb{L}^2(\mathbb{R})$ of the form (1) is called a wavelet system. However, wavelet construction based on a multiresolution structure has the advantage from the point of view effectiveness of computational algorithms, because it leads to the pyramidal scheme of wavelet decomposition and reconstruction (sf. [4]).

It is well-known that the problem of finding orthonormal wavelet bases, generated by a scaling function, can be reduced to solving the matrix equation

$$M(\omega)M^*(\omega) = I, \tag{2}$$

where

$$M(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) \end{pmatrix},$$

$m_0(\omega), m_1(\omega)$ are essentially bounded functions, and $m_0(-\omega) = \overline{m_0(\omega)}$, i.e., the Fourier series of these functions have real coefficients. It is known (see [4]) that for any scaling function $\varphi(x)$ and associated wavelet $\psi(x)$, generating an orthogonal wavelet basis, the corresponding symbols $m_0(\omega), m_1(\omega)$ satisfy (2). Any refinable function φ, whose symbol m_0 is solution to (2), generates a tight frame (see [5] for the case when m_0 is polynomial, and [2] for the general case).

Let us recall that a frame in a Hilbert space $\mathcal{H}$ is a family of its elements $\{f_k\}_{k\in\mathbb{Z}}$ such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2,$$

where optimal A and B are called frame constants. If $A = B$, the frame is called *a tight frame*. In the case when a tight frame has unit frame constants (for example, if it is an orthonormal basis) for any function $f \in \mathbb{L}^2(\mathbb{R})$ the expansion

$$f = \sum_{n\in\mathbb{Z}} \langle f_n, f\rangle f_n \tag{3}$$

is valid.

The frame $\left\{\{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}}\right\}_{l=1}^n$, where $\psi^l_{j,k}(x) = 2^{j/2}\psi_l(2^j x - k)$, generated by translates and dilations of finite number of functions, is called *an affine or wavelet frame.*

In the case when the symbol m_0 of a refinable function φ does not satisfy the equation

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1,$$

we cannot construct an orthonormal bases of V^1 of the form $\{\varphi(x-k), \psi(x-k)\}$. However, we can hope that there exists a collection of several framelets $\psi^1, \psi^2, \ldots, \psi^n \in V^1$, satisfying the following conditions:

1) the functions $\left\{\{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}}\right\}_{l=1}^n$ form a tight frame of the space $\mathbb{L}^2(\mathbb{R})$;

2) for algorithms of decomposition and reconstruction the recurrent formulae

$$\langle \varphi_{j,k}, f\rangle = c_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{h}_{k-2l}, \qquad \langle \psi^g_{j,k}, f\rangle = d^q_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{g}^q_{k-2l}, \quad 1 \leq q \leq n, \tag{4}$$

and

$$c_{j+1,l} = \sum_{k\in\mathbb{Z}} c_{j,k}h_{l-k} + \sum_{q=1}^n \sum_{k\in\mathbb{Z}} d^q_{j,k}g^q_{l-k}, \tag{5}$$

where g^q_k are coefficients of the expansions

$$m_0(\omega) = 2^{-1/2}\sum_{k\in\mathbb{Z}} h_k e^{-ik\omega}, \quad m_q(\omega) = 2^{-1/2}\sum_{k\in\mathbb{Z}} g^q_k e^{-ik\omega},$$

take place.

Let φ be a refinable function with a symbol m_0, $\hat{\psi}^k(\omega) = m_k(\omega/2)\hat{\varphi}(\omega/2) \in V^1$, where each symbol m_k is a 2π-periodic and essentially bounded function for $k = 1, 2, \ldots, n$. It is well-known that for constructing tight frames with property 2) the matrix

$$\mathcal{M}(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \ldots & m_n(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) & \ldots & m_n(\omega+\pi) \end{pmatrix}.$$

plays an important role. It is easy to see that the equality

$$\mathcal{M}(\omega)\mathcal{M}^*(\omega) = I \tag{6}$$

is equivalent to (4) and (5).

In our recent paper the following theorems were proved

Theorem A ([7]). *If (6) holds, then the functions* $\{\psi_j\}_{j=1}^k$ *generate a tight frame of* $\mathbb{L}^2(\mathbb{R})$.

Remark. For $n=1$ this theorem was proved in [2]. For an arbitrary n it was proved in [8] under some additional decay assumption for $\hat{\varphi}$. In [3] this and the next theorem was proved for a special case when m_0 is a trigonometric polynomial. In [8] Theorem A was called *the unitary extension principle.*

Theorem B ([7]). *Equation (6) has a solution if and only if*

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 \le 1 \ \textit{(a. e.)}. \tag{7}$$

Thus, the problem of constructing tight frames, generated by a refinable function, can be reduced to finding m_k, satisfying (6). It is clear that relation (6) can be re-written in the form

$$\mathbb{M}(\omega) := \mathcal{M}_\psi(\omega)\mathcal{M}_\psi^*(\omega) = \begin{pmatrix} 1-|m_0(\omega)|^2 & -m_0(\omega)\overline{m_0(\omega+\pi)} \\ -\overline{m_0(\omega)}m_0(\omega+\pi) & 1-|m_0(\omega+\pi)|^2 \end{pmatrix}, \tag{8}$$

where

$$\mathcal{M}_\psi(\omega) = \begin{pmatrix} m_1(\omega) & m_2(\omega) & \dots & m_n(\omega) \\ m_1(\omega+\pi) & m_2(\omega+\pi) & \dots & m_n(\omega+\pi) \end{pmatrix}.$$

Let us introduce the diagonal matrix $\Lambda(\omega)$ with eigenvalues of the matrix $\mathbb{M}(\omega)$ on the diagonal and the matrix $P(\omega)$ whose columns are the corresponding eigenvectors. Then

$$\Lambda(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1-|m_0(\omega)|^2-|m_0(\omega+\pi)|^2 \end{pmatrix}.$$

For those ω for which

$$B(\omega)\overline{B(\omega)} := |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 \ne 0$$

we define the matrix $P(\omega)$ in the form

$$P(\omega) = \begin{pmatrix} \overline{\left(\dfrac{e^{i\omega}m_0(\omega+\pi)}{B(\omega)}\right)} & \dfrac{m_0(\omega)}{B(\omega)} \\ -\overline{\left(\dfrac{e^{i\omega}m_0(\omega)}{B(\omega)}\right)} & \dfrac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix},$$

where $B(\omega)$ is a π-periodic function. At points, where $B(\omega)=0$, we define the matrix $P(\omega)$ as the identity matrix.

Thus, we have

$$\mathbb{M}(\omega) = P(\omega)\Lambda(\omega)P^*(\omega). \tag{9}$$

Now we can discribe all possible solutions to (6).

Theorem C ([7]). *Let a 2π-periodic function $m_0(\omega)$ satisfy (7). Then there exists a pair of 2π-periodic measurable functions m_1, m_2 which satisfy (6) for $n = 2$. Any solution of (6) can be represented in the form of the first row of the matrix*

$$\mathcal{M}_\psi(\omega) = P(\omega)D(\omega)Q(\omega), \tag{10}$$

where $D(\omega)$ is a diagonal matrix, $D(\omega)\overline{D(\omega)} = \Lambda(\omega)$, $Q(w)$ is an arbitrary unitary (a.e.) matrix with π-periodic measurable components.

Remark. To describe all possible solution to (6) for an arbitrary n we have to take an arbitrary $n \times n$ unitary matrix Q with π-periodic elements and a $2 \times n$ matrix D' which is extension of the matrix $D(\omega)$ by mean of filling all new columns with zeros.

Theorem D ([7]). *Let a trigonometric polynomial $m_0(\omega)$ of degree n satisfy (7). Then there exists a pair of trigonometric polynomials m_1, m_2 of the degree at most n satisfying (6).*

Remark. In [3] this theorem was proved without the guaranteed degree of the polynomials m_1, m_2.

Let now and in what follows a function $m_0(\omega)$ satisfy the condition

$$\overline{m_0(\omega)} = e^{il\omega} m_0(\omega), \quad l \in \mathbb{Z}, \tag{11}$$

then the corresponding refinable function $\varphi(x)$ are even after appropriate whole integer (l is even) or half integer (l is odd) shift of an argument. We shall call such funcions and their symbols *symmetric*. From here on without loss of generality we suppose $l = 0, 1$.

We call *antisymmetric* those functions which after an appropriate whole or half shift are odd. We are interested in framelets ψ_k, which are either symmetric or antisymmetric. Symbols of antisymmetric framelets satisfy the relation

$$\overline{m_k(\omega)} = -e^{il\omega} m_k(\omega), \quad l \in \mathbb{Z}. \tag{12}$$

In applications such systems are of great practical importance. The numerical algorithms for them have low computational complexity. And the problem of signal edges is solved very easily by means of even or odd extension.

