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2004:08

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PREPRINT SERIES

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UNIVERSITY OF SOUTH CAROLINA

# The Nonlinear Schrödinger Equation on the Half-Line

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## Abstract

Assuming that the solution  $q(x, t)$  of the nonlinear Schrödinger equation on the half-line exists, it has been shown that  $q(x, t)$  can be represented in terms of the solution of a matrix Riemann-Hilbert (RH) problem formulated in the complex  $k$ -plane. The jump matrix of this RH problem has explicit  $x, t$  dependence and it is defined in terms of the scalar functions  $\{a(k), b(k), A(k), B(k)\}$  referred to as spectral functions. The functions  $a(k)$  and  $b(k)$  are defined in terms of  $q_0(x) = q(x, 0)$ , while the functions  $A(k)$  and  $B(k)$  are defined in terms of  $g_0(t) = q(0, t)$  and  $g_1(t) = q_x(0, t)$ . The spectral functions are *not* independent but they satisfy an algebraic *global relation*. Here we first prove that if there exist spectral functions satisfying this global relation, then the function  $q(x, t)$  defined in terms of the above RH problem exists globally and solves the nonlinear Schrödinger equation, and furthermore  $q(x, 0) = q_0(x)$ ,  $q(0, t) = g_0(t)$  and  $q_x(0, t) = g_1(t)$ . We then show that given appropriate initial and boundary conditions, it is possible to construct such spectral functions through the solution of a nonlinear Volterra integral equation whose solution exists globally. We also show that for a *particular class* of boundary conditions it is possible to bypass this nonlinear equation and to compute the spectral functions using only the algebraic manipulation of the global relation; thus for this particular class of boundary conditions, which we call *linearizable*, the problem on the half-line can be solved as effectively as the problem on the line. An example of a linearizable boundary condition is  $q_x(0, t) - \rho q(0, t) = 0$  where  $\rho$  is a real constant.

# 1 Introduction

A general method for solving boundary value problems for two dimensional linear and integrable nonlinear PDE's was announced in [1] and further developed in [2, 3]. For nonlinear evolution equations on the half-line, the starting point of this method is the *simultaneous* spectral analysis of the two eigenvalue equations defining the associated Lax pair. Under the assumption of the existence of a solution  $q(x, t)$ , this yields  $q(x, t)$  in terms of the solution of a matrix Riemann-Hilbert (RH) problem formulated in the complex  $k$ -plane, where  $k$  is the spectral parameter of the two eigenvalue equations. The jump matrix of this RH problem has explicit  $x, t$  dependence and it is uniquely defined in terms of some functions of  $k$  called the *spectral functions*. These functions can be expressed in terms of the boundary values of  $q$  and of its spatial derivatives. However, these boundary values are in general related and only some of them can be prescribed as boundary conditions. The most difficult step in the solution of boundary value problems is the determination of those spectral functions which involve the unknown boundary values. This can be achieved using the fact that the spectral functions satisfy in the complex  $k$ -plane a simple algebraic *global relation*. For linear evolution equations this relation is linear and this step involves only algebraic manipulations [4], however for nonlinear equations this relation is nonlinear.

In this paper we present the rigorous implementation of the method of [1]–[3] to the nonlinear Schrödinger equation on the half-line. Furthermore, we identify a *particular class of boundary conditions* for which the global relation can be analyzed using only algebraic manipulations.

This paper is organized as follows: In § 2 we first review the general methodology introduced in [1]–[3]. Namely we *assume* that there exists a solution  $q(x, t)$  with sufficient smoothness and decay, and we indicate how this solution can be expressed through the solution of a  $2 \times 2$  matrix RH problem, which is uniquely characterized in terms of certain spectral functions satisfying an appropriate global relation. In § 3 and § 4 we study rigorously the RH problem: In § 3 we show that given initial data  $q(x, 0) = q_0(x)$  and *assuming* that there exists an *admissible set of boundary values*  $\{g_0(t), g_1(t)\}$ , it is possible to define an equivalent class of spectral functions. A set of boundary values is called admissible iff it gives rise to spectral functions satisfying the global relation obtained in § 2. In § 4 we *define*  $q(x, t)$  in terms of the solution of a matrix  $2 \times 2$  RH problem uniquely characterized in terms of the spectral functions defined in § 3. We then show that  $q(x, t)$  satisfies the nonlinear Schrödinger equation, and furthermore  $q(x, 0) = q_0(x)$ ,  $q(0, t) = g_0(t)$ ,  $q_x(0, t) = g_1(t)$ . In § 5 we show that given appropriate boundary conditions, the admissible set of boundary values can be uniquely constructed in terms of the given initial and boundary conditions through the solution of a nonlinear Volterra integral equation which can be solved globally. In § 6 we show that for a particular class of boundary conditions it is possible to bypass this nonlinear equation. These boundary conditions are determined by analyzing the transformations in the complex  $k$ -plane which leave the global relation invariant. In § 7 we discuss further these results.

## 2 The Exact 1-Form

The nonlinear Schrödinger equation

$$iq_t + q_{xx} - 2\lambda|q|^2q = 0, \quad \lambda = \pm 1, \quad (2.1)$$

admits the Lax pair [5] formulation [6]

$$\mu_x + ik[\sigma_3, \mu] = Q(x, t)\mu, \quad (2.2a)$$

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(x, t, k)\mu, \quad (2.2b)$$

where  $\sigma_3 = \text{diag}(1, -1)$ ,

$$Q(x, t) = \begin{bmatrix} 0 & q(x, t) \\ \lambda\bar{q}(x, t) & 0 \end{bmatrix}, \quad \tilde{Q}(x, t, k) = 2kQ - iQ_x\sigma_3 - i\lambda|q|^2\sigma_3. \quad (2.3)$$

Let  $\hat{\sigma}_3$  denote the commutator with respect to  $\sigma_3$ , then  $(\exp \hat{\sigma}_3)A$  can be computed easily:

$$\hat{\sigma}_3 A = [\sigma_3, A], \quad e^{\hat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3}, \quad (2.4)$$

where  $A$  is a  $2 \times 2$  matrix.

