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LATTICE POINTS CLOSE TO A SMOOTH CURVE AND SQUAREFULL NUMBERS IN SHORT INTERVALS

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ABSTRACT. We extend an approach of Swinnerton-Dyer and obtain new upper bounds for the number of lattice points close to a smooth curve. One consequence of these bounds is a new assymptotic result for the distribution of squarefull numbers in short intervals.

1. Introduction

A squarefull number is a positive integer n such that if p is a prime dividing n, then p^2 divides n. Let $Q(x)$ be the number of squarefull numbers which are $\leq x$. Bateman and Grosswald [1] proved that

$$Q(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + o\left(x^{1/6}\right).$$

Concerning the gaps between squarefull numbers, P.Shiu [17] proved that there exist infinitely many positive integers n such that there is no squarefull number between n^2 and $(n+1)^2$. On the other hand, since each perfect square is a squarefull number, Shiu's result answers completely the question of how large the gaps between squarefull numbers can be.

Another question is what is the distribution of squarefull numbers in short intervals. The result of Bateman and Grosswald implies

$$Q\left(x + x^{1/2+\theta}\right) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)}x^{\theta} \tag{1}$$

when $1/6 \leq \theta < 1/2$. Clearly (1) does not hold with $\theta = 0$. We would like to determine the smallest value of $\theta \in (0, 1/2)$ for which (1) holds. Smaller and smaller admissible values of θ were obtained by P.Shiu [18], P.G.Schmidt [16], C.-H.Jia [13], P.G.Schmidt [16], H.Liu [14], D.R.Heath-Brown [5], M.Filaseta and O.Trifonov [2],

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and M.Huxley and O.Trifonov [10], with the latest paper showing that (1) holds for any $\theta \in (1/8, 1/2)$. We improve on these results by proving

Theorem 1. *The asymptotic formula (1) holds for all* $19/154 = .1233\ldots < \theta < 1/2$.

The proof of Theorem 1 reduces to estimating the number of lattice points close to the curve $a^2b^3 = x$. Elementary estimates of this type were obtained in [7], [3], [9], and [8]. We improve on these estimates by extending an approach of Swinnerton-Dyer. In [20] Swinnerton-Dyer obtained upper bounds for the number of lattice points on a curve, provided the curve satisfies some simple analytic conditions. One way to get these bounds is to consider second divided differences of triples of lattice points on the curve. We extend the approach of Swinnerton-Dyer by working with third divided differences of quadruples of lattice points close to the smooth curve. This enables us to obtain

Theorem 2. *Let* $1 \leq C \leq M \leq T \leq M^2$, $f : [M, 2M] \mapsto R$ *have a continuous third derivative and*

$$0 < \frac{T}{CM^j} \leq |f^{(j)}(x)| \leq \frac{CT}{M^j}$$

for $j = 2, 3$ *and all* $x \in [M, 2M]$*. Let* $0 < \delta < 1/2$ *and* $\delta \leq (CM)^{-\frac{1}{2}}$*. Define* $S(f, \delta)$ *to be the set of all integer points* (x, y) *such that* $x \in [M, 2M]$ *and* $|f(x) - y| < \delta$*. Then for every* $\epsilon > 0$ *there exists a constant* $c(\epsilon) > 0$ *such that*

$$\begin{aligned} |S(f,\delta)| \quad &\ll \quad c(\epsilon)M^{\epsilon}C\big(M^{1/2}T^{4/27} + M^{12/25}T^{4/25} + M^{9/10}\delta^{4/15} + M^{12/13}\delta^{4/13} \\ &\qquad + M^{2/7}T^{2/7} + T/M + M^{1/2}T^{1/4}\delta^{1/4}\big) + C^5TM^{1/2}\delta^{5/2}. \end{aligned}$$

Theorem 2 is of interest in itself. We envision various applications of it in number theory problems. Theorem 2 gives best results when T is close to $M^{3/2}$. For instance one can get the following corollary.

Corollary *Let f and $S(f,\delta)$ be as in Theorem 2 with C an absolute constant, $0<\delta<(CM)^{-\frac{2}{3}}$, and let $T=M^{\frac{3}{2}}$. Then*

$$|S(f,\delta)| \ll c(\epsilon)M^{\frac{13}{18}+\epsilon}.$$

The best previous result, even when $\delta = 0$, was $|S| \ll M^{\frac{3}{4}}$ and $13/18 = .7222\ldots < .75$.

Since the proof of Theorem 2 consists of several independent steps, we present it in five sections. In Section 2 we state some well-known basic properties of divided differences; we prove some preliminary lemmas in Section 3; in Section 4 we investigate lattice points on quadratic major arcs; we estimate the number of integer solutions of a certain system of two equations in Section 5 and we prove Theorem 2 in Section 6. Section 7 is dedicated to the proof of Theorem 1.

Notation

We use the following notation throughout the whole paper:

C, T, and M are real numbers with $1 \le C \le M \le T \le M^2$.

δ is a real number in $(0, 1/8)$.

Let r be a positive integer. Define $\mathcal{F}_r$ to be the class of all real-valued functions which are defined on $I = [M, 2M]$, have a continuous r-th derivative on I and $\frac{T}{CM^r} \le |f^{(r)}(x)| \le \frac{CT}{M^r}$ for all $x \in I$.

Define $S(f,\delta)$ to be the set of all integer points (x, y) such that $x \in [M, 2M]$ and $|f(x) - y| < \delta$.

For any finite set S of points in the plane we denote by $\mathcal{P}(S)$ its projection on the x-axis, and by $|S|$ its cardinality.

We call the graph of any quadratic function $ax^2 + bx + c$ with a, b, and c real numbers a parabola.

$f(u) \ll g(u)$ will mean that there exists an absolute constant c such that $|f(u)| \le cg(u)$ for all $u \ge 1$. $f(u) \ll_\epsilon g(u)$ will mean that there exists a constant $c(\epsilon)$ such that $|f(u)| \le c(\epsilon)g(u)$ for all $u \ge 1$. Similarly, $f(u) \ll_C g(u)$ will mean that the implied constant in $\ll$ depends on C only.

For r and n positive integers, we denote by $d_r(n)$ the number of distinct solutions in positive integers of the equation $x_1 x_2 \cdots x_r = n$.

For x a real number we denote by $||x||$ the distance from x to the closest integer number.

We denote by R the set of real numbers, and by Z the set of integers.

To simplify the notation we adopt the convention $\epsilon \pm \epsilon = \epsilon$ and $c\epsilon = \epsilon$ whenever c is an absolute constant. There will be no division by ϵ or comparison of ϵ's in this paper.

2. Divided differences

Divided differences are in the foundation of our estimates. In this section we state some definitions and properties of divided differences which are well-known and can be found for example in [11].

Let f be a function defined on the reals and $x_0, x_1, \ldots, x_n$ be distinct real numbers. Then there is a unique polynomial $P(x)$ of degree n such that $P(x_j) = f(x_j)$ for $j = 0, 1, \ldots, n$. The coefficient of x^n in $P(x)$ is defined to be the n-th divided difference for f, and we denote it $f[x_0, x_1, \ldots, x_n]$. Denote $V = V(x_0, x_1, \ldots, x_n) = \prod_{0 \le k < j \le n}(x_j - x_k)$ and let $V_i = \prod(x_j - x_k)$ where the product is over all k and j such that $0 \le k < j \le n$, $k \ne i$, and $j \ne i$.

Lemma 1. $f[x_0, x_1, \ldots, x_n] = \dfrac{\sum_{i=0}^{n}(-1)^{n-i}V_i f(x_i)}{V}$.

Lemma 2. *Let* $m = \min(x_0, x_1, \ldots, x_n)$ *and* $M = \max(x_0, x_1, \ldots, x_n)$. *If* f *has continuous* n*-th derivative on* $[m, M]$ *then*

$$f[x_0, x_1, \ldots, x_n] = f^{(n)}(\xi)/n!$$

for some $\xi \in (m, M)$.

Lemma 3. $f[x_0, x_1, \ldots, x_n] = \dfrac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ f(x_0) & f(x_1) & \cdots & f(x_n) \end{vmatrix}}{V}$.