The natural question about possibility to choose (anti)symmetric framelets for a given symmetric refinable function φ arises. In the recent paper [3] positive answer this question was given.

Theorem E ([3]). *For any refinable function φ with a polynomial symbol m_0 there are 3 (anti)symmetric functions m_1, m_2, m_3, providing a solution to (6).*

However, many examples, when system (6) can be solved with 2 (anti)symmetric framelets, are known. For instance this is possible for the cases of piecewise-linear ([8]) and piecewise-quadratic ([3]) B-splines. Our goal is to present a criterion for existence of 2 (anti)symmetric framelets.

2 Main result

First, we introduce necessary definitions and notation.

The degree of a trigonometric polynomial $\sum_{j=l}^{k} a_k e^{ijx}$, where $a_l \neq 0$ and $a_k \neq 0$, is defined to be $k-l$.

We denote by $\mathcal{L}$ a set of all Laurent polynomials with real coefficients, and by $\mathcal{L}_n$ a set of Laurent polynomials with real coefficients of degree at most n, i.e.,

$$\mathcal{L}_n := \left\{ \sum_{j=l}^{k} a_j z^j \;\middle|\; l,k \in \mathbb{Z};\ a_j \in \mathbb{R};\ 0 \leq k-l \leq n \right\}.$$

We denote by $\deg(f)$ the degree of the Laurent polynomial f.

In what follows Laurent polynomials $h_k(z)$ are specified by the z-transform of the symbols $m_k(\omega)$, i.e., $h_k(e^{i\omega}) := m_k(\omega)$.

Theorem 1. *Let $h_0(z)$ is a symmetric Laurent polynomial of degree n, satisfying (7). Then two (anti)symmetric solutions to (6) exist if and only if all roots of the Laurent polynomial*

$$h(z) := 1 - h_0(z)h_0(1/z) - h_0(-z)h_0(-1/z) \tag{13}$$

have even multiplicity. Moreover, in this case polynomials m_1, m_2 of degree at most n can be chosen.

This theorem has a simple consequence for B-spline multiresolutions. Let us recall that B-splines is defined to be

$$B_0(x) := \begin{cases} 1, & x \in [0,1], \\ 0, & x \notin [0,1]; \end{cases} \qquad B_{n+1} := B_n * B_0.$$

Corollary 1. *For the refinable functions B_n two (anty-)symmetric solutions* **exist** *for $n = 0,1,2,6$ and do not exist for $n = 3,4,5,7,8,\ldots,50$.*

This corollary was obtained by direct computation of roots of the corresponding polynomials with Matlab.

Remark. For $n=0$ we have Haar's wavelets. The solution for $n=1$ was found by A.Ron and Z.Shen [8] and the solution for $n=2$ was found by C.Chui and W.He [3]. We consider the case $n=6$ in Section 4.

A symbol $h_0(z) = 1/2 + \sum_{n=1}^{N} a_n(z^{-2n+1} + z^{2n-1})$, $\sum_{n=1}^{N} a_n = 1/4$ is called *interpolatory.*

Corollary 2. *An interpolatory symbol h_0 admit (anti)symmetric solutions to (6) if and only if $h_0(z) = (z^{1-2N} + 2 + z^{2N-1})/4$.*

Remark It is clear that only for $N=1$ we have a real interpolatory refinable function, satisfying the condition $\varphi(0)=1$, $\varphi(n)=0$, $n \in \mathbb{Z}$, $n \neq 0$.

3 Proof of main result

3.1 Necessity

We assume that the polynomials m_0, m_1, m_2 provide a solution to (6), m_0 is symmetric and m_1, m_2 are either symmetric or antisymmetric. Let us introduce a matrix $N(z)$ which is z-transform of the matrix $\mathcal{M}(\omega)$. Then equation (6) can be rewritten in the form

$$N(z)N^T(1/z) = I.$$

Obviuosly we can extend the rectangular matrix up to a 3×3 para-unitary matrix

$$N'(z) = \begin{pmatrix} h_0(z) & h_1(z) & h_2(z) \\ h_0(-z) & h_1(-z) & h_2(-z) \\ a_0(z^2) & a_1(z^2) & a_2(z^2) \end{pmatrix}, \qquad N'(z)N'^T(1/z) = I.$$

The new row can be obtained as a vector product of the known rows. It guarantees that $a_i(z) \in \mathcal{L}$. We note that any other possible polynomial choices of the last row differ from this one by the factor $\pm z^k$, $k \in \mathbb{Z}$. We used the factor z so that elements of the last row depend only on even powers. Thus, we have

$$a_0(z^2) = z(h_1(1/z)h_2(-1/z) - h_2(1/z)h_1(-1/z)). \tag{14}$$

We denote by S the set of all (anti)symmetric polynomials. We shall use the subscripts e, o, w, h to denote subsets of S, consisting of respectively even, odd, whole and half symmetric polynomials of S and integer superscripts to denote a value k in the relation $f(1/z) = z^{-k}f(z)$. For instance, $S^4_{w,o}$ is the set of polynomial which are odd at a whole point and for any $f \in S^4_{w,o}$ we have $f(1/z) = z^{-4}f(z)$. Of course subscripts w and h is compatible correspondingly only with even and odd superscripts.

Let us prove that $a_0 \in S$. This is enough to prove the necessity. Indeed, since we have the identity

$$h_0(z)h_0(1/z) + h_0(-z)h_0(-1/z) + a_0(z^2)a_0(1/z^2) \equiv 1,$$

then the symmetry of a_0 implies that all non-zero roots of the polynomial

$$1 - h_0(z)h_0(1/z) - h_0(-z)h_0(-1/z) = a_0(z^2)a_0(1/z^2) = \pm z^{2k}a_0^2(z^2) \tag{15}$$

have even multiplicity. First, we note that polynomials from S satisfy the followng obvious properties:

(a1) $f, g \in S \Rightarrow fg \in S$;
(a2) $f, g \in S_e \Rightarrow fg \in S_e$;
(a3) $f, g \in S_o \Rightarrow fg \in S_e$;
(a4) $f \in S_o, g \in S_e \Rightarrow fg \in S_o$;
(a5) $f, g \in S_w \Rightarrow fg \in S_w$;
(a6) $f, g \in S_h \Rightarrow fg \in S_w$;
(a7) $f \in S_w, g \in S_h \Rightarrow fg \in S_h$;
(b1) $f(x) \in S_w \Leftrightarrow f(-x), f(1/x) \in S_w$;
(b2) $f(x) \in S_o \Leftrightarrow f(1/x) \in S_o$;

(b3) $f(x) \in S_{h,e} \Leftrightarrow f(-x) \in S_{h,o}$;
(b4) $f(x) \in S_{w,e}$ $(f(x) \in S_{w,o}) \Leftrightarrow f(-x) \in S_{h,e}$ $(f(x) \in S_{w,o})$;
(c) $f \in S_w \Rightarrow f(x) \pm f(-x) \in S$.

Now we consider the case $h_0 \in S_w$. Taking into account (a) — (c), and (14), we obtain that the symmetry of a_0 may be violated only if $h_2(z)h_1(-z) \in S_h$. In this case h_1 and h_2 belong to different classes S_w and S_h. We suppose that $h_1 \in S_w$, $h_2 \in S_h$.

Because of the orthogonality of the 1st and the 2nd rows of the matrix $N'(z)$, we have

$$h_0(z)h_0(-1/z) + h_1(z)h_1(-1/z) + h_2(z)h_2(-1/z) \equiv 0. \tag{16}$$

We see that, according to properties (a2), (b1), (c), the 1st and the 2nd summands in (16) belong to S_e, whereas, by (a4) and (b2), the 3rd one belongs to S_o. It means $h_2(z) \equiv 0$. Hence, we have $a_0 \equiv a_1 \equiv 0$, $a_2(z) = z^k$. Thus, we come to the case of one framelet. Of course this is impossible for $h_0 \in S_w$ (sf. [4], Chapter 8), though (14) gives us a permissible function $a_0(z) \equiv 0$.

We note that by the same reasons $h_1, h_2 \in S_w$. Actually, if $h_1, h_2 \in S_h$, then the 1st term in (16) is even whereas the 2nd the 3rd ones are odd. It means that $h_0(z)h_0(1/z) \equiv 0$.