Equations (2.2a)–(2.2b) can be rewritten as

$$d \left( e^{i(kx+2k^2t)\hat{\sigma}_3} \mu(x, t, k) \right) = W(x, t, k), \quad (2.5)$$

where the exact 1-form  $W$  is defined by

$$W = e^{i(kx+2k^2t)\hat{\sigma}_3} (Q\mu dx + \tilde{Q}\mu dt). \quad (2.6)$$

### 2.1 Bounded and Analytic Eigenfunctions

Let the equation (2.1) be valid for

$$0 < x < \infty, \quad 0 < t < T,$$

where  $T \leq \infty$  is a given positive constant; unless otherwise specified, we suppose that  $T < \infty$ . Assume that the function  $q(x, t)$  has sufficient smoothness and decay. A solution of equation (2.5) is given by

$$\mu_*(x, t, k) = I + \int_{(x_*, t_*)}^{(x, t)} e^{-i(kx+2k^2t)\hat{\sigma}_3} W(\xi, \tau, k), \quad (2.7)$$

where  $I$  is the  $2 \times 2$  identity matrix,  $(x_*, t_*)$  is an arbitrary point in the domain  $0 < \xi < \infty$ ,  $0 < \tau < T$ , and the integral is over a (piecewise) smooth curve from  $(x_*, t_*)$  to  $(x, t)$ . Since the 1-form  $W$  is exact,  $\mu_*$  is independent of the path of integration. The analyticity properties of  $\mu_*$  with respect to  $k$  depend on the choice of  $(x_*, t_*)$ . It was shown in [2] that for a polygonal domain there exists a canonical way of choosing the points  $(x_*, t_*)$ , namely they are the corners of the associated polygon. Thus we define three different solutions  $\mu_1$ ,

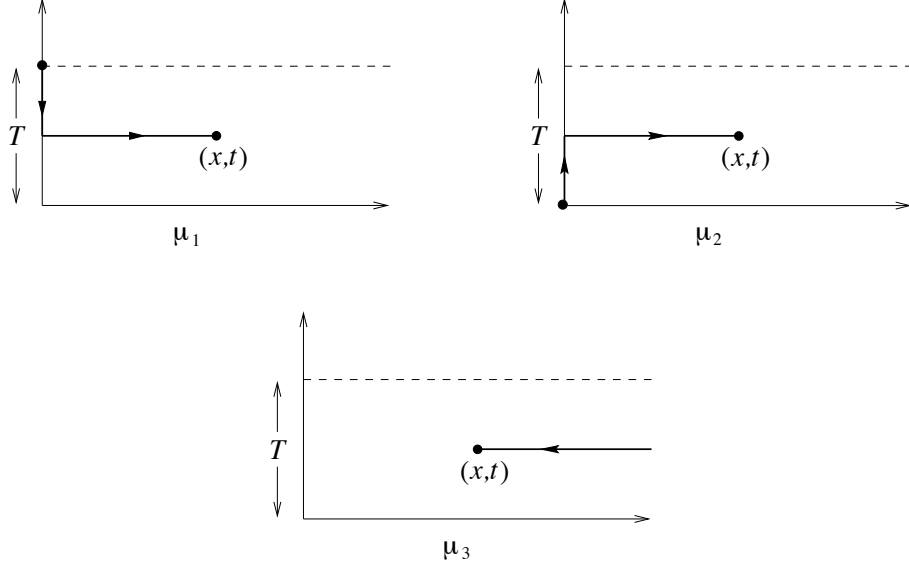


Figure 2.1: The solutions  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  of (2.5)

$\mu_2$ ,  $\mu_3$ , corresponding to  $(0, T)$ ,  $(0, 0)$ ,  $(\infty, t)$ , see Figure 2.1. Also we choose the particular contours shown in Figure 2.1.

This choice implies the following inequalities on the contours,

$$\begin{aligned}\mu_1 : \xi - x &\leq 0, & \tau - t &\geq 0, \\ \mu_2 : \xi - x &\leq 0, & \tau - t &\leq 0, \\ \mu_3 : \xi - x &\geq 0.\end{aligned}$$

The second column of the matrix equation (2.7) involves  $\exp[i2(k(\xi - x) + 2k^2(\tau - t))]$ . Using the above inequalities it follows that this exponential is bounded in the following regions of the complex  $k$ -plane:

$$\begin{aligned}\mu_1 : \{\operatorname{Im} k \leq 0 \cap \operatorname{Im} k^2 \geq 0\}, \\ \mu_2 : \{\operatorname{Im} k \leq 0 \cap \operatorname{Im} k^2 \leq 0\}, \\ \mu_3 : \{\operatorname{Im} k \geq 0\}.\end{aligned}$$

Thus the second column vectors of  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are bounded and analytic for  $\arg k$  in  $(\pi, 3\pi/2)$ ,  $(3\pi/2, 2\pi)$  and  $(0, \pi)$  respectively. We will denote these vectors with superscripts (3), (4) and (12) to indicate that they are bounded and analytic in the third quadrant, fourth quadrant and the upper half of the complex  $k$ -plane respectively. Similar conditions are valid for the first column vectors, thus

$$\mu_1(x, t, k) = (\mu_1^{(2)}, \mu_1^{(3)}), \quad \mu_2(x, t, k) = (\mu_2^{(1)}, \mu_2^{(4)}) \quad \text{and} \quad \mu_3(x, t, k) = (\mu_3^{(34)}, \mu_3^{(12)}). \quad (2.8)$$

We note that the functions  $\mu_1$  and  $\mu_2$  are entire functions of  $k$ . Equations (2.8) together with the estimate

$$\mu_j(x, t, k) = I + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \quad j = 1, 2, 3, \quad (2.9)$$

imply that the functions  $\mu_j$  are the fundamental eigenfunctions needed for the formulation of a RH problem in the complex  $k$ -plane. The jump matrix of this RH problem is uniquely defined in terms of the  $2 \times 2$ -matrix valued functions

$$s(k) = \mu_3(0, 0, k) \quad \text{and} \quad S(k) = [e^{2ik^2 T \hat{\sigma}_3} \mu_2(0, T, k)]^{-1}. \quad (2.10)$$

This is a direct consequence of the fact that any two solutions of (2.7) are simply related,

$$\mu_3(x, t, k) = \mu_2(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_3} \mu_3(0, 0, k), \quad (2.11)$$

$$\mu_1(x, t, k) = \mu_2(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_3} [e^{2ik^2 T \hat{\sigma}_3} \mu_2(0, T, k)]^{-1}. \quad (2.12)$$

The functions  $s(k)$  and  $S(k)$  follow from the evaluations at  $x = 0$  and  $t = T$  respectively of the function  $\mu_3(x, 0, k)$  and of  $\mu_2(0, t, k)$  which satisfy the following linear integral equations:

$$\mu_3(x, 0, k) = I + \int_{\infty}^x e^{ik(\xi-x)\hat{\sigma}_3} (Q\mu_3)(\xi, 0, k) d\xi, \quad (2.13)$$

$$\mu_2(0, t, k) = I + \int_0^t e^{2ik^2(\tau-t)\hat{\sigma}_3} (\tilde{Q}\mu_2)(0, \tau, k) d\tau. \quad (2.14)$$

It is also worth noticing that the matrix valued function  $S(k)$  can be alternatively defined by the equation,

$$S(k) = \mu_1(0, 0, k),$$

which is more convenient in the case when  $T = \infty$ .