3. Preliminary Lemmas

This section contains three lemmas. The idea of the first two lemmas is that if some mild conditions on δ hold then one can find a subset of $S(f,\delta)$ which has the following properties:
(i) it essentially has the same size as $S(f,\delta)$;
(ii) it is either strictly convex or strictly concave depending on the sign of f'';
(iii) no two elements of the set are "too close".
The first lemma of this section is contained in Huxley [7, pp.204–206].

Lemma 4. *Let $f \in \mathcal{F}_2$, $T \geq M$ and $0 < \delta^2 < \frac{1}{4}\min_{x\in I}|f''(x)|$. Then there exist non-intersecting sets S_1 and S_2 such that*
(i) $S(f,\delta) = S_1 \bigcup S_2$;
(ii) S_1 is either strictly convex or strictly concave set, depending on the sign of f'';
(iii) $|S_2| \ll C\delta M + 1$.

The next lemma is a modification of Theorem 3 in [22].

Lemma 5. *Let $f \in \mathcal{F}_2 \bigcap \mathcal{F}_3$ and $S_1 \subset S(f,\delta)$. Assume that S_1 is either strictly convex or strictly concave set. Let $\frac{1}{16\delta} > E > 0$. Define S_3 to be the set of integer points (x_j, y_j) in S_1 such that there exist elements of S_1, say (x_i, y_i) and (x_k, y_k), such that $x_i < x_j < x_k$, $x_j - x_i < E$, and $x_k - x_j < E$. Then*

$$|S_3| \ll \frac{C^3 T\delta E^3}{M} + \frac{CTE^5}{M^2}.$$

Proof of Lemma 5.
Let a_1 and a_2 be positive integers. Define $T(a_1,a_2)$ to be the set of all positive integers n such that n, $n+a_1$, and $n+a_1+a_2$ are consecutive elements in $\mathcal{P}(S_1)$. Then by the definition of S_3 we have

$$|S_3| = \sum_{a_1<E, a_2<E} |T(a_1,a_2)|.$$

We will derive an estimate for $|T(a_1,a_2)|$. Fix $a_1 < E$ and $a_2 < E$, and let $d = \gcd(a_1,a_2)$. Let x_0 and x_0+u be two consecutive elements of $T(a_1,a_2)$. Then $x_0, x_0+a_1, x_0+a_1+a_2, x_0+u, x_0+u+a_1$, and $x_0+u+a_1+a_2$ are all in $\mathcal{P}(S_1)$. Denote $x_1 = x_0+a_1$, $x_2 = x_0+a_1+a_2$. Therefore $f(x_0) = y_0+\theta_0\delta$, $f(x_1) = y_1+\theta_1\delta$, and $f(x_2) = y_2+\theta_2\delta$ for some integers y_0, y_1, y_2 and some real numbers θ_i with $|\theta_i| < 1$ for $i = 0,1,2$. Denote $x_0' = x_0+u$, $x_1' = x_1+u$, and $x_2' = x_2+u$. Again, we have $f(x_0') = y_0' + \theta_0'\delta$, $f(x_1') = y_1' + \theta_1'\delta$, and $f(x_2') = y_2' + \theta_2'\delta$ for some integers y_0', y_1', y_2' and some real numbers θ_i' with $|\theta_i'| < 1$ for $i = 0, 1, 2$. By Lemma 1

(2) $a_1a_2(a_1+a_2)f[x_0, x_1, x_2] = a_2f(x_0)-(a_1+a_2)f(x_1)+a_1f(x_2) = m+\theta\delta(a_1+a_2)$

where $m = a_2y_0 - (a_1 + a_2)y_1 + a_1y_2$ and $\theta = \frac{1}{a_1+a_2}(a_2\theta_0 - (a_1 + a_2)\theta_1 + a_1\theta_2)$. Obviously $|\theta| < 2$. Also, m is an integer and $d \mid m$. Since S_1 is either strictly convex or strictly concave then $m \neq 0$. (If $m = 0$ then the lattice points $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2) are in S_1 and lie on a straight line.)

Note that the right-hand-side of (2) is $> d/2$ in absolute value since $|m| \geq d$ and $|\theta\delta(a_1 + a_2)| < 4\delta E < 1/4$. Furthermore, Lemma 2 implies $|f[x_0, x_1, x_2]| = |f''(\xi)|/2 < \frac{CT}{2M^2}$ with $\xi \in I$. Therefore, if $|T(a_1, a_2)| > 0$, then

$$a_1a_2(a_1 + a_2) \geq \frac{M^2d}{CT}. \tag{3}$$

Similarly we have

$$a_1a_2(a_1 + a_2)f[x_0', x_1', x_2'] = m' + \theta'\delta(a_1 + a_2) \tag{4}$$

where $m' = a_2y_0' - (a_1+a_2)y_1' + a_1y_2'$ and $\theta' = \frac{1}{a_1+a_2}(a_2\theta_0' - (a_1+a_2)\theta_1' + a_1\theta_2')$. Again $|\theta'| < 2$, m' is a nonzero integer and $d \mid m'$. Define a function $h(x) = f(x+u) - f(x)$. Subtracting (2) from (4) and taking into account Lemma 1 we get

$$a_1a_2(a_1 + a_2)h[x_0, x_1, x_2] = (m' - m) + (\theta' - \theta)\delta(a_1 + a_2). \tag{5}$$

¿From Lemma 2, $h[x_0, x_1, x_2] = \frac{h''(\eta)}{2} = \frac{f''(\eta+u)-f''(\eta)}{2} = \frac{uf'''(\rho)}{2}$ for some η and ρ in I. We consider two cases.

Case I: $m \neq m'$.

Since $d \mid m' - m$ we have $|m' - m| \geq d$ in this case. Also $|(\theta' - \theta)\delta(a_1 + a_2)| < 8E\delta < 1/2$. Therefore the right-hand-side of (5) is $> d/2$ in absolute value. We get $u \geq \dfrac{d}{|f'''(\rho)|a_1a_2(a_1 + a_2)} \geq \dfrac{M^3d}{CTa_1a_2(a_1 + a_2)} := P$ in case I.

Case II: $m = m'$.

In this case $m = a_2y_0 - (a_1 + a_2)y_1 + a_1y_2 = a_2y_0' - (a_1 + a_2)y_1' + a_1y_2' = m'$, so $a_2(y_0 - y_1 - y_0' + y_1') = a_1(y_1 - y_2 - y_1' + y_2')$. Note that $y_0 - y_1 - y_0' + y_1' \neq 0$. If not, the slope of the straight line through (x_0, y_0) and (x_1, y_1) equals the slope of the straight line through (x_0', y_0') and (x_1', y_1') which contradicts the strict convexity (concavity) of S_1. Therefore $|y_0 - y_1 - y_0' + y_1'| \geq \dfrac{a_1}{d}$.

Furthermore $f(x_0) - f(x_1) - f(x_0') + f(x_1') = h(x_1) - h(x_0) = a_1 h'(\xi_1) = a_1 u f''(\eta_1)$ with $\xi_1, \eta_1 \in I$. Also, $f(x_0) - f(x_1) - f(x_0') + f(x_1') = y_0 - y_1 - y_0' + y_1' + (\theta_0 - \theta_1 - \theta_0' + \theta_1')\delta$. Thus $a_1 u |f''(\eta_1)| \geq \frac{a_1}{d} - 4\delta \geq \frac{a_1}{2d}$ $(\delta < 1/8)$. We obtain

$$u \geq \frac{M^2}{2CTd} := R.$$

On the other hand, from (5) it follows that $a_1 a_2 (a_1 + a_2) |h[x_0, x_1, x_2]| \leq 2\delta(a_1 + a_2)$, so $a_1 a_2 \frac{u|f'''(\rho)|}{2} \leq 2\delta$, and we get

$$u \leq \frac{4C\delta M^3}{a_1 a_2 T} := Q.$$

Thus, we proved that the elements of $T(a_1, a_2)$ reside in at most $\dfrac{M}{P} + 1$ bunches, each of length $\leq Q$. The distance between any two elements of $T(a_1, a_2)$ which are in the same bunch is $\geq R$. Therefore $|T(a_1, a_2)| \leq \left(\dfrac{M}{P} + 1\right)\left(\dfrac{Q}{R} + 1\right)$. Note that (3) implies $\dfrac{M}{P} = \dfrac{CTa_1 a_2 (a_1 + a_2)}{M^2 d} \geq 1$. Therefore

$$(6) \qquad |T(a_1, a_2)| \leq \frac{2M}{P}\left(\frac{Q}{R} + 1\right) = \frac{16C^3 \delta (a_1 + a_2) T}{M} + \frac{2CTa_1 a_2 (a_1 + a_2)}{M^2 d}.$$

Using (6) and $|S_3| = \sum_{a_1 < E, a_2 < E} |T(a_1, a_2)|$, the proof of the lemma follows. □

An essential component of Swinnerton-Dyer's approach is the strict convexity (concavity) of the set $S(f, \delta)$. Since we are working with third divided differences rather than second, we need some sort of three-convexity condition. To ensure strict three-convexity (concavity) of $S(f, \delta)$ one needs to impose too rigid (for the applications we envision) conditions on δ. Since we need only local three-convexity (concavity) we circumvent the above obstacle by using the following combinatorial lemma and the estimates from Section 4.