Moreover, it is easy to prove that $h_1 \in S_e$, $h_2 \in S_o$ or vise versa. The point is that for the functions h_1, h_2 with the same evenness, by (14), we get an even polynomial a_0. Thus, the polynomial $a_0(z^2)a_0(1/z^2)$ has positive coeffitients of the lowest and the highest powers, whereas the corresponding coefficients of the left-hand part of (15) are negative.

Now if we suppose $h_0 \in S_h$, in analogous way we obtain that the both functions h_1 and h_2 belong to S_h. Indeed, since the 1st term in (16) is odd then two other terms are odd. Hence, $h_1, h_2 \in S_h$. Easily to see that the case $h_1, h_2 \in S_{h,e}$ is not valid. It contradicts to the equality

$$h_0(z)h_0(1/z) + h_1(z)h_1(1/z) + h_2(z)h_2(1/z) = 1$$

due to the positivity of coefficients of the lowest and the highest powers of all terms. From(14) and (b3) we have $a_0 \in S_o$ for $h_1, h_2 \in S_o$ and $a_0 \in S_e$ for $h_1 \in S_e$, $h_2 \in S_o$.

Thus, we have 3 permissible cases:
1) $h_0 \in S_w$, $h_1 \in S_{w,e}$, $h_2 \in S_{w,o}$;
2) $h_0 \in S_h$, $h_1 \in S_{h,o}$, $h_2 \in S_{h,o}$;
2) $h_0 \in S_h$, $h_1 \in S_{h,e}$, $h_2 \in S_{h,o}$.

Examples for all of them will be given in Section 4.

3.2 Sufficiency

We assume that $m_0(\omega)$ is a trigonometric polynomial, then $\mathcal{B}(\omega) = |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$ and $\mathcal{A}(\omega) = 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2$ are also non-negative trigonometric polynomials So according to Riesz lemma, we can take π-periodic polynimials $A(\omega)$ and $B(\omega)$ such that $|A(\omega)|^2 = \mathcal{A}(\omega)$, $|B(\omega)|^2 = \mathcal{B}(\omega)$. Since $h(z)$ is a symmetric polynomial of even power and its roots have even multiplicity, we can take an (anti)symmetric polynomial $A(\omega)$. The choice of a_0 is not unique, in what follows we suppose for definiteness that either $a_0 \in S_w^0$ or $a_0 \in S_h^1$. We assume that the choice of matrices $P(\omega)$ and $D(\omega)$ in Theorem C corresponds the last assumptions.

Our further proof repeats for the most part reasoning from [7].

In fact, we cannot control the choice of the matrices $P(\omega)$ and $D(\omega)$ in (10). So we need to choose a unitary rational π-periodic matrix $Q(\omega)$ such that $\mathcal{M}_\psi(\omega)$ consists of trigonometric polynomials.

Let us apply z-transform to (10). In what follows we consider the Laurent polynomials $b(e^{2i\omega}) = B(\omega)$, $a_0(e^{2i\omega}) = A(\omega)$. After the same change of variable the matrix $P(\omega)$ becomes

$$H(z) = \begin{pmatrix} \dfrac{\frac{1}{z}h_0\left(-\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{h_0(z)}{b(z^2)} \\ -\dfrac{\frac{1}{z}h_0\left(\frac{1}{z}\right)}{b(1/z^2)} & \dfrac{h_0(-z)}{b(z^2)} \end{pmatrix}.$$

We put the last representation of the matrix $H(z)$ through procedure of reduction. If polynomials $h_0(z)$, $h_0(-z)$, $b(z^2)$ are divisible by $z - z_0$, that in view of the symmetry of $h(z)$ they are also divisible by $z + z_0$. Obviously, since $h_0(1) \neq 0$, then $z_0 \neq \pm 1$. We cancel the fractions of $H(z)$ by $(1/z^2 - z_0^2)(z^2 - z_0^2)$. After all possible cancellations we obtain the same matrix $H'(z) = H(z)$ but its elements are expressed in terms of new functions $h_0'(z)$ and $b'(z)$. It is clear that $b'(z^2)b'(1/z'^2) = h_0'(z)h_0'(1/z) + h_0'(-z)h_0'(-1/z)$ and numerators of the matrix $H'(z)$ do not vanish simultaneously. Indeed, since the determinant of $H(z)$ is equal to $1/z$, if for some z_0 we have $h_0(z_0) = h_0(-z_0) = h_0(1/z_0) = h_0(-1/z_0) = 0$, then either $b(z_0^2) = 0$ or $b(1/z_0^2) = 0$. It means that the reduction of $H(z)$ can be continued. We note that because the coefficients of $h_0(z)$ and $b(z)$ are real, the polynomials $h_0'(z)$ and $b'(z)$ also have real coefficients. Moreover, $h_0'(z)$ is symmetric.

After taking z-transform the elements $q_{11}(z^2)$, $q_{12}(z^2)$, $q_{21}(z^2)$, $q_{22}(z^2)$ of the matrix $Q(\omega)$ satisfy the relations

$$q_{22}(z) = q_{11}(1/z)z^N, \quad q_{12}(z) = -q_{21}(1/z)z^N, \quad N \in \mathbb{Z}.$$

Here, without loss of generality, we may suppose $N = 0$, because any other choice leads to the integer shift of one of the basic framelets.

To reduce poles of the matrix $H'(z)$ after multiplication by $Q(\omega)$ we suppose that

$$q_{11}(z) = \frac{g_1(z)}{b'(z)}, \qquad q_{21}(z) = \frac{g_2(z)}{b'(1/z)},$$

where g_1, g_2 are Laurent polynomials.

Let $\mathcal{R} = \{\pm z_1^{\pm 1}, \pm z_2^{\pm 1}, \ldots, \pm z_n^{\pm 1}\}$ be a set of all different roots of the polynomial $b'(z^2)b'(1/z^2)$. We denote by k_j the multiplicity of the root z_j. It is clear that all four roots $\pm z_j^{\pm 1}$ have the same multiplicity. So the degree of polynomial $b'(z^2)b'(1/z^2)$ is equal to $4\sum k_j = 4k$, where k is the degree of polynomial b'.

To prove the theorem we need to find polynomials g_1, g_2 which satisfy equations

$$\frac{1}{z}h_0'\left(-\frac{1}{z}\right) g_1(z^2) + a_0(z^2)h_0'(z)g_2(z^2) = b'(z^2)b'(1/z^2)f_1(z); \tag{17}$$

$$-\frac{1}{z}h_0'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right)+a_0(z^2)h_0'(z)g_1\left(\frac{1}{z^2}\right)=b'(z^2)b'(1/z^2)f_2(z); \tag{18}$$

$$-\frac{1}{z}h_0'\left(\frac{1}{z}\right)g_1(z^2)+a_0(z^2)h_0'(-z)g_2(z^2)=b'(z^2)b'(1/z^2)f_3(z); \tag{19}$$

$$\frac{1}{z}h_0'\left(\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right)+a_0(z^2)h_0'(-z)g_1\left(\frac{1}{z^2}\right)=b'(z^2)b'(1/z^2)f_4(z). \tag{20}$$

where $f_1, f_2, f_3, f_4 \in \mathcal{L}$. Moreover, we need to satisfy the condition of the unitarity of the matrix $Q(\omega)$. Hence,

$$g_1(z)g_1(1/z)+g_2(z)g_2(1/z)=b'(z)b'(1/z). \tag{21}$$

Now we leave aside equation (21) and prove the existence of polynomials $g_1, g_2 \in \mathcal{L}_k$, satisfying (17) – (20). First we choose templates for the polynomials g_1 and g_2. There are 12 different cases which depend on type of symmetry of the polynomials h_0, a_0, and on evenness of the number k. They can be classified into three groups with 4 cases in every group.