## 2.2 The Spectral Functions

The fact that  $Q$  and  $\tilde{Q}$  are traceless together with (2.9) imply  $\det \mu_j(x, t, k) = 1$  for  $j = 1, 2, 3$ . Thus

$$\det s(k) = \det S(k) = 1. \quad (2.15)$$

From the symmetry properties of  $Q$  and  $\tilde{Q}$  it follows that

$$(\mu(x, t, k))_{11} = \overline{(\mu(x, t, \bar{k}))_{22}}, \quad (\mu(x, t, k))_{21} = \overline{\lambda \mu(x, t, \bar{k})_{12}},$$

and thus

$$s_{11}(k) = \overline{s_{22}(\bar{k})}, \quad s_{21}(k) = \overline{\lambda s_{12}(\bar{k})}, \quad S_{11}(k) = \overline{S_{22}(\bar{k})}, \quad S_{21}(k) = \overline{\lambda S_{12}(\bar{k})}.$$

We will use the following notation for  $s$  and  $S$ :

$$s(k) = \begin{bmatrix} \overline{a(\bar{k})} & b(k) \\ \overline{\lambda b(\bar{k})} & a(k) \end{bmatrix}, \quad S(k) = \begin{bmatrix} \overline{A(\bar{k})} & B(k) \\ \overline{\lambda B(\bar{k})} & A(k) \end{bmatrix}. \quad (2.16)$$

The definitions of  $\mu_j(0, t, k)$ ,  $j = 1, 2$ , and of  $\mu_2(x, 0, k)$  imply that these functions have larger domains of boundedness,

$$\mu_1(0, t, k) = \left( \mu_1^{(24)}(0, t, k), \mu_1^{(13)}(0, t, k) \right), \quad (2.17a)$$

$$\mu_2(0, t, k) = \left( \mu_2^{(13)}(0, t, k), \mu_2^{(24)}(0, t, k) \right), \quad (2.17b)$$

$$\mu_2(x, 0, k) = \left( \mu_2^{(12)}(x, 0, k), \mu_2^{(34)}(x, 0, k) \right). \quad (2.17c)$$

The definitions of  $s(k)$ ,  $S(k)$  and the notation (2.16) imply

$$\begin{bmatrix} b(k) \\ a(k) \end{bmatrix} = \mu_3^{(12)}(0, 0, k), \quad \begin{bmatrix} -e^{-4ik^2T} B(k) \\ \overline{A(\bar{k})} \end{bmatrix} = \mu_2^{(24)}(0, T, k), \quad (2.18)$$

where the vectors  $\mu_3^{(12)}(x, 0, k)$  and  $\mu_2^{(24)}(0, t, k)$  satisfy the following ODEs:

$$\begin{aligned} \partial_x \mu_3^{(12)}(x, 0, k) + 2ik \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mu_3^{(12)}(x, 0, k) \\ = Q(x, 0) \mu_3^{(12)}(x, 0, k), \quad 0 \leq \arg k \leq \pi, \quad 0 < x < \infty, \\ \lim_{x \rightarrow \infty} \mu_3^{(12)}(x, 0, k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \partial_t \mu_2^{(24)}(0, t, k) + 4ik^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mu_2^{(24)}(0, t, k) \\ = \tilde{Q}(0, t, k) \mu_2^{(24)}(0, t, k), \quad \arg k \in [\pi/2, \pi] \cup [3\pi/2, 2\pi], \quad 0 < t < T, \\ \mu_2^{(24)}(0, 0, k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \quad (2.20)$$

The above definitions imply the following properties:

$a(k), b(k)$

$a(k), b(k)$  are defined and analytic for  $\arg k \in (0, \pi)$ .

$$|a(k)|^2 - \lambda |b(k)|^2 = 1, \quad k \in \mathbb{R}.$$

$$a(k) = 1 + O\left(\frac{1}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \quad (2.21)$$

$A(k), B(k)$

$A(k), B(k)$  are entire functions bounded for  $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ . If  $T = \infty$ , the functions  $A(k)$  and  $B(k)$  are defined and analytic in the quadrants  $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ .

$$A(k) \overline{A(\bar{k})} - \lambda B(k) \overline{B(\bar{k})} = 1, \quad k \in \mathbb{C} \quad (k \in \mathbb{R} \cup i\mathbb{R}, \text{ if } T = \infty).$$

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad k \rightarrow \infty. \quad (2.22)$$

All of the above properties, except for the property that  $B(k)$  is bounded for  $\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]$ , follow from the analyticity and boundedness of  $\mu_3(x, 0, k)$ ,  $\mu_2(0, t, k)$ , from the conditions of unit determinant, and from the large  $k$  asymptotics of these eigenfunctions. Regarding  $B(k)$  we note that  $B(k) = B(T, k)$ , where  $B(t, k) = -\exp(4ik^2t) (\mu_2^{(24)}(0, t, k))_1$ . Equations (2.20) imply a linear Volterra integral equation for the vector  $\exp(4ik^2t) \mu_2^{(24)}(0, t, k)$ , from which it immediately follows that  $B(t, k)$  is an entire function of  $k$  bounded for  $\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]$ .

## 2.3 The RH Problem

Equations (2.11) and (2.12) can be rewritten in the form

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad (2.23)$$

where the matrices  $M_-$ ,  $M_+$  and  $J$  are defined as follows

$$\begin{aligned} M_+ &= \left( \frac{\mu_2^{(1)}}{a(k)}, \mu_3^{(12)} \right), \arg k \in [0, \frac{\pi}{2}]; & M_- &= \left( \frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(12)} \right), \arg k \in [\frac{\pi}{2}, \pi]; \\ M_+ &= \left( \mu_3^{(34)}, \frac{\mu_1^{(3)}}{d(\bar{k})} \right), \arg k \in [\pi, \frac{3\pi}{2}]; & M_- &= \left( \mu_3^{(34)}, \frac{\mu_2^{(4)}}{a(\bar{k})} \right), \arg k \in [\frac{3\pi}{2}, 2\pi]; \end{aligned} \quad (2.24)$$

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}; \quad (2.25)$$

$$J(x, t, k) = \begin{cases} J_4, & \arg k = 0 \\ J_1, & \arg k = \frac{\pi}{2} \\ J_2 = J_3 J_4^{-1} J_1, & \arg k = \pi \\ J_3, & \arg k = \frac{3\pi}{2}; \end{cases} \quad (2.26)$$

with

$$J_1 = \begin{bmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{bmatrix}, \quad J_4 = \begin{bmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \lambda\bar{\gamma}(k)e^{2i\theta} & 1 - \lambda|\gamma(k)|^2 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{bmatrix}; \quad (2.27)$$

$$\theta(x, t, k) = kx + 2k^2t; \quad \gamma(k) = \frac{b(k)}{\bar{a}(k)}, \quad k \in \mathbb{R}; \quad \Gamma(k) = \frac{\lambda\overline{B(\bar{k})}}{a(k)d(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \quad (2.28)$$

The contour for this RH problem is depicted in Figure 2.2.

**Remark 2.1.** The function  $\Gamma(k)$  is a meromorphic function in the upper half-plane (in the second quadrant if  $T = \infty$ ).

The matrix  $M(x, t, k)$  defined by equations (2.24) is in general a meromorphic function of  $k$  in  $\mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$ . The possible poles of  $M$  are generated by the zeros of  $a(k)$ , of  $d(k)$ , and by the complex conjugate of these zeros.

Assume that:

1.  $a(k)$  has  $n$  simple zeros  $\{k_j\}_1^n$ ,  $n = n_1 + n_2$ , where  $\arg k_j \in (0, \frac{\pi}{2})$ ,  $j = 1, \dots, n_1$ ;  $\arg k_j \in (\frac{\pi}{2}, \pi)$ ,  $j = n_1 + 1, \dots, n_1 + n_2$ .
2.  $d(k)$  has  $\Lambda$  simple zeros  $\{\lambda_j\}_1^\Lambda$ , where  $\arg \lambda_j \in (\frac{\pi}{2}, \pi)$ ,  $j = 1, \dots, \Lambda$ .
3. None of the zeros of  $a(k)$  for  $\arg k \in (\frac{\pi}{2}, \pi)$ , coincides with a zero for  $d(k)$ .