Lemma 6. *Let k be a positive integer and let $T = \{(x_1, y_1), \dots, (x_{4k}, y_{4k})\}$ be a set of points in the plane with $x_1 < x_2 < \cdots < x_{4k}$. Then either there exists a parabola that passes through at least k elements of T, or there exists a subset of 5 distinct points of T such that no parabola passes through any 4 of these points.*

Proof of Lemma 6.

Denote by P_{123} the parabola which passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). If k or more elements of T lie on P_{123} then we are done. So, assume that $|P_{123} \bigcap T| \leq k-1$. Then there are elements of T which are not on P_{123}. Let (x_l, y_l) be one of them.

Let P_{12l} be the parabola through the points (x_1, y_1), (x_2, y_2), and (x_l, y_l), P_{13l} - the parabola through (x_1, y_1), (x_3, y_3), and (x_l, y_l), and P_{23l} - the parabola through (x_2, y_2), (x_3, y_3), and (x_l, y_l).

If $|P_{12l} \bigcap T| \geq k$, or $|P_{13l} \bigcap T| \geq k$, or $|P_{23l} \bigcap T| \geq k$, then we are done. Assume otherwise. Then $|(P_{123} \bigcup P_{12l} \bigcup P_{13l} \bigcup P_{23l}) \bigcap T| \leq 4k-4$ (actually one can easily replace $4k-4$ by $4k-12$). Thus there exists an element of T, say (x_m, y_m), which is not on $P_{123} \bigcup P_{12l} \bigcup P_{13l} \bigcup P_{23l}$.

Let $U = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_l, y_l), (x_m, y_m)\}$. Then U is a 5 element subset of T and no parabola passes through any 4 elements of U. $\square$

4. Lattice points on quadratic major arcs

To be able to use Lemma 6 we need a tool to deal with parabolas that contain lots of lattice points. To this end we use the concept of quadratic major arcs which was introduced by M.Huxley in [8]. Most of the material in this section is based on the papers [9] and [8].

Let $S = S(f, \delta)$, and let $W = \max\limits_{q \leq M^3} 2^{\omega(q)}$ where $\omega(q)$ is the number of distinct prime divisors of q. We call the set of points $\mathcal{A} = \{(x, P(x)) | x \in [\alpha, \beta]\}$ a quadratic major arc if $P(x)$ is a quadratic polynomial in x, $(\alpha, P(\alpha)) \in S$, $(\beta, P(\beta)) \in S$, and $\mathcal{A}$ contains at least $10W$ elements of S. In other words, a quadratic major arcs is a part of parabola that contains at least $10W$ lattice points which are close to the graph of f. Let $\mathcal{Q}$ be a union of quadratic major arcs whose projections on the x-axis do not intersect. The aim of this section is to obtain an upper bound for $|\mathcal{Q} \bigcap S|$. We assume that $f \in \mathcal{F}_2 \bigcap \mathcal{F}_3$ and $\delta^2 \leq \frac{1}{4} \min\limits_{x \in [M, 2M]} |f''(x)|$.

First, we use the following Lemma which was proved in [9].

Lemma 7. *Let $P(x)$ be a quadratic trinomial and $f(x) \in \mathcal{F}_2 \bigcap \mathcal{F}_3$. Then the set $\{x : |f(x) - P(x)| < \delta\}$ is a union of at most three disjoint open intervals and the length of each of these intervals does not exceed $\frac{M}{2} \left(\frac{C\delta}{T}\right)^{1/3}$.*

Let $\mathcal{A} = \{(x, P(x))|x \in [\alpha, \beta]\}$ be a quadratic major arc in $\mathcal{Q}$. Let J_1, J_2, and J_3 be disjoint open intervals such that $\{x : |f(x) - P(x)| < \delta\} = J_1 \bigcup J_2 \bigcup J_3$ (one or two of these intervals could be the empty set). Denote $\{(x, P(x))|x \in J_k\}$ by R_k for $k = 1, 2, 3$. Clearly, $\mathcal{A} \bigcap S \subset R_1 \bigcup R_2 \bigcup R_3$. Let $s \in \{1, 2, 3\}$ be such that $|R_s \bigcap S| = \max_{k=1,2,3} |R_k \bigcap S|$. Since $|\mathcal{A} \bigcap S| \geq 10W$ we have $|J_s \bigcap S| \geq 3W + 1$. *W.l.o.g.* assume $s = 1$.

Note that if a parabola passes through the distinct lattice points (x_i, y_i), (x_j, y_j), and (x_k, y_k), then from Lagrange's interpolation formula follows $P(x) = \frac{ax^2+bx+c}{q}$ where a, b, c, and q are integers, $\gcd(a, b, c) = 1$, and $q|(x_j - x_i)(x_k - x_i)(x_k - x_j)$. Then the lattice points on the graph of $y = P(x)$ are all points of the form $(n, P(n))$ with n an integer solution of the congruence $ax^2 + bx + c \equiv 0 \pmod{q}$. Next we make use of a standard lemma on congruences, stated in [8].

Lemma 8. *The solutions of the quadratic congruence $ax^2 + bx + c \equiv 0$ (mod q) form a union of residue classes to a modulus q' with $q'|q$, $q|q'^2$. The number of such residue classes modulo q' is at most $2^{\omega(q'^2/q)}$.*

Note that since $q'^2/q < q < M^3$, W is an upper bound for the number of residue classes in the above lemma. Let D be the set of lattice points in $R_1 \bigcap S$ which are in the residue class modulo q' containing the most elements of $R_1 \bigcap S$. Then $|R_1 \bigcap S| \leq W|D|$, and $|D| \geq |R_1 \bigcap S|/W > 3$.

Suppose $\mathcal{Q} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \cdots \bigcup \mathcal{A}_l$ with $\mathcal{P}(\mathcal{A}_i) \bigcup \mathcal{P}(\mathcal{A}_j) = \emptyset$ when $i \neq j$. For each $\mathcal{A}_i$, $i = 1, 2, \ldots, l$ we perform the procedure we did on M and obtain a corresponding set of lattice points D_i. The sets we get have the following properties:

(i) $|\mathcal{A}_i \bigcap S| \leq 3W|D_i|$ and $|D_i| \geq 4$ for $i = 1, 2, \ldots, l$;

(ii) All points in D_i lie on a parabola $P_i(x) = \dfrac{a_i x^2 + b_i x + c_i}{q_i}$ where a_i, b_i, c_i, and q_i are integers with $\gcd(a_i, b_i, c_i) = 1$, and $q_i > 0$;

(iii) We have $D_i = \{(x_1^{(i)}, P_i(x_1^{(i)})), \ldots, (x_{k_i}^{(i)}, P_i(x_{k_i}^{(i)}))\}$ where $x_1^{(i)}, x_2^{(i)}, \cdots, x_{k_i}^{(i)}$ form an arithmetic progression to some modulus q_i' with $q_i \mid q_i'^2$;

(iv) $|P_i(x) - f(x)| < \delta$ for all $x \in [x_1', x_{k_i}']$, and $x_{k_i}' - x_1' \leq \frac{M}{2}\left(\frac{C\delta}{T}\right)^{1/3}$.

Now we make use of Lemma 22 from [8] which provides an upper bound for the number of lattice points on the D_i's with a fixed leading coefficient a/q.