1) $h_0 \in S_w$, a_0 is odd.
   (a) $a_0 \in S^1_{h,o}$, k is odd, $g_1 \in S^1_{h,e}$, $g_2 \in S^{-1}_{h,e}$, $\deg(g_1)=\deg(g_2)=k$;
   (b) $a_0 \in S^0_{w,o}$, k is odd, $g_1 \in S^0_{w,o}$, $g_2 \in S^{-1}_{h,e}$, $\deg(g_1)=k-1$, $\deg(g_2)=k$;
   (c) $a_0 \in S^1_{h,o}$, k is even, $g_1 \in S^2_{w,e}$, $g_2 \in S^0_{w,o}$, $\deg(g_1)=\deg(g_2)=k$;
   (d) $a_0 \in S^0_{w,o}$, k is even, $g_1 \in S^1_{h,o}$, $g_2 \in S^0_{w,e}$, $\deg(g_1)=\deg(g_2)=k$;

2) $h_0 \in S_h$, a_0 is odd.
   (a) $a_0 \in S^1_{h,o}$, k is odd, $g_1 \in S^2_{w,e}$, $g_2 \in S^{-1}_{h,e}$, $\deg(g_1)=k-1$, $\deg(g_2)=k$;
   (b) $a_0 \in S^0_{w,o}$, k is odd, $g_1 \in S^1_{h,e}$, $g_2 \in S^{-1}_{h,e}$, $\deg(g_1)=\deg(g_2)=k$;
   (c) $a_0 \in S^1_{h,o}$, k is even, $g_1 \in S^1_{h,e}$, $g_2 \in S^{-2}_{w,e}$, $\deg(g_1)=k-1$, $\deg(g_2)=k$;
   (d) $a_0 \in S^0_{w,o}$, k is even, $g_1 \in S^2_{w,e}$, $g_2 \in S^0_{w,e}$, $\deg(g_1)=\deg(g_2)=k$;

3) $h_0 \in S_h$, a_0 is even.
   (a) $a_0 \in S^1_{h,e}$, k is odd, $g_1 \in S^2_{w,e}$, $g_2 \in S^{-1}_{h,o}$, $\deg(g_1)=k-1$, $\deg(g_2)=k$;
   (b) $a_0 \in S^0_{w,e}$, k is odd, $g_1 \in S^1_{h,e}$, $g_2 \in S^{-1}_{h,o}$, $\deg(g_1)=\deg(g_2)=k$;
   (c) $a_0 \in S^1_{h,e}$, k is even, $g_1 \in S^1_{h,e}$, $g_2 \in S^{-2}_{w,o}$, $\deg(g_1)=k-1$, $\deg(g_2)=k$;
   (d) $a_0 \in S^0_{w,e}$, k is even, $g_1 \in S^2_{w,e}$, $g_2 \in S^0_{w,o}$, $\deg(g_1)=\deg(g_2)=k$;

For the cases 2d) and 3b) we have $k+2$ unknown parameters and for others cases we have $k+1$ parameters. Obviously for all cases the left-hand parts of (17) – (20) are (anti)symmetric.

First we show that there exist polynomials g_1 and g_2, satisfying equations (17) – (20) at points of the set $\mathcal{R}$. As it usually is in the case of a root $\tilde{z}$ of multiplicity $\tilde{k}$, we require that the left-hand parts of (17) – (20) are divisible by $(z-z^0)^k$.

Equations (17) – (20) give us $16k$ homogeneous linear equations for $k+1$ or $k+2$ unknown coeffitients of polynomials g_1 and g_2. We shall prove that at most k of them are linearly independent. The proof of this fact we conduct in 4 steps. Three of these steps are based on the following lemma.

Lemma 1. *Let $a_1(z)$, $a_2(z)$, $a_3(z)$, $a_4(z)$, $b_1(z)$, $b_2(z)$ $c_1(z)$, $c_2(z)$ be Laurent polynomials, $|a_1(z_0)|^2 + |a_2(z_0)|^2 \neq 0$, l is a positive integer. If*

$$a_1(z)b_1(z) + a_2(z)b_2(z) = (z - z_0)^l c_1(z), \tag{22}$$

$$a_1(z)a_4(z) - a_2(z)a_3(z) = (z - z_0)^l c_2(z), \tag{23}$$

then we have

$$a_3(z)b_1(z) + a_4(z)b_2(z) = (z - z_0)^l c(z), \tag{24}$$

where $c(z) \in \mathcal{L}$.

Proof. Let us assume for definiteness that $a_1(z_0) \neq 0$. We express b_1 from (22) and a_4 from (23). Using the obtained representations, we have

$$\begin{aligned} &a_3(z)b_1(z) + a_4(z)b_2(z) = \\ &\qquad a_3(z)\frac{(z-z_0)^l c_1(z) - a_2(z)b_2(z)}{a_1(z)} + b_2(z)\frac{(z-z_0)^l c_2(z) + a_2(z)a_3(z)}{a_1(z)} = \\ &\qquad\qquad (z-z_0)^l \frac{a_3(z)c_1(z) + b_2(z)c_2(z)}{a_1(z)} =: (z-z_0)^l c(z). \end{aligned}$$

□

In the first step we prove that for every $\tilde{z} \in \mathcal{R}$ only one equation of the pairs $\{(17), (19)\}$ and $\{(18), (20)\}$ should be retained. Indeed, on the one hand

$$\det \begin{vmatrix} \frac{1}{z}h_0'\left(-\frac{1}{z}\right) & a_0(z^2)h_0'(z) \\ -\frac{1}{z}h_0'\left(\frac{1}{z}\right) & a_0(z^2)h_0'(-z) \end{vmatrix} = \frac{1}{z}a_0(z^2)b'(z^2)b'(1/z^2) = (z - \tilde{z})^{\tilde{k}} c_1(z), \quad c_1(z) \in \mathcal{L},$$

on the other hand, since $a_0(z)a_0(1/z) = 1 - b(z)b(1/z)$, $a_0(\tilde{z}^2) \neq 0$ for any $\tilde{z} \in \mathcal{R}$. Hence, the last matrix at point $\tilde{z}$ has at least one non-zero element. We assume for definiteness that the first row contains non-zero element. Then by Lemma 1, if g_1 and g_2 at the point $\tilde{z}$ satisfy (17) with multiplicity $\tilde{k}$, then they also satisfy (19) at least with the same multiplicity. So at the point $\tilde{z}$ we can exclude equation (19) from consideration. In the same manner we eliminate one of equations (18) and (20).

In the second step we reject equations, corresponding to the roots $\tilde{z}$ and $1/\tilde{z}$. Now for two roots $\tilde{z}$ and $1/\tilde{z}$ we have $4\tilde{k}$ equations. It turns out that at most $2\tilde{k}$ of them are linearly independent. We show that we can keep only equations of the form (17) and (19). Indeed, let us assume that in the previous step we kept equation (17) for $\tilde{z} \in \mathcal{R}$ and equation (18) for $1/\tilde{z}$. Now we prove that linear equations generated by (18) for $1/\tilde{z}$ can be omitted. We apply the change of variable $z \mapsto 1/z$ to (18). Then the left-hand part of (18) becomes

$$a_0(1/z^2)h_0'(1/z)g_1(z^2) - zh_0'(-z)g_2(z^2). \tag{25}$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h_0'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ a_0\left(\frac{1}{z^2}\right)h_0'\left(\frac{1}{z}\right) & -zh_0'(-z) \end{vmatrix} = b'(z^2)b'(1/z^2)(b''(z^2)b''(1/z^2)h_0'(z)h_0'(1/z)-1),$$

where $b''(z)=b(z)/b'(z)$, is divisible by $(z-\tilde{z})^{\tilde{k}}$, expression (25) is also divisible by $(z-\tilde{z})^{\tilde{k}}$ and the left-hand part of (18) is divisible by $(z-1/\tilde{z})^{\tilde{k}}$.

Dependence of the equations, generated by (20), is obtained by the same reasons. Indeed, after transform $z \mapsto 1/z$ the left-hand part of (20) is equal to

$$a_0(1/z^2)h_0'(-1/z)g_1(z^2)+zh_0'(z)g_2(z^2).$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h_0'\left(-\frac{1}{z}\right) & a_0(z^2)h_0'(z) \\ a_0\left(\frac{1}{z^2}\right)h_0'(-1/z) & zh_0'(z) \end{vmatrix} = b(z^2)b(1/z^2)h_0'(z)h_0'(-1/z),$$

then the left-hand part of (20) is divisible by $(z-1/\tilde{z})^{\tilde{k}}$.

In the third step we prove that equations, corresponding $\tilde{z}$ and $-\tilde{z}$, are linear dependent.

Let us assume that we have chosen equation (17) for the both roots $\pm\tilde{z}$. After substitution $z \mapsto -z$ the right-hand part of (17) is transformed to

$$-\frac{1}{z}h_0'\left(\frac{1}{z}\right)g_1(z^2)+a_0(z^2)h_0'(-z)g_2(z^2).$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h_0'\left(-\frac{1}{z}\right) & a_0(z^2)h_0'(z) \\ -\frac{1}{z}h_0'\left(\frac{1}{z}\right) & a_0(z^2)h_0'(-z) \end{vmatrix} = \frac{1}{z}a_0(z^2)b'(z^2)b'(1/z^2)$$

is divisible by $(z-\tilde{z})^{\tilde{k}}$, then the equations for $-\tilde{z}$ are linear dependent of the equations for $\tilde{z}$.

In the case, when we take equation (17) for $\tilde{z}$ and equation (19) for $-\tilde{z}$, the corresponding linear equations coincide.