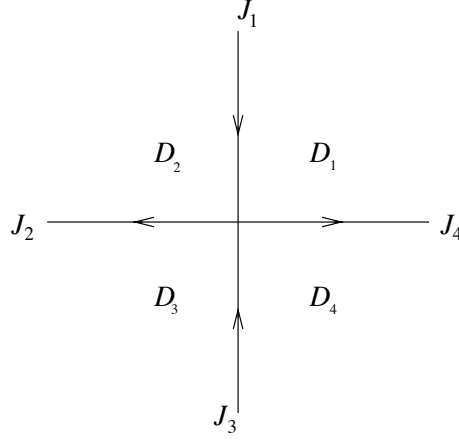


Figure 2.2: The contour for the RH problem

In order to evaluate the associate residues we introduce the following notation:

$[A]_1$  (resp.  $[A]_2$ ) denote the first (resp. second) column of  $A$  and  $\dot{a}(k) = \frac{da}{dk}$ .

The following formulae are valid:

$$\operatorname{Res}_{k_j} [M(x, t, k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)} [M(x, t, k_j)]_2, \quad j = 1, \dots, n_1, \quad (2.29a)$$

$$\operatorname{Res}_{\bar{k}_j} [M(x, t, k)]_2 = \frac{\lambda}{\dot{a}(k_j)b(k_j)} e^{-2i\theta(\bar{k}_j)} [M(x, t, \bar{k}_j)]_1, \quad j = 1, \dots, n_1, \quad (2.29b)$$

$$\operatorname{Res}_{\lambda_j} [M(x, t, k)]_1 = \frac{\lambda B(\bar{\lambda}_j)}{a(\lambda_j)d(\lambda_j)} e^{2i\theta(\lambda_j)} [M(x, t, \lambda_j)]_2, \quad j = 1, \dots, \Lambda, \quad (2.29c)$$

$$\operatorname{Res}_{\bar{\lambda}_j} [M(x, t, k)]_2 = \frac{B(\bar{\lambda}_j)}{a(\lambda_j)d(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)} [M(x, t, \bar{\lambda}_j)]_1, \quad j = 1, \dots, \Lambda, \quad (2.29d)$$

where

$$\theta(k_j) = k_j x + 2k_j^2 t. \quad (2.30)$$

**Remark 2.2.** The column  $[\mu_3(x, 0, k_j)]_2$  is a nontrivial vector solution of (2.2a). Therefore,  $a(k)$  and  $b(k)$  can not have common zeros and hence  $b(k_j) \neq 0$ . Similar arguments together with the third assumption above imply that  $B(\bar{\lambda}_j) \neq 0$ .

In order to derive equation (2.29a) we note that the second column of equation (2.11) is

$$\mu_3^{(12)} = a\mu_2^{(4)} + b\mu_2^{(1)} e^{-2i\theta}.$$

Recalling that  $\mu_2$  is an entire function and evaluating this equation at  $k = k_j$ ,  $j = 1, \dots, n_1$ , we find

$$\mu_3^{(12)}(k_j) = b(k_j) e^{-2i\theta(k_j)} \mu_2^{(1)}(k_j), \quad (2.31)$$

where for simplicity of notation we have suppressed the  $x, t$  dependence. Thus

$$\operatorname{Res}_{k_j} [M]_1 = \frac{\mu_2^{(1)}(k_j)}{\dot{a}(k_j)} = \frac{e^{2i\theta(k_j)} \mu_3^{(12)}(k_j)}{\dot{a}(k_j)b(k_j)},$$

which is equation (2.29a), since  $\mu_3^{(12)}(k_j) = [M]_2(k_j)$ .

In order to derive equation (2.29c) we note that the first column of equation  $M_- = M_+ J_1$ , yields

$$a\mu_1^{(2)} = d\mu_2^{(1)} + \lambda \bar{B} e^{2i\theta} \mu_3^{(12)}.$$

Evaluating this equation at  $k = \lambda_j$  (each term has an analytic continuation into the second quadrant) and using

$$\text{Res}_{\lambda_j} [M]_1 = \frac{\mu_1^{(2)}(\lambda_j)}{d(\lambda_j)}, \quad [M]_2 = \mu_3^{(12)},$$

we find equation (2.29c).

**Remark 2.3.** By extending  $q_0(x)$  to the whole axis,  $q_0(x) = 0$ ,  $x < 0$ , we can identify the set  $\{k_j\}_1^n$  of zeros of  $a(k)$  as the discrete spectrum of the Dirac operator associated with the nonlinear Schrödinger equation considered on the whole axis (cf. [7]). If  $\lambda = 1$  this operator is selfadjoint. This implies the emptiness of the set  $\{k_j\}_1^n$  when  $\lambda = 1$ . However, we do not have a similar argument for the function  $d(k)$ . Therefore, in order to ensure the solvability of the Riemann-Hilbert problem in the defocusing case we shall *assume* that  $d(k)$  has no zeros if  $\lambda = 1$ , see § 4. The asymptotic results presented in appendix B suggest that the solvability condition does not hold in the defocusing case if  $d(k)$  has zeros. Thus we conjecture that solitons do not exist for  $\lambda = 1$ .

## 2.4 The Global Relation

We now show that the spectral functions are not independent but satisfy an important global relation. Indeed, the integral of the 1-form  $W(x, t, k)$  around the boundary of the domain  $\{(\xi, \tau) : 0 < \xi < \infty, 0 < \tau < t\}$  vanishes. Let  $W$  be defined by (2.6) with  $\mu = \mu_3$ . Then

$$\begin{aligned} & \int_{\infty}^0 e^{ik\xi\sigma_3} (Q\mu_3)(\xi, 0, k) d\xi + \int_0^t e^{2ik^2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(0, \tau, k) d\tau \\ & + e^{2ik^2t\hat{\sigma}_3} \int_0^{\infty} e^{ik\xi\hat{\sigma}_3} (Q\mu_3)(\xi, t, k) d\xi \\ & = \lim_{x \rightarrow \infty} e^{ikx\hat{\sigma}_3} \int_0^t e^{2ik^2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(x, \tau, k) d\tau. \end{aligned} \quad (2.32)$$