Lemma 9. *Let $q \le 1/(2\delta)$ be a positive integer. Denote by q'' the least positive integer such that $q \mid q''^2$. Then*

$$\sum_{\frac{a_i}{q_i}=\frac{a}{q}} |D_i| \ll \frac{C^{1/3}\delta^{1/3}M}{T^{1/3}q''} + \frac{C^{4/3}\delta M^2 q^{1/3}}{T^{2/3}q''^2} + \frac{C\delta^{3/2}M^3q^{1/2}}{Tq''^3}.$$

The only difference between Lemma 9 and Lemma 22 in [8] is the notation (we have $\bigcup D_i$ instead of S', and T/M^2 instead of Δ).

Let $A \le 1/(2\delta)$ be a parameter whose value we specify latter on. Property (iii) of the D_i's implies

$$\sum_{q_i' \ge A} |D_i| \le \frac{M}{A} + 1.$$

The next lemma is a slight variation of Lemma 23 in [8].

Lemma 10. *We have*

$$\sum_{q_i' < A} |D_i| \ll C^{4/3}\delta^{1/3}T^{2/3}AM^{-1} + C^{7/3}\delta T^{1/3}A^{5/3} + C^2\delta^{3/2}AM + C\delta M + 1.$$

Proof of Lemma 10.

First, if $\dfrac{a_i}{q_i} = 0$ then the elements of D_i lie on a straight line. In [7, pp.205-206] M.Huxley proved that there are $\ll C\delta M + 1$ elements of $S(f,\delta)$ which lie on straight lines each of which contains at least three elements of $S(f,\delta)$. This estimate holds provided $f \in \mathcal{F}_2 \bigcap \mathcal{F}_3$, $T \ge M$, and $\delta^2 \le \frac{1}{4}\min_{x\in[M,2M]} |f''(x)|$ (and we assumed f satisfies these conditions in the beginning of the section).

Now, let $\dfrac{a_i}{q_i} \ne 0$. Consider the divided difference $f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}]$. From Lemma 1, the properties of D_i, and $x_j^{(i)} \in S(f,\delta)$ for $j = 1,2,3$, we get $f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}] = \frac{n_i+\theta_i\delta}{2q_i'^2}$ where $|\theta_i| < 4$, n_i is an integer, and $\frac{n_i}{2q_i'^2} = \frac{a_i}{q_i}$. Since $a_i \ne 0$, $n_i \ne 0$. Therefore $|n_i + \theta_i\delta| \ge |n_i|/2 \ge 1/2$. From Lemma 2 $f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}] = f''(\xi_i)/2$ for some $\xi_i \in [M, 2M]$. Thus $\frac{CT}{2M^2} \ge \frac{|f''(\xi_i)|}{2} \ge \frac{|n_i|}{4q_i'^2} = \frac{|a_i|}{2q_i}$, and $q_i \ge \frac{M^2}{CT}$. Therefore $\frac{a_i}{q_i}$ is within $\frac{2\delta}{q_i'^2} \le \frac{2CT\delta}{M^2}$ of $f''(\xi_i)/2$. Since the range of f'' is an interval of length $< \frac{CT}{M^2}$, the range of the non-zero $\frac{a_i}{q_i}$'s is an interval of length at most $\frac{2CT}{M^2}$. Therefore for each $q < A$ there are $< \frac{2qCT}{M^2} + 1 < \frac{3qCT}{M^2}$ admissible values of a. Let $q = st^2$, where s is squarefree. Then $q'' = st$, and $q'' \le q_i' < A$. Using Lemma 9 we get

$$\sum_{q_i'<A, a_i\ne 0} |D_i| \ll \sum_{st<A} \frac{st^2CT}{M^2}\left(\frac{C^{1/3}\delta^{1/3}M}{T^{1/3}st} + \frac{C^{4/3}\delta M^2(st^2)^{1/3}}{T^{2/3}(st)^2} + \frac{C\delta^{3/2}M^3(st^2)^{1/2}}{T(st)^3}\right).$$

Since $\sum_{st<A} t \ll A^2$, $\sum_{st<A} s^{-2/3}t^{2/3} \ll A^{5/3}$, and $\sum_{st<A} s^{-3/2} \ll A$, we get the statement of the lemma. □

Therefore we have

$$\sum |D_i| \ll C^{4/3}\delta^{1/3}T^{2/3}AM^{-1} + C^{7/3}\delta T^{1/3}A^{5/3} + C^2\delta^{3/2}AM + C\delta M + 1 + \frac{M}{A}$$

for any $0 < A \le 1/(2\delta)$. Optimizing with respect to A (e.g. Lemma 2.4 of Graham and Kolesnik [4]) we obtain

(7)

$$|\mathcal{Q}\bigcap S| \le 3W\sum |D_i| \ll W\left(C^{2/3}\delta^{1/6}T^{1/3} + C^{7/8}\delta^{3/8}T^{1/8}M^{5/8} + C\delta^{3/4}M + 1\right).$$

Note that we omit the term $C\delta M$ since it is absorbed by the term $C\delta^{3/4}M$.

5. On the number of integer solutions of a system of two equations

Let A and B be positive real numbers. One of the key elements in the proof of Theorem 2 is an upper bound for the number of integer solutions of the system

$$\begin{aligned} D_1 + D_3 &\ne D_2 + D_4 \\ z_1D_1 + z_3D_3 &= z_2D_2 + z_4D_4 \\ z_1^2D_1 + z_3^2D_3 &= z_2^2D_2 + z_4^2D_4 \end{aligned} \tag{8}$$

which satisfy the conditions

$$0 < z_1 < z_2 < z_3 < z_4 \le A, \quad 0 < |D_j| \le B \text{ for } j = 1, 2, 3, 4 \tag{9}$$

and

$$\gcd(z_1, z_2, z_3) | \gcd(D_1, D_2, D_3, D_4). \tag{10}$$

Our main tool will be Corollary 2 of Heath-Brown [6].

Lemma 11. (Heath-Brown) *Let q be a nonsingular integral ternary quadratic form with matrix $\mathcal{M}$. Let $\Delta = |\det(\mathcal{M})|$, and assume that $\Delta \ne 0$. Write Δ_0 for the highest common factor of the 2×2 minors of $\mathcal{M}$. Then the number of primitive* *integer solutions of* $q(\mathbf{x}) = 0$ *in the box* $|x_i| \le R_i$ *is*

$$\ll_\epsilon \left\{ 1 + \left(\frac{R_1 R_2 R_3 \Delta_0^2}{\Delta} \right)^{1/3+\epsilon} \right\} d_3(\Delta) \tag{11}$$

for any $\epsilon > 0$.

Now we prove

Lemma 12. *The number of integer solutions of the system (8) which satisfy (9), (10), and* $\gcd(z_1, z_2, z_3) = d$ *is*

$$\ll_\epsilon B^\epsilon \left(\frac{A^{1+\epsilon} B^{8/3}}{d^{11/3}} + \frac{B^4}{d^4} \right).$$

Proof of Lemma 12.
¿From (8) $z_4 = \frac{z_1 D_1 - z_2 D_2 + z_3 D_3}{D_4}$. Substituting in the last equation of (8) we get

$$(D_1^2 - D_1 D_4) z_1^2 + (D_2^2 + D_2 D_4) z_2^2 + (D_3^2 - D_3 D_4) z_3^2 \tag{12}$$
$$-2 D_1 D_2 z_1 z_2 + 2 D_1 D_3 z_1 z_3 - 2 D_2 D_3 z_2 z_3 = 0$$

which can be treated as a ternary quadratic form in z_1, z_2, and z_3. The absolute value of its determinant is $\Delta = | - D_0 D_1 D_2 D_3 D_4^2| \neq 0$ where we have denoted $D_0 = (D_1 + D_3) - (D_2 + D_4)$. From (8) and (9) we get $0 < |D_0| \le 4B$. Also,

$$\begin{aligned} \Delta_0 \;=\; \gcd(\;& D_1 D_2 D_3 D_4,\ D_1 D_2 D_4 (D_1 - D_2 - D_4), \\ & D_1 D_3 D_4 (D_4 - D_1 - D_3),\ D_2 D_3 D_4 (D_3 - D_2 - D_4)). \end{aligned}$$

Therefore $\Delta_0 = |D_4| \Delta_1$ where $\Delta_1 = \gcd(D_1 D_2 D_3, D_0 D_1 D_2, D_0 D_1 D_3, D_0 D_2 D_3)$, and $\dfrac{\Delta_0^2}{\Delta} = \dfrac{\Delta_1^2}{|D_0 D_1 D_2 D_3|}$.
Let p be a prime dividing $D_0 D_1 D_2 D_3$. Let $p^{\alpha_j} || D_j$ for $j = 0, 1, 2, 3$. Then $p^\alpha || D_0 D_1 D_2 D_3$ where $\alpha = \sum_{j=0}^3 \alpha_j$. Let $\beta = \alpha - \max(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Clearly $p^\beta || \Delta_1$. From the definition of α and β we have $\beta \le \frac{3\alpha}{4}$. Thus, if $\alpha = 1$, $\beta = 0$, and if $\alpha = 2$, $\beta \le 1$.