Finally, we note that because the left-hand parts of (17) – (20) belong to S, the linear equations for $\tilde{z}, 1/\tilde{z} \in \mathcal{R}$ coincide.

Thus, we have proved the existence of a pair of polynomials $g_1, g_2 \in \mathcal{L}_n$, satisfying equations (17) – (20) on all of $\mathcal{R}$. Although the polynomial $b'(z^2)b'(1/z^2)$ can have complex roots, it is easy to check that we can choose polynomials g_1, g_2 with real coefficients. Indeed, if z_0 is a root of $b'(z^2)b'(1/z^2)$, then $\bar{z}_0$ is also a root. Coefficients of the equations, corresponding these roots, differ in complex conjugation. So instead of them we can consider real equations, corresponding to real and imagine parts of the initial equations.

Thus, we have k homogeneous linear equation for $k+1$ or $k+2$ unknown values. Let us take any non-degenerate solution of the system. Now we prove that the corresponding pair of polynomials g_1 and g_2 of degree at most k, satisfying (17) – (20) and the relation

$$g_1^2(1) + g_2^2(1) = b'^2(1), \tag{26}$$

satisfies also the equation

$$g_1(z)g_1(1/z) + g_2(z)g_2(1/z) = b'(z)b'(1/z). \tag{27}$$

Indeed, let us assume for definiteness that $\tilde{z} \in \mathcal{R}$ and $|h'(-1/\tilde{z})|^2 + |a(z^2)h'(z)|^2 \neq 0$. By (18), we have

$$\det \begin{vmatrix} \frac{1}{z}h_0'\left(-\frac{1}{z}\right) & a_0(z^2)h_0'(z) \\ g_1\left(\frac{1}{z^2}\right) & g_2\left(\frac{1}{z^2}\right) \end{vmatrix} = \frac{1}{z}h_0'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) - a_0(z^2)h_0'(z)g_1\left(\frac{1}{z^2}\right) =$$

$$- b'(z^2)b'(1/z^2)f_2(z).$$

Thus, by Lemma 1 and from (17), the expression $g_1(z)g_1(1/z) + g_2(z)g_2(1/z)$ is divisible by $(z-\tilde{z})^{\tilde{k}}$. It means that polynomials in the left-hand and right-hand parts of (27) have $2k$ common zeros. It remains to normalize the left-hand polynomial according to (26). The normalization is impossible only in the case when $g_1(1) = g_2(1) = 0$. However, it implies that the left-hand part of (27) has $2k+1$ zeros. It follows from this that $g_1(z)g_1(1/z) + g_2(z)g_2(1/z) \equiv 0$. Hence, $g_1(z) \equiv g_2(z) \equiv 0$. It contradicts to the assumption that at least one of the polynomials g_1 and g_2 is non-degenerate.

3.3 Proof of Corollary 2

Indeed, for interpolatory symbols we have

$$h_0(z) = 1/2 + \sum_{n=1}^{N} a_n(z^{-2n+1} + z^{2n-1}) =: 1/2 + \tilde{h}_0(z),$$

$$a_0(z)a_0(1/z) = 1-(1/2+\tilde{h}_0(z))^2-(1/2-\tilde{h}_0(z))^2 = 2(1/2+\tilde{h}_0(z))(1/2-\tilde{h}_0(z)) = 2h_0(z)h_0(-z).$$

The functions $h_0(z)$ and $h_0(-z)$ have distinct roots. Hence, by Theorem 1, we have to find those interpolatory symbols which have roots of even multiplicity. Such polynomials have to be non-negative. Otherwise, they have roots of odd multiplicity on the circle $|z| = 1$. Thus, by Riesz lemma, $h_0(z) = p(z)p(1/z)$, where $p \in S$. On the other hand, $h_0(z) + h_0(-z) = 1$. It implies the equality $p(z)p(1/z) + p(-z)p(-1/z) = 1$, which is valid (see [4], Chapter 8) only for $p(z) = z^M(1+z^{2N-1})/2$, $M, N \in \mathbb{Z}$.

4 Examples

In this section we discribe all possible refinable functions of the class S with degrees of their symbols up to 5, satisfying criterion for the existence of 2 (anti)symmetric framelets and give examples of their symbols and graphs. Besides, we construct 2 framelets corresponding to the B-spline B_6. The degree of its symbol is equal to 7.

The case $\deg(h_0) = 1$ corresponds to the Haar wavelets. It gives an example of an orthonormal basis with symmetry.

4.1 The case $\deg(h_0) = 2$

This case is also trivial. A.Ron and Z.Shen [8] presented the construction of 2 framelets associated with the piecewise linear B-spline B_1 with the symbol $h_0(z) = z^{-1}/4 + 1/2 + z/4$ (see Fig. 1). The symbols of framelets can be represented in the form $h_1(z) = -z^{-1}/4 + 1/2 - z/4$ and $h_1(z) = (z^{-1} - z)/2\sqrt{2}$. Easily to see that this is a unique example of a refinable function for which the unitary extension principle is applicable. Indeed, let $h_0(z) = \alpha z^{-1} + \beta + \alpha z$. On the one hand, $2\alpha + \beta = 1$. On the other hand, by Theorem B, we have $h_0^2(1) + h_0^2(-1) \leq 1$. Hence, $h_0(-1) = 0$ that implies $\beta - 2\alpha = 0$. Hence, $\beta = 1/2$, $\alpha = 1/4$.

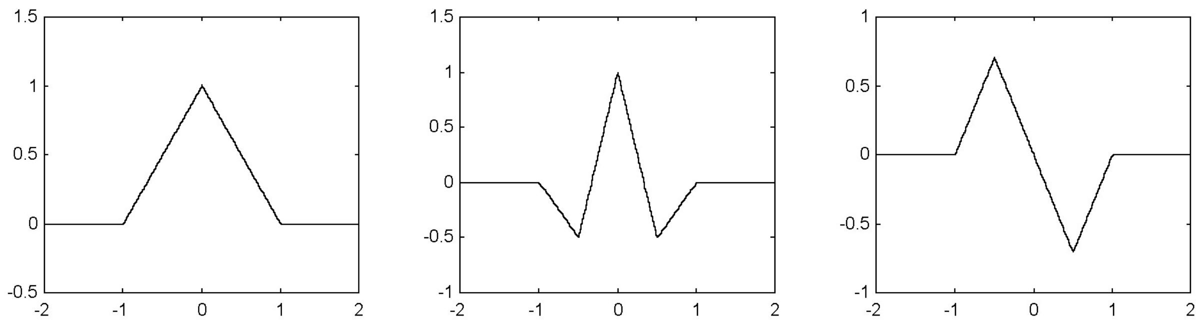

Fig. 1. $\deg(h_0) = 2$

4.2 The case $\deg(h_0) = 3$

As we know at the moment only one example of a refinable function, admiting 2 framelets is known for this case. C.Chui and W.He [3] have done it for the B-spline B_2, $h_0(z) = z^{-1}/8 + 3(1+z)/8 + z^2/8$. Then the framelets are defined by the symbols $h_1(z) = \sqrt{3}(1-z)/4$, $h_2(z) = (z^{-1} + 3 - 3z - z^2)/8$ (see Fig.3).

An arbitrary symbol $h_0(z) \in S_h$ of degree 3, satisfying the condition $h_0(1) = 1$, can be represented in the form

$$h_0(z) = \alpha z^{-1} + \beta + \beta z + \alpha z^2, \quad \alpha + \beta = 1/2. \tag{28}$$

We prove that such symbols, admit constructing 2 antisymmetric framelets if and only if $\alpha \geq 0$, $\beta \geq 0$.

It follows from (28) that

$$\begin{aligned} h(z) = 1 - h_0(z)h_0(1/z) - h_0(-z)h_0(-1/z) &= -4\alpha\beta z^{-2} + (1 - 4(\alpha^2 + \beta^2) - 4\alpha\beta z^2 = \\ &- 4\alpha\beta z^{-2} + 8\alpha\beta - 4\alpha\beta z^2 = -4\alpha\beta z^{-2}(z^2-1)^2 = 4\alpha\beta(z - 1/z)(1/z - z). \end{aligned}$$

Hence, $h(z)$ is positive for $|z| = 1$, if and and only if $ab \geq 0$. If $\alpha = 0$, $\beta = 1/2$ we the Haar wavelets. If $\alpha = 1/2$, $\beta = 0$ we have a tight frame with a unique framelet. All other choice of positive parameters leads to a framelet system with two antisymmetric framelets with

$$h_1(z) = -\alpha z^{-1} - \beta + \beta z + \alpha z^2, \qquad h_2(z) = 2\sqrt{\alpha\beta}(-1 + z)$$

Several examples of graphs of such refinable functions and framelets are illustrated by Figures 2–6.