Using the definition of  $s(k)$  in (2.10) it follows from (2.13) that the first term of this equation equals  $s(k) - I$ . Equation (2.11) evaluated at  $x = 0$  gives

$$\mu_3(0, \tau, k) = \mu_2(0, \tau, k) e^{-2ik^2\tau\hat{\sigma}_3} s(k),$$

thus

$$e^{2ik^2\tau\hat{\sigma}_3} (\tilde{Q}\mu_3)(0, \tau, k) = [e^{2ik^2\tau\hat{\sigma}_3} (\tilde{Q}\mu_2)(0, \tau, k)] s(k);$$

this equation together with (2.14) imply that the second term of (2.32) equals

$$[e^{2ik^2t\hat{\sigma}_3} \mu_2(0, t, k) - I] s(k).$$

Hence assuming that  $q$  has sufficient decay as  $x \rightarrow \infty$  equation (2.32) becomes

$$-I + S(t, k)^{-1}s(k) + e^{2ik^2t\hat{\sigma}_3} \int_0^\infty e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi, t, k) d\xi = 0, \quad (2.33)$$

where the first and second columns of this equation are valid for  $\arg k$  in the lower and the upper half of the complex  $k$ -plane respectively and  $S(t, k)$  is defined by

$$S(t, k) = [e^{2ik^2t\hat{\sigma}_3}\mu_2(0, t, k)]^{-1}.$$

Letting  $t = T$  and noting that  $S(k) = S(T, k)$ , equation (2.33) becomes

$$-I + S(k)^{-1}s(k) + e^{2ik^2T\hat{\sigma}_3} \int_0^\infty e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi, T, k) d\xi = 0.$$

The (12) component of this equation is

$$\begin{aligned} B(k)a(k) - A(k)b(k) &= e^{4ik^2T}c^+(k) \quad \arg k \in [0, \pi], \\ c^+(k) &= \int_0^\infty e^{ik\xi}(Q\mu_3)_{12}(\xi, T, k) dk. \end{aligned} \quad (2.34)$$

### 3 The Spectral Functions

The analysis of § 2 motivates the following definitions for the spectral functions.

**Definition 3.1.** (*The spectral functions  $a(k)$  and  $b(k)$* ).

Given  $q_0(x) \in S(\mathbb{R}^+)$ , we define the map

$$\mathbb{S} : \{q_0(x)\} \mapsto \{a(k), b(k)\} \quad (3.1)$$

as follows:

$$a(k) = \varphi_2(0, k), \quad b(k) = \varphi_1(0, k), \quad \text{Im } k \geq 0, \quad (3.2)$$

where the vector  $\varphi(x, k) = (\varphi_1, \varphi_2)^t$  is the unique solution of

$$\varphi_{1_x} + 2ik\varphi_1 = q_0(x)\varphi_2, \quad (3.3a)$$

$$\varphi_{2_x} = \lambda\bar{q}_0(x)\varphi_1, \quad \text{Im } k \geq 0, \quad 0 < x < \infty, \quad (3.3b)$$

$$\lim_{x \rightarrow \infty} \varphi = (0, 1)^t. \quad (3.3c)$$

The functions  $a$  and  $b$  are well defined. Indeed, equations (3.3) are equivalent to the Volterra linear integral equation,

$$\varphi_1(x, k) = - \int_x^\infty e^{-2ik(x-y)} q_0(y) \varphi_2(y, k) dy, \quad \text{Im } k \geq 0, \quad (3.4a)$$

$$\varphi_2(x, k) = 1 - \lambda \int_x^\infty \bar{q}_0(y) \varphi_1(y, k) dy, \quad \text{Im } k \geq 0. \quad (3.4b)$$

The spectral functions  $a(k)$  and  $b(k)$  have the following properties:

**Properties of  $a(k)$  and  $b(k)$**

- (i)  $a(k)$  and  $b(k)$  are analytic for  $\text{Im } k > 0$  and continuous and bounded for  $\text{Im } k \geq 0$ .
- (ii)  $a(k) = 1 + O(\frac{1}{k})$ ,  $b(k) = O(\frac{1}{k})$ ,  $k \rightarrow \infty$ .
- (iii)  $|a(k)|^2 - \lambda|b(k)|^2 = 1$ ,  $k \in \mathbb{R}$ .
- (iv) The map  $\mathbb{Q} : \{a(k), b(k)\} \mapsto \{q_0(k)\}$ , inverse to  $\mathbb{S}$ , is defined as follows:

$$q_0(x) = 2i \lim_{k \rightarrow \infty} (kM^{(x)}(x, k))_{12}, \quad (3.5)$$

where  $M^{(x)}(x, k)$  is the unique solution of the following RH problem:

$$\bullet \quad M^{(x)}(x, k) = \begin{cases} M_-^{(x)}(x, k), & \text{Im } k \leq 0, \\ M_+^{(x)}(x, k), & \text{Im } k \geq 0, \end{cases} \quad (3.6a)$$

is a sectionally meromorphic function.

$$\bullet \quad M_-^{(x)}(x, k) = M_+^{(x)}(x, k)J^{(x)}(x, k), \quad k \in \mathbb{R}, \quad (3.6b)$$

where

$$J^{(x)}(x, k) = \begin{bmatrix} 1 & -\frac{b(k)}{\bar{a}(k)}e^{-2ikx} \\ \frac{\lambda\bar{b}(k)}{a(k)}e^{2ikx} & \frac{1}{|a|^2} \end{bmatrix}. \quad (3.6c)$$

$$\bullet \quad M^{(x)}(x, k) = I + O(\frac{1}{k}), \quad k \rightarrow \infty. \quad (3.6d)$$

- We assume that if  $\lambda = -1$ ,  $a(k)$  has  $n$  simple zeros  $\{k_j\}_1^n$ ,  $n = n_1 + n_2$ , where  $\arg k_j \in (0, \frac{\pi}{2})$ ,  $j = 1, \dots, n_1$ ;  $\arg k_j \in (\frac{\pi}{2}, \pi)$ ,  $j = n_1 + 1, \dots, n_1 + n_2$ .
- If  $\lambda = -1$ , the first column of  $M_+^{(x)}$  has simple poles at  $k = k_j$ ,  $j = 1, \dots, n$  and the second column of  $M_-^{(x)}$  has simple poles at  $k = \bar{k}_j$ , where  $\{k_j\}_1^n$  are the simple zeros of  $a(k)$ ,  $\text{Im } k > 0$ . The associated residues are given by

$$\begin{aligned} \text{Res}_{k_j} [M^{(x)}(x, k)]_1 &= \frac{e^{2ik_j x}}{\dot{a}(k_j)b(k_j)} [M^{(x)}(x, k_j)]_2, \\ \text{Res}_{\bar{k}_j} [M^{(x)}(x, k)]_2 &= \frac{\lambda e^{-2i\bar{k}_j x}}{\dot{a}(k_j)\bar{b}(k_j)} [M^{(x)}(x, \bar{k}_j)]_1. \end{aligned} \quad (3.6e)$$

- (v) We have

$$\mathbb{S}^{-1} = \mathbb{Q}. \quad (3.7)$$

*Proof.* (i)–(iii) follow from the definition; the derivation of (iv), (v) is given in the Appendix A.  $\square$

**Remark 3.2.** The properties of  $a(k)$  and  $b(k)$  imply that  $a(k)$  can be expressed in terms of  $b(k)$ . Indeed, if  $a(k) \neq 0$ , for  $\text{Im } k \geq 0$ , then

$$a(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln(1 + \lambda |b(k')|^2) \frac{dk'}{k' - k} \right\}, \quad \text{Im } k > 0.$$

Also, the upper-half plane analyticity of  $b(k)$  implies that

$$b(k) = \int_0^{\infty} \hat{b}(s) e^{iks} ds,$$

where  $\hat{b}(s)$  is a complex valued function of Schwartz type on  $\mathbb{R}^+$  (if the same behavior is assumed for  $q_0(x)$ ). Thus, if  $a(k) \neq 0$ , the maps  $\mathbb{S}$  and  $\mathbb{Q}$  define the bijection

$$q_0(x) \longleftrightarrow b(k). \quad (3.8)$$

If  $\lambda = -1$  and  $a(k)$  has zeros, the equation for  $a(k)$  must be replaced by

$$a(k) = \prod_{j=1}^n \frac{k - k_j}{\overline{k - k_j}} \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln(1 + \lambda |b(k')|^2) \frac{dk'}{k' - k} \right\}, \quad \text{Im } k > 0,$$

and a discrete component,  $\{k_j\}$ , must be added to the right hand side of (3.8).