Each nonzero integer t can be represented in the form $t = \pm a(t) b^2(t) c(t)$ where $a(t)$ and $b(t)$ are squarefree numbers, $c(t)$ is a cubefull number, and $\gcd(a(t), b(t)) = \gcd(b(t), c(t)) = \gcd(a(t), c(t)) = 1$. Hence, if $|D_0 D_1 D_2 D_3| = t$ then $\Delta_1 \le b(t) c^{3/4}(t)$ and $\dfrac{\Delta_0^2}{\Delta} \le \dfrac{c^{1/2}(t)}{a(t)}$.

We estimate the number of solutions $N(G)$ of (12) which satisfy (9), $G \le |D_0D_1D_2D_3| < 2G$, and $\gcd(z_1,z_2,z_3)=1$. Lemma 11 holds. Also, for a fixed integer $t \in [G, 2G)$ there are $16d_4(t) \ll_\epsilon t^\epsilon \ll_\epsilon G^\epsilon \ll_\epsilon B^{4\epsilon}$ quadruples (D_0, D_1, D_2, D_3) with $|D_0D_1D_2D_3| = t$. Applying Lemma 11 we get

$$N(G) \ll_\epsilon B^\epsilon \sum_{G \le t < 2G} \left(1 + A^{1+\epsilon} \frac{c^{1/6}(t)}{a^{1/3}(t)}\right),$$

where we have used $c(t) \le t \le G \le B^4$ and $d_3(\Delta) \ll_\epsilon \Delta^\epsilon \ll_\epsilon B^{6\epsilon}$.

Now we prove that for each $G \ge 1$,

$$\Sigma := \sum_{G \le t < 2G} \frac{c^{1/6}(t)}{a^{1/3}(t)} \ll G^{2/3}. \tag{13}$$

For k a non-negative integer define $I_k = [2^k, 2^{k+1})$ and let $\mathcal{C}_3$ be the set of cubefull numbers. We have $\Sigma \le \sum_{k,l} \sum_{b \in I_k} \sum_{c \in I_l \bigcap \mathcal{C}_3} U(G, b, c)$ where the first summation is over all non-negative integers k and l with $2k + l \le \log_2 16G$ and $U(G, b, c) = \sum_a \frac{c^{1/6}}{a^{1/3}}$ where the summation is over all squarefree integers a such that $G \le ab^2c < 2G$. Clearly, there are at most $\frac{2G}{b^2c}$ integers a with the above property and for these $a \ge G/(b^2c)$. Thus $U(G, b, c) \le \dfrac{2G^{2/3}}{b^{4/3}c^{1/2}} \le \dfrac{2G^{2/3}}{2^{4k/3}2^{l/2}}$. Since there are 2^k integers in I_k, and $\ll 2^{l/3}$ cubefull integers in I_l (e.g. see [12]) we get $\Sigma \ll \sum_{k,l} \dfrac{G^{2/3}}{2^{k/3}2^{l/6}} \ll G^{2/3}$.

Using (13) we obtain $N(G) \ll_\epsilon B^\epsilon \left(G + A^{1+\epsilon} G^{2/3}\right)$. Summing the above estimate with $G = 2^m$ for $0 \le m \le 4 \log_2 B$ we get that the number of primitive solutions of (12) which satisfy (9), $\gcd(z_1, z_2, z_3) = 1$, and $0 < |D_0D_1D_2D_3| \le B^4$ is $\ll_\epsilon B^\epsilon \left(B^4 + A^{1+\epsilon} B^{8/3}\right)$. Thus, we proved the lemma when $\gcd(z_1, z_2, z_3) = 1$.

Now, let $\gcd(z_1, z_2, z_3) = d > 1$. From (10) $d | D_j$ for $j = 0, \dots, 4$. Let $D'_j = D_j/d$ and $z'_j = z_j/d$. Then $\gcd(z'_1, z'_2, z'_3) = 1$, $0 < z'_j \le A/d$, for $j = 1, 2, 3$ and $0 < |D'_j| \le B/d$ for $j = 1, 2, 3, 4$. Equation (12) holds with the z_j replaced by z'_j and the D_j replaced by D'_j. Applying the estimate for the case $d = 1$ we get the statement of the lemma. □

6. Proof of Theorem 2

First, by Lemma 4 $S(f, \delta) = S_1 \bigcup S_2$, where S_1 is either strictly convex or strictly concave set, depending on the sign of f'', and $|S_2| \ll C\delta M + 1$. Next, let $E <$

$1/(16\delta)$ be a positive parameter whose value we specify later on. We apply Lemma 5 with E to S_1 and obtain a subset S_3 that contains the elements of S_1 which are "close" to both of their neighbors. (The point $(x_i, y_i) \in S_3$ if $(x_i, y_i) \in S_1$ and there exist $(x_j, y_j), (x_k, y_k) \in S_1$ with $x_j < x_i < x_k$ and $x_i - x_j < E$, $x_k - x_i < E$.) We discard all elements of S_3 from S_1, and then discard every other element of the remaining set. We denote what is left by S_4. Clearly $|S_1| \leq 2|S_4| + |S_3|$. Furthermore, S_4 has the property that if (x_i, y_i) and (x_j, y_j) are distinct elements of S_4 then $|x_j - x_i| \geq E$.

Next, we split the set S_4 into disjoint subsets, each containing $40W$ consecutive elements of S_4 (the last subset may contain less elements). According to Lemma 6 from each subset we can either select $10W$ points which are on one and the same parabola (a quadratic major arc), or we can select a quintuple of points such that no four of them lie on the same parabola. Therefore there exists disjoint sets S_5 and S_6 such that:

(i) $S_4 = S_5 \bigcup S_6$;

(ii) $S_5 = \bigcup_{j=1}^{k} \{(x_1^{(j)}, y_1^{(j)}), (x_2^{(j)}, y_2^{(j)}), (x_3^{(j)}, y_3^{(j)}), (x_4^{(j)}, y_4^{(j)}), (x_5^{(j)}, y_5^{(j)})\}$
with $x_1^{(1)} < x_2^{(1)} < \cdots < x_5^{(1)} < x_1^{(2)} < \cdots < x_5^{(2)} < \cdots < x_1^{(k)} < \cdots x_5^{(k)}$.

We call the set $\{(x_1^{(j)}, y_1^{(j)}), (x_2^{(j)}, y_2^{(j)}), (x_3^{(j)}, y_3^{(j)}), (x_4^{(j)}, y_4^{(j)}), (x_5^{(j)}, y_5^{(j)})\}$ the j-th quintuple, and each quintuple has the property that no four of its elements lie on the same parabola;

(iii) S_6 is a union of quadratic major arcs whose projections on the x-axis do not intersect;

(iv) $|S_4| \leq 8W|S_5| + 4|S_6| + 40W$.

Therefore

$$|S(f, \delta)| \ll C\delta M + |S_3| + W + |S_5| + |S_6|. \tag{14}$$

Note that from (7) we have

$$|S_6| \ll W \left(C^{2/3}\delta^{1/6}T^{1/3} + C^{7/8}\delta^{3/8}T^{1/8}M^{5/8} + C\delta^{3/4}M + 1 \right). \tag{15}$$

Using Lemma 5 one can get an estimate for $|S_3|$. What is left is to get an estimate for $|S_5|$. Consider the j-th quintuple of consecutive integer points in S_5, $\{(x_i^{(j)}, y_i^{(j)}) \quad i = 1, \ldots, 5\}$. Let A be a positive fixed parameter whose value we specify later on. There are $\le \frac{M}{A}+1$ quintuples with $x_5^{(j)} - x_1^{(j)} > A$. Now, let us estimate the number of quintuples with $x_5^{(j)} - x_1^{(j)} \le A$. We will concentrate on just one quintuple and to simplify the notation denote $x_i^{(j)}$ by x_i and $y_i^{(j)}$ by y_i for $i = 1, \dots, 5$. Define $a_i = x_{i+2} - x_{i+1}$ and $b_i = y_{i+2} - y_{i+1}$ for $i = 0, 1, 2, 3$. We have $a_0 + a_1 + a_2 + a_3 \le A$. Also $a_i \ge E$ for $i = 0, 1, 2, 3$.