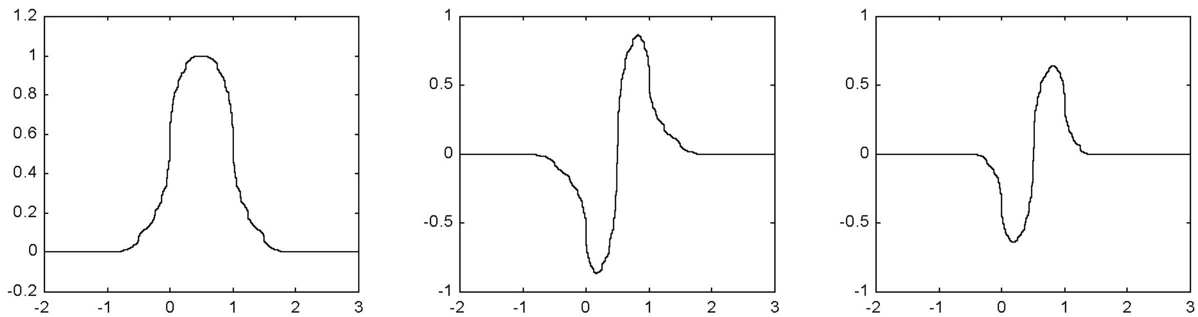

Fig. 2. $\deg(h_0) = 3$, $\beta = 7/16$

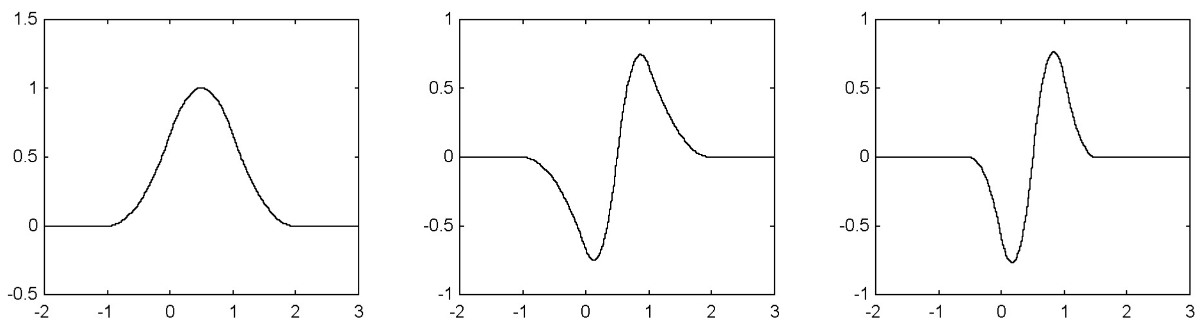

Fig. 3. $\deg(h_0) = 3$, $\beta = 3/8$ (piecewise quadratic splines)

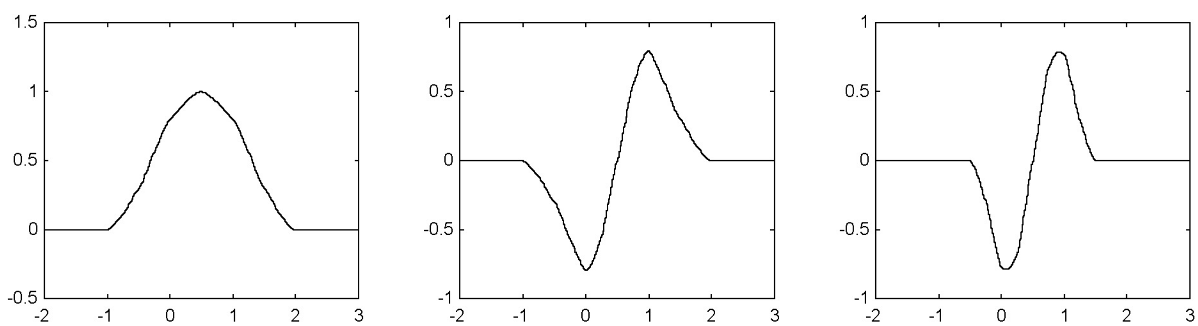

Fig. 4. $\deg(h_0) = 3$, $\beta = 5/16$

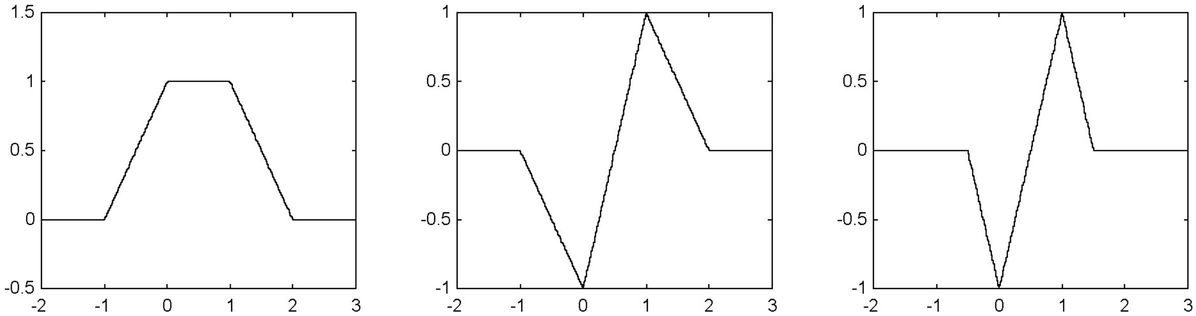

Fig. 5. $\deg(h_0) = 3$, $\beta = 1/4$ (piecewise linear splines)

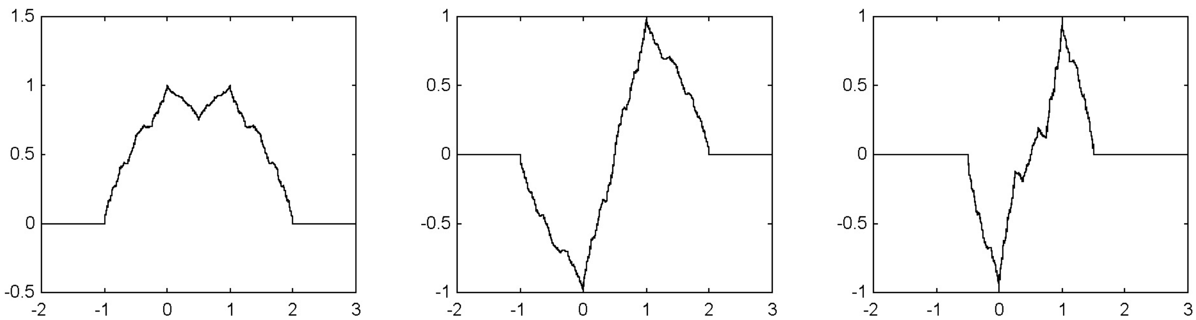

Fig. 6. $\deg(h_0) = 3$, $\beta = 3/16$

4.3 The case $\deg(h_0) = 4$

For this case an arbitrary admissible symbol h_0 has the form

$$h_0(z) = -\alpha z^{-2} + 0.25z^{-1} + (0.5 + 2\alpha) + 0.25z^{-1} - \alpha z^2, \quad \alpha \neq 0.$$

Thus,

$$\begin{aligned} &h(z) = \\ &-2\alpha^2 z^{-4} + (8\alpha^2 + 2\alpha - 1/8)z^{-2} - (12\alpha^2 + 4\alpha - 1/4) + (8\alpha^2 + 2\alpha - 1/8)z^2 - 2\alpha^2 z^4 = \\ &\qquad (z - 1/z)(1/z - z)(2\alpha^2 z^{-2} - (4\alpha^2 + 2\alpha - 1/8) + 2\alpha^2 z^2) \end{aligned} \tag{29}$$

On the one hand, the last factor in (29) has to have roots of even degree. On the other hand, if z_0 is its root, then $-z_0$ and $\pm z^{-1}$ are also its roots. It means that only ± 1, $\pm i$ are permissible roots. It follows from this that $4\alpha^2 + 2\alpha - 1/8 = \pm 4\alpha^2$. However, the positive sign in this relatoin leads to the negative function $h(z)$. So we have just two solutions

$$\alpha_{1,2} = \frac{-1 \pm \sqrt{2}}{4}$$

with the correspondig framelet symbols

$$h_1(z) = -0.25z^{-1} + 0.5 - 0.25z, \qquad h_2(z) = \alpha_i z^{-2} - 0.25z^{-1} + 0.25z - \alpha_i z^{-2}.$$

Figures 7 and 8 show plots of basic framelets.