**Definition 3.3.** (The spectral functions  $A(k)$  and  $B(k)$ )

Let

$$\tilde{Q}(t, k) = 2k \begin{bmatrix} 0 & g_0(t) \\ \lambda \overline{g_0(t)} & 0 \end{bmatrix} - i \begin{bmatrix} 0 & g_1(t) \\ \lambda \overline{g_1(t)} & 0 \end{bmatrix} \sigma_3 - i\lambda |g_0(t)|^2 \sigma_3, \quad \lambda = \pm 1. \quad (3.9)$$

Let  $g_0(t)$  and  $g_1(t)$  be smooth functions. The map

$$\tilde{\mathbb{S}} : \{g_0(t), g_1(t)\} \rightarrow \{A(k), B(k)\} \quad (3.10)$$

is defined as follows:

$$\begin{bmatrix} -e^{-4ik^2 T} B(k) \\ \overline{A(k)} \end{bmatrix} = \Phi(T, k), \quad k \in \mathbb{C}, \quad (3.11)$$

where the vector  $\Phi(t, k) = (\Phi_1, \Phi_2)^t$  is the unique solution of

$$\Phi_{1_t} + 4ik^2 \Phi_1 = \tilde{Q}_{11} \Phi_1 + \tilde{Q}_{12} \Phi_2, \quad (3.12a)$$

$$\Phi_{2_t} = \tilde{Q}_{21} \Phi_1 + \tilde{Q}_{22} \Phi_2, \quad 0 < t < T, \quad k \in \mathbb{C}, \quad (3.12b)$$

$$\Phi(0, k) = (0, 1)^t. \quad (3.12c)$$

The functions  $A(k)$  and  $B(k)$  are well defined, since equations (3.12) are equivalent to the linear Volterra integral equations

$$\Phi_1(t, k) = \int_0^t e^{-4ik^2(t-\tau)} (\tilde{Q}_{11}\Phi_1 + \tilde{Q}_{12}\Phi_2)(\tau, k) d\tau, \quad (3.13a)$$

$$\Phi_2(t, k) = 1 + \int_0^t (\tilde{Q}_{21}\Phi_1 + \tilde{Q}_{22}\Phi_2)(\tau, k) d\tau. \quad (3.13b)$$

If  $T = \infty$ , we assume that the functions  $g_0(t)$  and  $g_1(t)$  belong to  $S(\mathbb{R}_+)$ , and we use the alternative, based on the solution  $\mu_1(0, t, k)$ , definition of the spectral functions  $A(k)$  and  $B(k)$ . In other words, we put

$$\begin{bmatrix} B(k) \\ A(k) \end{bmatrix} = \tilde{\Phi}(0, k), \quad \text{Im } k^2 \geq 0,$$

where the vector  $\tilde{\Phi}(t, k) = (\tilde{\Phi}_1, \tilde{\Phi}_2)^t$  is the unique solution of

$$\begin{aligned} \tilde{\Phi}_{1t} + 4ik^2\tilde{\Phi}_1 &= \tilde{Q}_{11}\tilde{\Phi}_1 + \tilde{Q}_{12}\tilde{\Phi}_2, \\ \tilde{\Phi}_{2t} &= \tilde{Q}_{21}\tilde{\Phi}_1 + \tilde{Q}_{22}\tilde{\Phi}_2, \quad t > 0, \quad \text{Im } k^2 \geq 0, \\ \lim_{t \rightarrow \infty} \tilde{\Phi}(t, k) &= (0, 1)^t. \end{aligned}$$

In the case  $T < \infty$ , this definition is equivalent to (3.11). Note also, that the functions  $\tilde{\Phi}_1(t, k)$  and  $\tilde{\Phi}_2(t, k)$  satisfy the system of linear Volterra integral equations,

$$\begin{aligned} \tilde{\Phi}_1(t, k) &= - \int_t^\infty e^{-4ik^2(t-\tau)} (\tilde{Q}_{11}\tilde{\Phi}_1 + \tilde{Q}_{12}\tilde{\Phi}_2)(\tau, k) d\tau, \\ \tilde{\Phi}_2(t, k) &= 1 - \int_t^\infty (\tilde{Q}_{21}\tilde{\Phi}_1 + \tilde{Q}_{22}\tilde{\Phi}_2)(\tau, k) d\tau. \end{aligned}$$

Therefore, in the case  $T = \infty$  the spectral functions  $A(k)$  and  $B(k)$  are well defined and analytic for  $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$  only.

The spectral functions  $A(k)$  and  $B(k)$  have the following properties:

### Properties of $A(k)$ and $B(k)$

- (i)  $A(k), B(k)$  are entire functions bounded for  $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ . If  $T = \infty$ , the functions  $A(k)$  and  $B(k)$  are defined only for  $k$  in these quadrants.
- (ii)  $A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right)$ ,  $B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right)$ ,  $k \rightarrow \infty$ .
- (iii)  $A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1$ ,  $k \in \mathbb{C}$  ( $k \in \mathbb{R} \cup i\mathbb{R}$ , if  $T = \infty$ ).

(iv) The map  $\tilde{\mathbb{Q}} : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t)\}$ , inverse to  $\tilde{\mathbb{S}}$ , is defined as follows:

$$\begin{aligned} g_0(t) &= 2i \lim_{k \rightarrow \infty} (kM^{(t)}(t, k))_{12}, \\ g_1(t) &= \lim_{k \rightarrow \infty} \left[ 4 \left( k^2 M^{(t)}(t, k) \right)_{12} + 2ig_0(t)k \left( M^{(t)}(t, k) \right)_{22} \right], \end{aligned} \quad (3.14)$$

where  $M^{(t)}(t, k)$  is the unique solution of the following RH problem:

$$\bullet \quad M^{(t)}(t, k) = \begin{cases} M_+^{(t)}(t, k), & \arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}] \\ M_-^{(t)}(t, k), & \arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi], \end{cases} \quad (3.15a)$$

is a sectionally meromorphic function.

$$\bullet \quad M_-^{(t)}(t, k) = M_+^{(t)}(t, k)J^{(t)}(t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad (3.15b)$$

where

$$J^{(t)}(t, k) = \begin{bmatrix} 1 & -\frac{B(k)}{A(k)}e^{-4ik^2t} \\ \frac{\lambda B(\bar{k})}{A(k)}e^{4ik^2t} & \frac{1}{A(k)A(\bar{k})} \end{bmatrix}. \quad (3.15c)$$

$$\bullet \quad M^{(t)}(t, k) = I + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \quad (3.15d)$$