Consider the following matrix

$$\mathcal{R} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_5 \\ x_1^2 & x_2^2 & \cdots & x_5^2 \\ x_1^3 & x_2^3 & \cdots & x_5^3 \\ y_1 & y_2 & \cdots & y_5 \end{bmatrix}.$$

Denote $D_{s-1} = \det(\mathcal{R}_{4,s})$ for $s = 1, 2, 3, 4, 5$ (here $\mathcal{R}_{4,s}$ is the $(4, s)$ minor of the matrix $\mathcal{R}$). Thus, we defined a mapping $\mathcal{H}$ from the set of quintuples in S_5 to the set of thirteentuples $(a_0, \dots, a_3, b_0, \dots, b_3, D_0, \dots, D_4)$. Note, that the mapping is one-to-one. Indeed, if the i-th and the j-th quintuples map to the same thirteentuple $(a_0, \dots, b_0, \dots, D_4)$, then $\dfrac{y_2^{(i)} - y_1^{(i)}}{x_2^{(i)} - x_1^{(i)}} = \dfrac{b_0}{a_0} = \dfrac{y_2^{(j)} - y_1^{(j)}}{x_2^{(j)} - x_1^{(j)}}$ which contradicts the strong convexity (concavity) of S_5. (We have $S_5 \subset S_4 \subset S_1$ and S_1 is strictly convex (concave).) Thus, $|S_5| = |\mathcal{H}(S_5)|$. So, instead of estimating the size of S_5 we estimate the number of thirteentuples in the image of S_5.

Next, note that the following identities hold.

$$\begin{array}{ccccccccccc} D_0 & - & D_1 & + & D_2 & -D_3 & + & D_4 & = & 0 \\ x_1 D_0 & - & x_2 D_1 & + & x_3 D_2 & -x_4 D_3 & + & x_5 D_4 & = & 0 \\ x_1^2 D_0 & - & x_2^2 D_1 & + & x_3^2 D_2 & -x_4^2 D_3 & + & x_5^2 D_4 & = & 0 \\ x_1^3 D_0 & - & x_2^3 D_1 & + & x_3^3 D_2 & -x_4^3 D_3 & + & x_5^3 D_4 & = & -\det(\mathcal{R}) \end{array}$$

Also, if $(x_1, x_2, x_3, x_4, x_5)$ satisfies the above system of equations so does $(x_1 + c, x_2 + c, x_3 + c, x_4 + c, x_5 + c)$ where c is any real number. Take $c = -x_1$ and denote $z_j = x_{j+1} - x_1$, $j = 1, 2, 3, 4$. (We have $z_1 = a_0$, $z_2 = a_0 + a_1$, $z_3 = a_0 + a_1 + a_2$, $z_4 = a_0 + a_1 + a_2 + a_3 \le A$, $E \le z_1 < z_2 < z_3 < z_4$, and $E \le z_j - z_{j-1}$ for $j = 2, 3, 4$.)

Then

$$
\begin{array}{rcccccccl}
- & z_1D_1 & + & z_2D_2 & -z_3D_3 & + & z_4D_4 & = & 0 \\
- & z_1^2D_1 & + & z_2^2D_2 & -z_3^2D_3 & + & z_4^2D_4 & = & 0 \\
- & z_1^3D_1 & + & z_2^3D_2 & -z_3^3D_3 & + & z_4^3D_4 & = & -\det(\mathcal{R})
\end{array}
$$

Lemma 3 implies $D_j = 0$ if and only if the four points of the quintuple minus (x_j, y_j) are on the same parabola. Since no four points of the quintuple are on the same parabola we have $D_j \neq 0$ for $j = 0, \ldots, 4$. We can also bound from above the absolute values of the D_j's. For instance, from Lemma 3 follows

$$f[x_1, x_2, x_3, x_4]V(x_1, x_2, x_3, x_4) = D_4 + \theta\delta A^3$$

where $|\theta| < 4$. Therefore

$$|D_4| \le A^6 \frac{CT}{M^3} + 4\delta A^3 := B.$$

Working similarly, one can prove $0 < |D_j| \le B$ for each $j = 0, \ldots, 4$. Let $d = \gcd(z_1, z_2, z_3)$. From the definition of z_1, z_2, z_3, $\gcd(z_1, z_2, z_3) = \gcd(a_0, a_1, a_2)$. Next, we prove that $d|D_j$ for $j = 0, \ldots, 4$.

By definition $D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$. Subtracting the third column from the fourth, the second from the third, the first from the second, and expanding with respect to the first row we get $D_4 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_0(x_1+x_2) & a_1(x_2+x_3) & a_2(x_3+x_4) \\ b_0 & b_1 & b_2 \end{vmatrix}$. Subtracting the first row multiplied by $x_2 + x_3$ from the second row, and expanding with respect to the second row we get

$$D_4 = a_0(a_0 + a_1)(a_1b_2 - a_2b_1) - a_2(a_1 + a_2)(a_0b_1 - a_1b_0). \tag{16}$$

Clearly, $d^3|D_4$.

Applying the same column operations and similar row operations to the determinants defining D_1, D_2, and D_3 we get $D_1 = (a_0 + a_1)(a_0 + a_1 + a_2)((a_0 + a_1)b_3 - (b_0 + b_1)a_3) - a_3(a_2 + a_3)((a_0 + a_1)b_2 - a_2(b_0 + b_1))$,

$D_2 = a_0(a_0 + a_1 + a_2)((a_1 + a_2)b_3 - a_3(b_1 + b_2)) - a_3(a_1 + a_2 + a_3)((a_1 + a_2)b_0 - a_0(b_1 + b_2))$, and $D_3 = a_0(a_0+a_1)(a_1(b_2+b_3)-(a_2+a_3)b_1)-(a_2+a_3)(a_1+a_2+a_3)(a_0b_1-a_1b_0)$.

¿From the above formulas follows that $d|D_1$, $d|D_2$, and $d|D_3$. Since $0 \neq D_0 = D_1 - D_2 + D_3 - D_4$, then $d|D_0$ as well, and $D_1 + D_3 \neq D_2 + D_4$.

Consider the projection

$$(a_0, \dots, a_3, b_0, \dots, b_3, D_0, \dots, D_4) \to (a_0, \dots, a_3, D_0 \dots, D_4)$$

defined on $\mathcal{H}(S_5)$. Note that all ninetuples we get from the projection satisfy the conditions (8), (9), (10). Assume that

$$\gcd(a_0, a_1, a_2) = d. \tag{17}$$

¿From Lemma 12, the number of ninetuples in the projection of $\mathcal{H}(S_5)$ which satisfy $a_0 + a_1 + a_2 + a_3 \leq A$ is

$$\ll_\epsilon B^\epsilon \left(\frac{A^{1+\epsilon} B^{8/3}}{d^{11/3}} + \frac{B^4}{d^4} \right).$$

Next, we estimate the number of distinct thirteentuples in $\mathcal{H}(S_5)$ which have the same a's and D's (project to the same ninetuple).

Claim: If $\gcd(a_0, a_1, a_2) = d$ and

$$A \leq \min(\frac{1}{8\delta}, \frac{1}{2}M^{5/7}(CT)^{-2/7}, \frac{1}{3}M^{1/2}(\delta CT)^{-1/4}, M^2/(CT)) \tag{18}$$

then there are $\ll d$ elements of $\mathcal{H}(S_5)$ of the form

$$(a_0, a_1, a_2, a_3, \dots, D_0, D_1, D_2, D_3, D_4).$$

Suppose the i-th quintuple maps to $(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, D_0, D_1, D_2, D_3, D_4)$ and the j-th quintuple ($j > i$), to $(a_0, a_1, a_2, a_3, b_0', b_1', b_2', b_3', D_0, D_1, D_2, D_3, D_4)$. From (16) follows that $D_4 = a_2(a_1 + a_2)d_1 - a_0(a_0 + a_1)d_2 = a_2(a_1 + a_2)d_1' - a_0(a_0 + a_1)d_2'$ where $d_1 = a_1b_0 - a_0b_1$, $d_2 = a_2b_1 - a_1b_2$, $d_1' = a_1b_0' - a_0b_1'$, and $d_2' = a_2b_1' - a_1b_2'$.