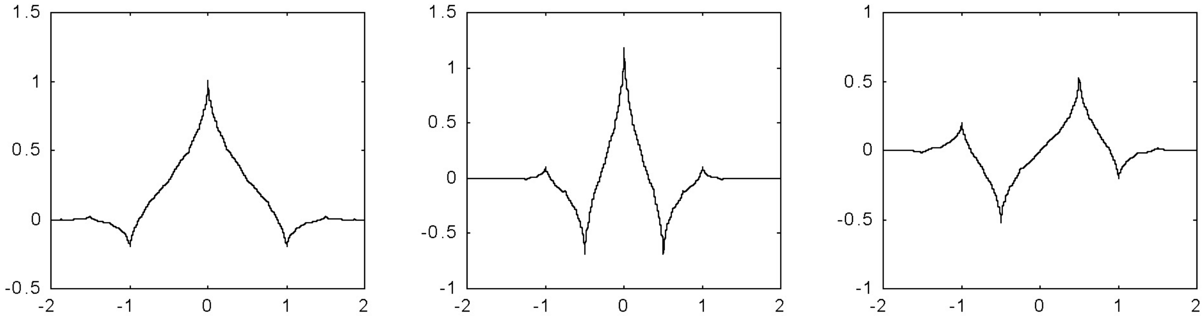

Fig. 7. $\deg(h_0) = 4$, $\alpha = 0.0517767$

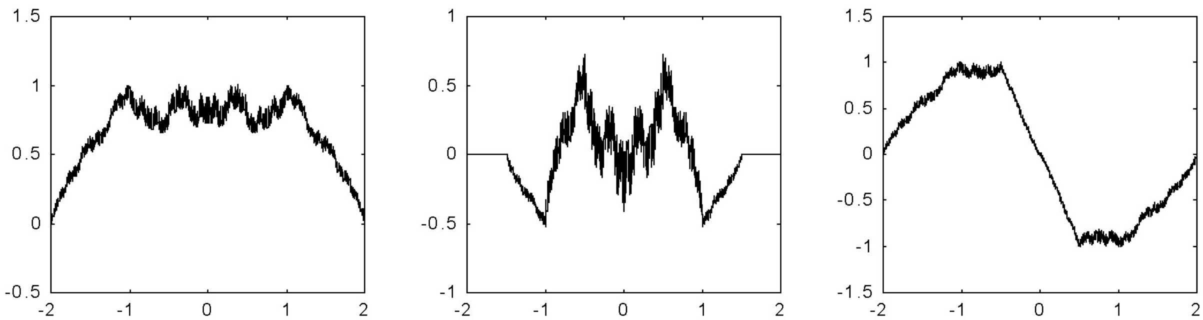

Fig. 8. $\deg(h_0) = 4$, $\alpha = -0.3017767$

4.4 The case $\deg(h_0) = 5$

Here we consider symbols, satisfying the relation

$$h_0(z) = \alpha z^{-2} + \beta z^{-1} + \gamma + \gamma z + \beta z^{-1} + \alpha z^3,$$

where $\alpha + \beta + \gamma = 1/2$, $\alpha \neq 0$.

We have

$$\begin{aligned} h(z) = \\ & -4\alpha\beta z^{-4} - 4(\alpha+\beta)\gamma z^{-2} + (1 - 4(\alpha^2+\beta^2+\gamma^2)) - 4(\alpha+\beta)\gamma z^2 - 4\alpha\beta z^4 = \\ & (z - 1/z)(1/z - z)(4\alpha\beta z^{-2} + 4(2\alpha\beta + \alpha\gamma + \beta\gamma) + 4\alpha\beta z^2). \quad (30) \end{aligned}$$

Thus, the parameters have to satisfy either

$$8\alpha\beta + 4\gamma(\alpha+\beta) = 8\alpha\beta \quad (31)$$

or

$$8\alpha\beta + 4\gamma(\alpha+\beta) = -8\alpha\beta. \quad (32)$$

First consider equation (31). It is reduced to the relation $\gamma(\alpha+\beta) = 0$ that implies two possibilities: (a) $\alpha + \beta = 0$, $\gamma = 1/2$; (b) $\alpha + \beta = 1/2$, $\gamma = 0$. Solution (a) is not valid because in this case α and β have different signs that leads to the negative polinomial $h(z)$

(for $|z| = 1$). Solution (b) is valid when $\alpha > 0$, $\beta \geq 0$. Every such a pair of the parameters gives two odd framelets. Their symbols can be written in the form

$$h_1(z) = 2\sqrt{\alpha\beta}(1 - z), \quad h_2(z) = \alpha(z^{-3} - z^4) + \beta(z^{-2} + z^3).$$

Figures 9–11 shows several examples of their plots.

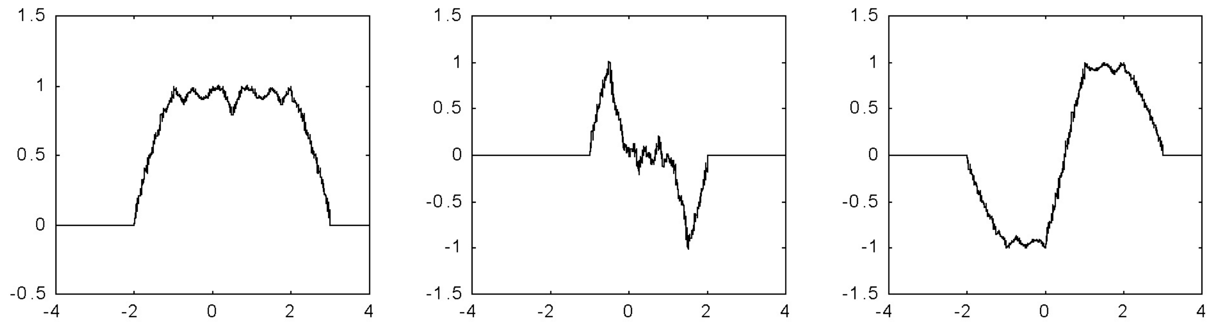

Fig. 9. $\deg(h_0) = 5$, $\beta = 0.2$

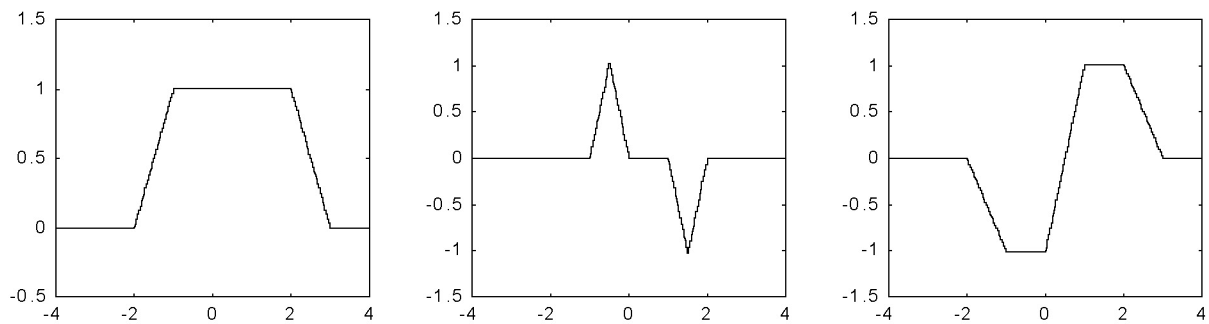

Fig. 10. $\deg(h_0) = 5$, $\beta = 0.25$

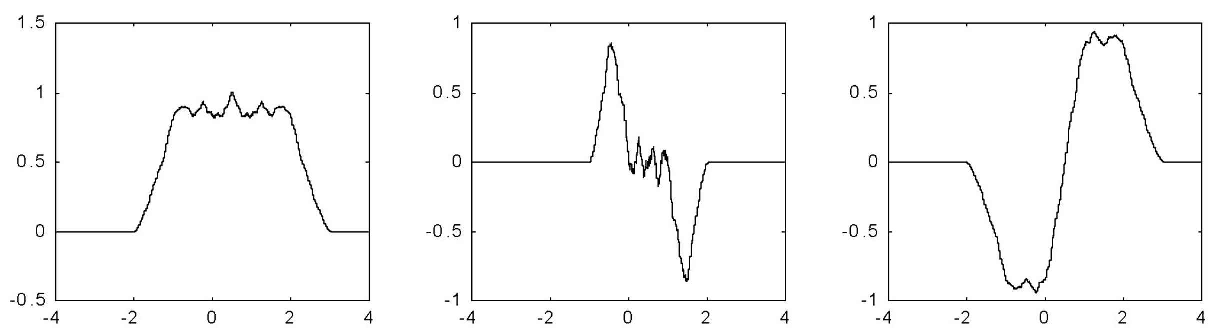

Fig. 11. $\deg(h_0) = 5$, $\beta = 0.3$

Now we consider equation (32). Let fix γ, then $\beta = 1/2 - \alpha - \gamma$. We substitute β to (32) and solve it with respect to α. Then

$$\alpha_{1,2} = \frac{1 - 2\gamma \pm \sqrt{1 - 2\gamma}}{4};$$

$$\beta_{1,2} = \frac{1 - 2\gamma \mp \sqrt{1 - 2\gamma}}{4}.$$

Hence, $\gamma \leq 1/2$. We consider four cases (a) $\gamma = 1/2$; (b) $0 < \gamma < 1/2$; (c) $\gamma = 0$; (d) $\gamma < 0$.