- We assume that  $A(k)$  has  $N$  simple zeros  $\{K_j\}_1^N$ ,  $\arg K_j \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ . The first column of  $M_+^{(t)}(t, k)$  has simple poles at  $k = K_j$ ,  $j = 1, \dots, N$ , and the second column of  $M_-^{(t)}(t, k)$  has simple poles at  $k = \bar{K}_j$ , where  $\{K_j\}_1^N$  are the simple zeros of  $A(k)$ ,  $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ . The associated residues are given by

$$\begin{aligned} \operatorname{Res}_{K_j} [M^{(t)}(t, k)]_1 &= \frac{\exp[4iK_j^2t]}{A(K_j)B(K_j)} [M^{(t)}(t, K_j)]_2, \quad j = 1, \dots, N, \\ \operatorname{Res}_{\bar{K}_j} [M^{(t)}(t, k)]_2 &= \frac{\lambda \exp[-4i\bar{K}_j^2t]}{A(\bar{K}_j)B(\bar{K}_j)} [M^{(t)}(t, \bar{K}_j)]_1, \quad j = 1, \dots, N. \end{aligned} \quad (3.15e)$$

(v) We have

$$\tilde{\mathbb{S}}^{-1} = \tilde{\mathbb{Q}}. \quad (3.16)$$

*Proof.* (i)–(iii) follow from the definition; the derivation of (iv), (v) is given in the Appendix A.  $\square$

**Remark 3.4.** The properties of  $A(k)$  and  $B(k)$  imply that  $A(k)$  can be expressed in terms of  $B(k)$ . Indeed, if  $A(k) \neq 0$ , then

$$A(k) = \prod_{j=1}^N \frac{k - K_j}{k - \bar{K}_j} \exp \left\{ \frac{1}{2i\pi} \int_{\mathcal{L}} \ln(1 + \lambda B(k') \overline{B(\bar{k}')} ) \frac{dk'}{k' - k} \right\},$$

for  $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ , where the contour  $\mathcal{L}$  is the union of the real and the imaginary axis with the orientation shown in Figure 2.2. Also,

$$B(\pm k) = \int_0^\infty \hat{B}_\pm(s) e^{ik^2 s} ds, \quad \arg k \in \left[0, \frac{\pi}{2}\right].$$

Thus, the maps  $\tilde{\mathbb{S}}$  and  $\tilde{\mathbb{Q}}$  define the bijection

$$\{g_0(t), g_1(t)\} \longleftrightarrow \{B(k), K_1, \dots, K_N, N < \infty\}. \quad (3.17)$$

The global relation suggests the following notion of an admissible set of boundary values.

**Definition 3.5.** (An admissible set of functions)

Given  $q_0(x) \in S(\mathbb{R}^+)$  define  $a(k)$  and  $b(k)$  according to Definition 3.1. Suppose that there exist smooth functions  $g_0(t)$  and  $g_1(t)$ , such that :

(i) The associated  $A(k)$ ,  $B(k)$  defined according to Definition 3.3 satisfy the relation

$$a(k)B(k) - b(k)A(k) = e^{4ik^2 T} c^+(k), \quad \arg k \in [0, \pi], \quad (3.18)$$

where  $c^+(k)$  is analytic for  $\text{Im } k > 0$  and continuous and bounded for  $\text{Im } k \geq 0$  and  $c^+(k) = O\left(\frac{1}{k}\right)$ ,  $k \rightarrow \infty$ .

(ii) The functions  $q(0, t) = g_0(t)$ ,  $q_x(0, t) = g_1(t)$  and  $q(x, 0) = q_0(x)$  are compatible with the NLS equation at  $x = t = 0$ , i.e., they satisfy

$$\begin{aligned} g_0(0) &= q_0(0), \quad g_1(0) = q_0'(0), \quad ig_0'(0) + q_0''(0) - 2\lambda(|q_0|^2 q_0)(0) = 0, \\ ig_1'(0) + q_0'''(0) - 2\lambda(|q_0|^2 q_0)'(0) &= 0, \dots \end{aligned}$$

(The exact number of conditions depends on the regularity of the solution that is to be constructed using  $g_0$  and  $g_1$ .) Then we call  $\{g_0(t), g_1(t)\}$  an admissible set of functions with respect to  $q_0(x)$ .

**Remark 3.6.** If  $T = \infty$ , then the functions  $g_0(t)$  and  $g_1(t)$  are assumed to belong in  $S(\mathbb{R}_+)$ , and the global relation (3.18) transforms into <sup>1</sup>.

$$a(k)B(k) - b(k)A(k) = 0, \quad \arg k \in \left[0, \frac{\pi}{2}\right]. \quad (3.19)$$

---

<sup>1</sup>We note that this condition is similar to the restrictions on the scattering data that appears in the boundary problem for the elliptic version of the sine-Gordon equation [8]



## 4 The Riemann-Hilbert Problem

**Theorem 4.1.** *Let  $q_0(x) \in S(\mathbb{R}^+)$ . Suppose that the set of functions  $g_0(t)$  and  $g_1(t)$  are admissible with respect to  $q_0(x)$ , see Definition 3.5. Define the spectral functions  $a(k)$ ,  $b(k)$ ,  $A(k)$  and  $B(k)$ , in terms of  $q_0(x)$ ,  $g_0(t)$ ,  $g_1(t)$  according to Definitions 3.1 and 3.3. Assume that:*

- (i) *If  $\lambda = -1$ ,  $a(k)$  has at most  $n$  simple zeros  $\{k_j\}_1^n$ ,  $n = n_1 + n_2$ , where  $\arg k_j \in (0, \frac{\pi}{2})$ ,  $j = 1, \dots, n_1$ ;  $\arg k_j \in (\frac{\pi}{2}, \pi)$ ,  $j = n_1 + 1, \dots, n_1 + n_2$ .*
- (ii) *If  $\lambda = -1$ , the function  $d(k)$  (cf. (2.25)) has at most  $\Lambda$  simple zeros  $\{\lambda_j\}_1^\Lambda$ , where*

$$\arg \lambda_j \in \left(\frac{\pi}{2}, \pi\right), \quad j = 1, \dots, \Lambda.$$

*If  $\lambda = 1$ , the function  $d(k)$  has no zeros in the second quadrant.*

- (iii) *None of the zeros of  $a(k)$  for  $\arg k \in (\frac{\pi}{2}, \pi)$ , coincides with a zero of  $d(k)$ .*

Define  $M(x, t, k)$  as the solution of the following  $2 \times 2$  matrix RH problem:

- *$M$  is sectionally meromorphic in  $k \in \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$ .*
- *The first column of  $M$  has simple poles at  $k_j$ ,  $j = 1, \dots, n_1$  and  $\lambda_j$ ,  $j = 1, \dots, \Lambda$ ; the second column of  $M$  has simple poles at  $\bar{k}_j$ ,  $j = 1, \dots, n_1$  and  $\bar{\lambda}_j$ ,  $j = 1, \dots, \Lambda$ . The associated residues satisfy the relations in (2.29).*
- *$M$  satisfies the jump condition*