The first step in proving the claim will be to show $d_1 = d_1'$ and $d_2 = d_2'$. We have $(d_1 - d_1')a_2(a_1 + a_2) = (d_2 - d_2')a_0(a_0 + a_1)$. Let $a = \gcd(a_2(a_1 + a_2), a_0(a_0 + a_1))$. Then $\frac{a_0(a_0 + a_1)}{a} | d_1 - d_1'$. Using Lemma 1 we obtain

$$|f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}] - \frac{d_1}{a_0a_1(a_0 + a_1)}| \leq \frac{2\delta}{a_0a_1}. \tag{19}$$

So, $|d_1| \le 2\delta(a_0+a_1)+a_0a_1(a_0+a_1)\frac{CT}{M^2}$ and using the same argument one obtains the same upper bound for $|d_1'|$. If we assume $d_1 \ne d_1'$ we get

$$\frac{a_0(a_0+a_1)}{a} \le 4\delta(a_0+a_1)+2a_0a_1(a_0+a_1)\frac{CT}{M^2}, \text{ and } 1 \le \frac{4a\delta}{a_0}+2aa_1\frac{CT}{M^2}.$$

Since $A\delta < \frac{1}{8}$, and $a \le a_0(a_0+a_1) \le a_0A$, we obtain $a \ge \dfrac{M^2}{4CTA}$. From (16) follows that $a|D_4$. Using that

$$B = A^6\frac{CT}{M^3}+4\delta A^3 \ge |D_4| \ge a \ge \frac{M^2}{4CTA}$$

we obtain a contradiction with (18). Therefore $d_1 = d_1'$ and $d_2 = d_2'$.

Similarly, we prove that D_4 determines uniquely $d_3 = a_2(b_0+b_1)-b_2(a_0+a_1)$ and $d_4 = b_0(a_1+a_2)-a_0(b_1+b_2)$. Note that D_4 also equals

$$(a_0+a_1)a_2(b_0(a_1+a_2)-a_0(b_1+b_2))-a_0(a_1+a_2)(a_2(b_0+b_1)-b_2(a_0+a_1)).$$

Therefore D_4 determines uniquely d_1, d_2, and $a_0b_2-a_2b_0$. Thus $\dfrac{b_i}{a_i}-\dfrac{b_j}{a_j}=\dfrac{b_i'}{a_i}-\dfrac{b_j'}{a_j}$ for $0 \le i < j \le 2$, so $b_i' = b_i + a_it/d$ for $i = 0,1,2$ and some integer t. Also, since $D_1, b_0, b_1, b_2, a_0, a_1, a_2$, and a_3 determine uniquely b_3 (see the formula for D_1), the claim will be proved if we can show that there are $\ll d$ possible choices for t.

Lemma 2 implies that there exists $\xi_i \in (x_1^{(i)}, x_3^{(i)})$ such that $f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}] = f''(\xi_i)/2$. Using (19) we get

$$|f''(\xi_i)-\frac{2d_1}{a_0a_1(a_0+a_1)}| \le \frac{4\delta}{a_0a_1}.$$

Similarly, there exists $\xi_j \in (x_1^{(j)}, x_3^{(j)})$ such that $|f''(\xi_j)-\dfrac{2d_1}{a_0a_1(a_0+a_1)}| \le \dfrac{4\delta}{a_0a_1}$. Thus, $|f''(\xi_i)-f''(\xi_j)| = |(\xi_i-\xi_j)f'''(\eta)| \le \dfrac{8\delta}{a_0a_1}$ where $\eta \in (x_1^{(i)}, x_3^{(j)})$, so

$$|\xi_i-\xi_j| \le \frac{8CM^3\delta}{Ta_0a_1}, \text{ and } |x_1^{(i)}-x_3^{(j)}| \le 2A+\frac{8CM^3\delta}{Ta_0a_1}.$$

Therefore if a quintuple maps to thirteentuple of the type

$$(a_0, a_1, a_2, a_3, \ldots, D_0, D_1, D_2, D_3, D_4)$$

then all x-coordinates of the points in the quintuple are in an interval J of length $\le 2A+\dfrac{8CM^3\delta}{Ta_0a_1}$.

Since $\dfrac{y_2^{(i)} - y_1^{(i)}}{x_2^{(i)} - x_1^{(i)}} = \dfrac{b_0}{a_0} = f'(\eta_i) + \dfrac{2\theta_i\delta}{a_0}$ with $|\theta_i| < 1$, $\eta_i \in J$, and $\dfrac{b_0'}{a_0} = f'(\eta_j) + \dfrac{2\theta_j\delta}{a_0}$ with $|\theta_j| < 1$, $\eta_j \in J$, we get

$$\frac{|b_0' - b_0|}{a_0} = \frac{|t|}{d} \le |f'(\eta_j) - f'(\eta_i)| + \frac{2|\theta_j - \theta_i|\delta}{a_0} \le |J|\frac{CT}{M^2} + \frac{4\delta}{a_0}.$$

Therefore there are $\le 1 + \dfrac{4d\delta}{a_0} + \dfrac{2CTAd}{M^2} + \dfrac{8C^2Md\delta}{a_0a_1}$ choices for t. Note that $d \le a_0$ and from (18) $CTA \le M^2$. Pick $E = \min\left(1/(17\delta), C(M\delta)^{1/2}\right)$. Since $1 \le C \le M$ and $\delta < (CM)^{-1/2}$, we have $E \ll C(M\delta)^{1/2}$. Also, since $a_0 \ge E$ and $a_1 \ge E$ we get that there are $\ll d$ choices for t which proves the claim.

Now, using the Claim, Lemma 12, and summing over d we get

$$|S_5| \ll_\epsilon B^\epsilon \left(A^{1+\epsilon}B^{8/3} + B^4\right) + M/A + 1,$$

and substituting for B,

(20)
$$|S_5| \ll M^\epsilon \left(C^{8/3}A^{17}T^{8/3}M^{-8} + A^9\delta^{8/3} + C^4A^{24}T^4M^{-12} + A^{12}\delta^4\right) + M/A + 1.$$

Note that the above estimate holds for any positive A which satisfies (18). Optimizing with respect to A we get

$$\begin{aligned} |S_5| \quad \ll \quad & CM^\epsilon \left(M^{1/2}T^{4/27} + M^{9/10}\delta^{4/15} + M^{12/25}T^{4/25} + M^{12/13}\delta^{4/13}\right) \\ & + M\delta + (CTM)^{2/7} + M^{1/2}(C\delta T)^{1/4} + CT/M. \end{aligned}$$

Recall (14). Since $E < 1/(16\delta)$ Lemma 5 holds and we obtain $|S_3| \ll C^5TM^{1/2}\delta^{5/2}$. Recall (15). Using $1 \le M \le T \le M^2$ we get

$$|S_6| \ll C\left(T^{1/3}M^{-1/12} + T^{1/8}M^{7/16} + M^{5/8}\right) \ll CM^{1/2}T^{4/27}.$$

Substituting the upper bounds for $|S_3|$, $|S_5|$, and $|S_6|$ in (14) and noting that $C\delta M$ and W get absorbed by larger terms we obtain the theorem. ∎

7. Proof of Theorem 1.

W.l.o.g we can assume $19/154 < \theta \le .129$ since the theorem has been already established in [10] when $1/8 < \theta < 1/2$. We also assume that x is sufficiently large ($x \ge 10^7$ will do).

Next, we reduce the problem to estimating the number of lattice points close to certain curves by using Lemma 5 from [10].

Lemma 13. *Let $x > 1$, $\epsilon > 0$, $\sqrt{x} \le \epsilon^3 h \le \epsilon^5 x$, and $h = x^{1/2+\theta}$. Then*

$$Q(x+h) - Q(x) = \frac{\zeta(\frac{3}{2})}{2\zeta(3)} x^\theta + O\left(\epsilon x^\theta\right) + O(R_1 + R_2)$$

where R_1 is the number of pairs of positive integers (m,k) with $\epsilon x^\theta < m \le x^{1/5}$, $m^2k^3 \in (x, x+h]$, and R_2 is the number of pairs of positive integers (m,k) with $\epsilon x^\theta < k \le x^{1/5}$, $m^2k^3 \in (x, x+h]$.