The case (a) corresponds to the Haar wavelets. In the case (b) we have two framelets of different evenness. The case (c) is well-known. It gives one framelet. The case (d) is invalid, because it leads to the negative h.

We give examples for 3 sets of parameters.

1) $\alpha = 0.011680$, $\beta = -0.010680$, $\gamma = 0.499$, (see Fig. 12).

$$h_1(z) = -0.260919(z^{-2} + z^3) + 0.238581(z^{-1} + z^2) + 0.022338(1 + z);$$

$$h_2(z) = 0.261180(z^{-2} - z^3) - 0.238819(z^{-1} - z^2);$$

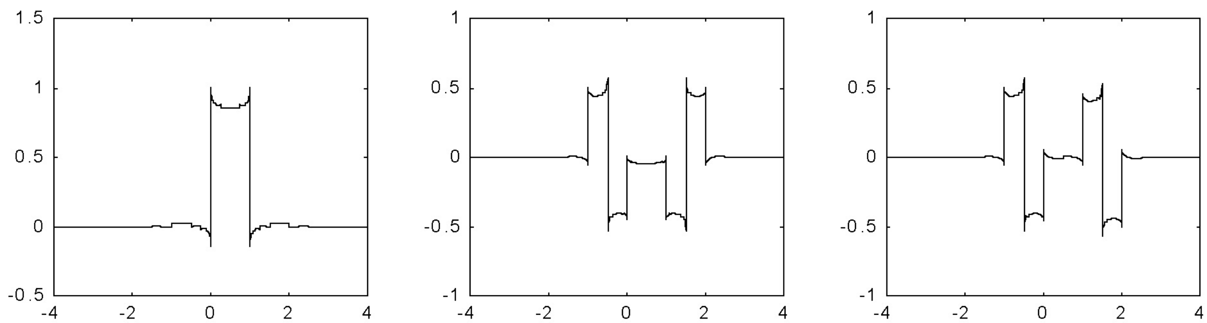

Fig. 12. $\deg(h_0) = 5$, $\alpha = 0.011680$, $\beta = -0.010680$, $\gamma = 0.499$

2) $\alpha = -0.043402$, $\beta = 0.068402$, $\gamma = 0.475$, (see Fig. 13).

$$h_1(z) = -0.189184(z^{-2} + z^3) + 0.298156(z^{-1} + z^2) - 0.108972(1 + z);$$

$$h_2(z) = 0.194098(z^{-2} - z^3) - 0.305901(z^{-1} - z^2);$$

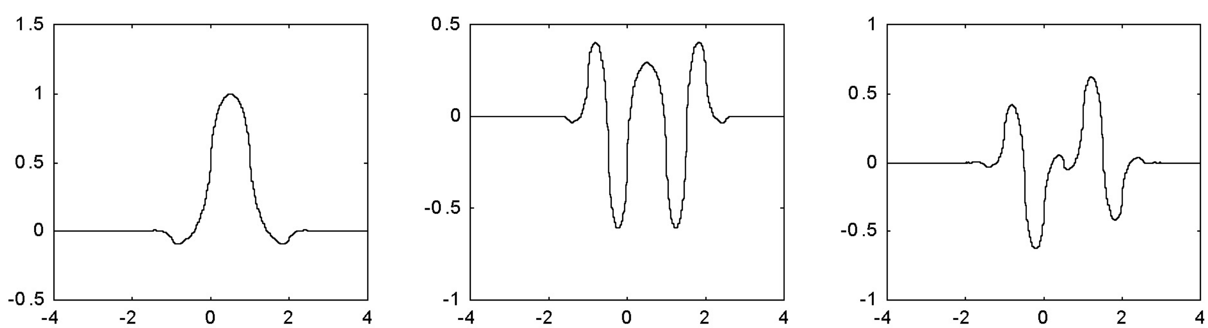

Fig. 13. $\deg(h_0) = 5$, $\alpha = -0.043402$, $\beta = 0.068402$, $\gamma = 0.475$

3) $\alpha = -0.051777$, $\beta = 0.301777$, $\gamma = 0.25$, (see Fig. 14).

$$h_1(z) = 0.051777(z^{-2} + z^3) - 0.301777(z^{-1} + z^2) - 0.25(1 + z);$$

$$h_2(z) = 0.073223(z^{-2} - z^3) - 0.426776(z^{-1} - z^2);$$

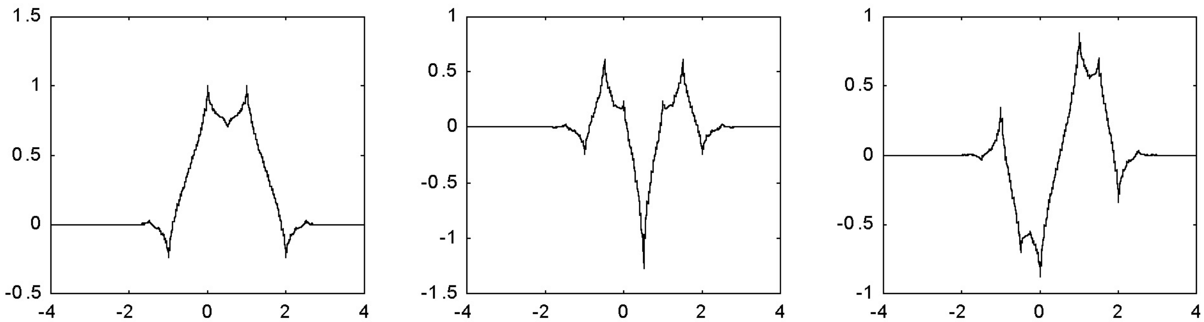

Fig. 14. $\deg(h_0) = 5$, $\alpha = -0.051777$, $\beta = 0.301777$, $\gamma = 0.25$

4.5 B-spline B_6

We give here symbols of framelets generated by the B-spline B_6 with the symbol

$$h_0(z) = z^{-3}\left(\frac{1+z}{2}\right)^7.$$

Symbols of framelets are defined by the relations

$$h_1(z) = -0.007813(z^{-3} - z^4) - 0.054687(z^{-2} + z^{-1} + z^2 - z^3) + 0.492187(1 - z);$$

$$h_2(z) = 0.041340(-z^{-2} + z^3) + 0.289379(-z^{-1} + z^2) + 0.248039(-1 + z).$$

Their plots are presented by Figure 15.

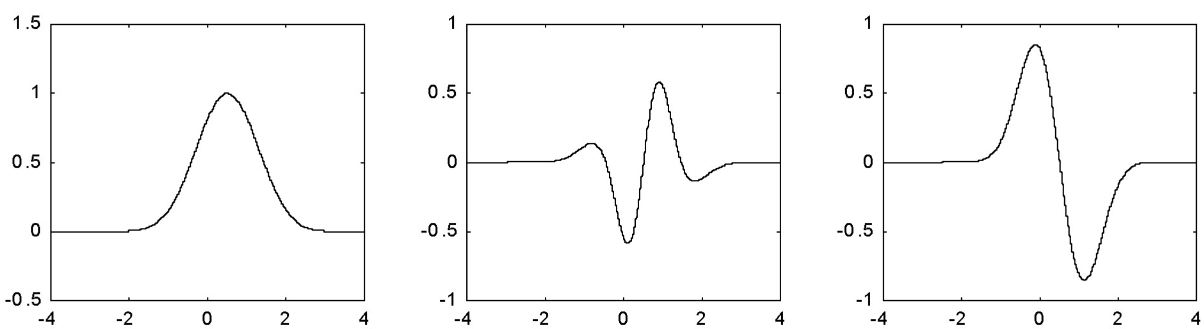

Fig. 15. $\deg(h_0) = 7$, The framelets associated with the spline B_6

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