We use Lemma 13 with $h = x^{1/2+\theta}$, and $\epsilon = x^{-.01}$. First, we estimate R_1. Let $S_1(A,B)$ be the number of integers m in the interval $(A,B]$ for which there exists an integer k with $m^2k^3 \in (x, x+h]$. Then $R_1 = S_1(x^{\theta-.01}, x^{1/5})$.

It is easy to check that $S_1(M, 2M) \le |\{m \in (M, 2M] \bigcap Z : ||f(m)|| < \delta\}|$ where $f(m) = x^{1/3} m^{-2/3}$ and $\delta = x^{\theta - 1/6} M^{-2/3}$. Indeed, if $m \in S_1(M, 2M)$ then $f(m) = (x/m^2)^{1/3} < k \le ((x+h)/m^2)^{1/3}$ for some integer k, and $||f(m)|| < ((x+h)^{1/3} - x^{1/3}) m^{-2/3}$.

To estimate the size of $S_1(M, 2M)$ we use Theorem 2 from [10].

Lemma 14. *Let $f \in \mathcal{F}_2 \bigcap \mathcal{F}_3$, $1 \le M \le T$, and $0 \le \delta \le \frac{1}{2} C^{-1/2} T^{1/2} M^{-1}$. Then*

$$|S(f,\delta)| \ll 1 + T^{3/10} M^{3/10} (\log M)^{1/2} + T^{4/11} M^{2/11} (\log M)^{5/11}$$

$$+ \delta^{1/8} T^{3/8} M^{1/4} (\log M)^{5/8} + \delta^{1/7} T^{1/7} M^{4/7} (\log M)^{5/7} + \delta^{2/5} T^{1/5} M^{3/5} \log M + \delta M.$$

From Lemma 14 with $f(m) = \left(x/m^2\right)^{1/3}$, $\delta = x^{\theta-1/6} M^{-2/3}$, and $T = x^{1/3} M^{-2/3}$ we get

$$S_1(M, 2M) \ll x^{.12} \log x \quad \text{ for all } M \in [x^{\theta-.01}, x^{1/5}].$$

Summing over $O(\log x)$ intervals of the form $(M, 2M]$ we obtain

$$R_1 \ll x^{.12} \log^2 x.$$

Next , we estimate R_2. Let $S_2(A,B)$ be the number of integers k in the interval $(A,B]$ for which there exists an integer m with $m^2k^3 \in (x, x+h]$. Then $R_2 = S_2(x^{\theta-.01}, x^{\frac{1}{5}})$.

Again, it is easy to check that $S_2(M,2M) \leq |\{k \in (M,2M] \bigcap Z : ||f_1(k)|| < \delta_1\}|$ where $f_1(k) = x^{1/2}k^{-3/2}$ and $\delta_1 = x^{\theta}M^{-3/2}$.

We estimate $S_2(M,2M)$ for $M \in (x^{\theta-.01}, x^{.138}]$ via exponential sums. We use a handy lemma of Stečkin [19] (one can find a proof of the lemma in [21, pp.290–292]).

Lemma 15. *Let $g : R \to R$ be a 1-periodic function of bounded variation on the interval $[0,1]$. Let f be any real-valued function defined on R, $a < b$ be real numbers with $b - a > 1$, and N be a positive real number. Then*

$$|\sum_{a<k\leq b} g(f(k)) - (b-a)\int_0^1 g(t)dt| \ll \sum_{l=1}^{N} \frac{|S_l|}{l} + \frac{b-a}{N},$$

where $S_l = \sum_{a<k\leq b} e^{2\pi i l f(k)}$, the summation is over all integers $k \in (a,b]$, and the constant in $\ll$ depends only on the variation of g. Furthermore, if the measure of the support of g in $[0,1]$ is $\ll 1/N$ then

$$|\sum_{a<k\leq b} g(f(k)) - (b-a)\int_0^1 g(t)dt| \ll \frac{1}{N}\sum_{l=1}^{N} |S_l| + \frac{b-a}{N}.$$

The proof of the above lemma is via L_1, one-sided approximation by trigonometric polynomials. We apply Lemma 15 with $f(k) = x^{1/2}k^{-3/2}$,

$$g(x) = \begin{cases} 1 & \text{if} \quad ||x|| < \delta_1 \\ 0 & \text{if} \quad ||x|| \geq \delta_1 \end{cases},$$

$a = M$, and $b = 2M$. Then $S_2(M,2M) \ll \sum_{l=1}^{N} \frac{|S_l|}{l} + \frac{M}{N}$. To estimate S_l we use the exponent pair $\left(\frac{1}{20}, \frac{33}{40}\right) = AABAAB(0,1)$ (e.g. see [4]) and obtain $|S_l| \ll \left(lx^{1/2}M^{-5/2}\right)^{1/20} M^{33/40}$. Set $N = M^{2/7}x^{-1/42}$. We get $S_2(M,2M) \ll x^{1/42}M^{5/7}$. Splitting the interval $(x^{\theta-.01}, x^{.138}]$ to subintervals of the form $(M,2M]$ and applying the last estimate to each of these subintervals we obtain

$$S_2(x^{\theta-.01}, x^{.138}) \ll x^{.123}.$$

When $x^{0.138} \leq M \leq x^{0.158}$ we use Corollary 1 of [22].

Lemma 16. *Let $f \in \mathcal{F}_2 \bigcap \mathcal{F}_3$ and $0 \leq \delta \leq \frac{1}{2}C^{-1/2}T^{1/2}M^{-1}$. Then*

$$|S(f,\delta)| \ll_C M^{1/2}T^{1/6} + \delta M + \delta^{1/4}T^{1/4}M^{1/2}.$$

Applying Lemma 16 with f_1, δ_1, and $T = x^{1/2}M^{-3/2}$ we get

$$S_2(M, 2M) \ll x^{1/12}M^{1/4} + x^{\theta}M^{-1/2} + x^{1/8+\theta/4}M^{-1/4}.$$

This implies

$$S_2(x^{0.138}, x^{0.158}) \ll x^{.123}.$$

The next range is $M \in (x^{.158}, x^{41/231})$ $(41/231 = .177\ldots)$. This is the critical range. All conditions of Theorem 2 hold with f_1, δ_1, and $T = x^{1/2}M^{-3/2}$. We get $S_2(M, 2M) \ll_\epsilon M^\epsilon(x^{2/27}M^{5/18} + x^{2/25}M^{6/25} + x^{4\theta/15}M^{1/2} + x^{4\theta/13}M^{6/13} + x^{1/7}M^{-1/7} + x^{1/2}M^{-5/2} + x^{(2\theta+1)/8}M^{-1/4}) + x^{(1+5\theta)/2}M^{-19/4}$. This implies

$$S_2(x^{.158}, x^{41/231}) \ll_\epsilon x^{\frac{19}{154}+\epsilon}.$$

Finally, we estimate $S_2(M, 2M)$ when $M \in (x^{41/231}, x^{1/5}]$ by using Lemma 14 with f_1 and δ_1. We obtain $S_2(M, 2M) \ll \log M(x^{3/20}M^{-3/20} + x^{2/11}M^{-4/11} + x^{(3+2\theta)/16}M^{-1/2} + x^{(1+2\theta)/14}M^{1/7} + x^{(1+4\theta)/10}M^{-3/10} + x^{\theta}M^{-1/2})$, which in turn implies

$$S_2(x^{41/231}, x^{1/5}) \ll x^{19/154}\log x.$$

So, $R_2 = S_2(x^{\theta-.01}, x^{.138}) + S_2(x^{.138}, x^{.158}) + S_2(x^{.158}, x^{41/231}) + S_2(x^{41/231}, x^{1/5}) \ll_\epsilon x^{\frac{19}{154}+\epsilon}$. Combining Lemma 13 and the estimates for R_1 and R_2 we get

$$Q\left(x + x^{1/2+\theta}\right) - Q(x) = \frac{\zeta(\frac{3}{2})}{2\zeta(3)}x^{\theta} + O\left(x^{\theta-.01}\right) + O_\epsilon\left(x^{19/154+\epsilon}\right).$$

Setting $\epsilon = \frac{1}{2}(\theta - \frac{19}{154})$ we complete the proof of the theorem. ∎

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