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad (4.1)$$

*where  $M$  is  $M_-$  for  $\arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$ ,  $M$  is  $M_+$  for  $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ , and  $J$  is defined in terms of  $a$ ,  $b$ ,  $A$  and  $B$  by equations (2.25)–(2.28), see Figure 2.2.*

- *At  $\infty$  we have*

$$M(x, t, k) = I + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \quad (4.2)$$

*Then  $M(x, t, k)$  exists and is unique.*

*Define  $q(x, t)$  in terms of  $M(x, t, k)$  by*

$$q(x, t) = 2i \lim_{k \rightarrow \infty} (kM(x, t, k))_{12}. \quad (4.3)$$

*Then  $q(x, t)$  solves the NLS equation (2.1). Furthermore,*

$$q(x, 0) = q_0(x), \quad q(0, t) = g_0(t) \quad \text{and} \quad q_x(0, t) = g_1(t).$$

*Proof.* If  $\lambda = 1$  the function  $a(k) \neq 0$  for  $\text{Im } k > 0$  (see Remark 2.3), and by assumption  $d(k) \neq 0$  for  $\arg k \in (\frac{\pi}{2}, \pi)$ . In this case the unique solvability of the RH problem is a consequence of the existence of a “vanishing lemma”, i.e. the RH obtained from the above RH by replacing (4.2) with  $M = O(\frac{1}{k})$ ,  $k \rightarrow \infty$ , has only the trivial solution. The vanishing lemma can be established using the symmetry properties of  $J$ , see [9]. If  $\lambda = -1$ ,  $a(k)$  and  $d(k)$  can have zeros; this “singular” RH problem can be mapped to a “regular” RH problem (i.e. to a RH problem for holomorphic functions) coupled with a system of algebraic equations, see [9]. The unique solvability of the relevant algebraic equations and the proof of the associated vanishing lemma are based on the symmetry properties of  $J$ , see [9].

*Proof that  $q(x, t)$  solves the NLS.*

It is straightforward to prove that if  $M$  solves the above RH problem and if  $q(x, t)$  is defined by (4.4) then  $q(x, t)$  solves the NLS equation. This proof is based on ideas from the dressing method, see [10].

*Proof that  $q(x, 0) = q_0(x)$ .*

Define  $M^{(x)}(x, k)$  by:

$$M^{(x)} = M(x, 0, k), \quad \arg k \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]; \quad (4.4a)$$

$$M^{(x)} = M(x, 0, k)J_1^{-1}(x, 0, k), \quad \arg k \in [\frac{\pi}{2}, \pi]; \quad (4.4b)$$

$$M^{(x)} = M(x, 0, k)J_3(x, 0, k), \quad \arg k \in [\pi, \frac{3\pi}{2}]. \quad (4.4c)$$

We first discuss the case that the sets  $\{k_j\}$  and  $\{\lambda_j\}$  are empty. The function  $M^{(x)}$  is sectionally meromorphic in  $\mathbb{C} \setminus \mathbb{R}$ . Furthermore,

$$\begin{aligned} M_-^{(x)}(x, k) &= M_+^{(x)}(x, k)J^{(x)}(x, k), \quad k \in \mathbb{R}, \\ M^{(x)}(x, k) &= I + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \end{aligned}$$

where  $J^{(x)}(x, t)$  is defined in (3.6c). Thus according to (3.5),

$$q_0(x) = 2i \lim_{k \rightarrow \infty} k(M^{(x)}(x, k))_{12}. \quad (4.5)$$

Comparing this equation with equation (4.3) evaluated at  $t = 0$ , we conclude that  $q_0(x) = q(x, 0)$ .

We now discuss the case that the sets  $\{k_j\}$  and  $\{\lambda_j\}$  are *not* empty. The first column of  $M(x, t, k)$  has poles at  $\{k_j\}_1^{n_1}$  for  $\arg k \in (0, \frac{\pi}{2})$ , and has poles at  $\{\lambda_j\}_1^\Lambda$  for  $\arg k \in (\frac{\pi}{2}, \pi)$ . On the other hand the first column of  $M^{(x)}(x, k)$  should have poles at  $\{k_j\}_1^n$ ,  $n = n_1 + n_2$ . We will now show that the transformations defined by (4.4) map the former poles to the latter ones. Since  $M^{(x)} = M(x, 0, k)$  for  $\arg k \in [0, \frac{\pi}{2}]$ ,  $M^{(x)}$  has poles at  $\{k_j\}_1^{n_1}$  with the correct residue condition. Letting  $M = (M_1, M_2)$ , equation (4.4b) can be written as

$$M^{(x)}(x, k) = (M_1(x, 0, k) - \Gamma(k)e^{2ikx}M_2(x, 0, k), M_2(x, 0, k)).$$

The residue condition at  $\lambda_j$  implies that  $M^{(x)}$  has *no* poles at  $\lambda_j$ ; on the other hand this equation shows that  $M^{(x)}$  has poles at  $\{k_j\}_{n_1+1}^n$  with residues given by

$$\operatorname{Res}_{k_j} [M^{(x)}(x, k)]_1 = -\operatorname{Res}_{k_j} \Gamma(k) e^{2ik_j x} [M^{(x)}(x, k_j)]_2, \quad j = n_1 + 1, \dots, n,$$

which using the definition of  $\Gamma(k)$  (and the equation  $d(k_j) = -\lambda b(k_j) \overline{B(\bar{k}_j)}$ ) becomes the residue condition of (3.6e). Similar considerations apply to  $\bar{k}_j$  and  $\bar{\lambda}_j$ .

*Proof that  $q(0, t) = g_0(t)$  and  $q_x(0, t) = g_1(t)$ .*

Let  $M^{(1)}(x, t, k), \dots, M^{(4)}(x, t, k)$  denote  $M(x, t, k)$  for  $\arg k \in [0, \frac{\pi}{2}], \dots, [\frac{3\pi}{2}, 2\pi]$ . Recall that  $M$  satisfies

$$\begin{aligned} M^{(2)} &= M^{(1)} J_1, & M^{(2)} &= M^{(3)} J_2, \\ M^{(4)} &= M^{(1)} J_4, & M^{(4)} &= M^{(3)} J_3, \quad (J_2 = J_3 J_4^{-1} J_1) \end{aligned} \quad (4.6)$$

on the respective parts of the contour  $\mathcal{L} = \mathbb{R} \cup i\mathbb{R}$  (cf. Figure 2.2).

Let  $M^{(t)}(t, k)$  be defined by

$$M^{(t)}(t, k) = M(0, t, k) G(t, k), \quad (4.7)$$

where  $G$  is given by  $G^{(1)}, \dots, G^{(4)}$  for  $\arg k \in [0, \frac{\pi}{2}], \dots, [\frac{3\pi}{2}, 2\pi]$ . Suppose we can find matrices  $G^{(1)}$  and  $G^{(2)}$  holomorphic for  $\operatorname{Im} k > 0$  (and continuous for  $\operatorname{Im} k \geq 0$ ), matrices  $G^{(3)}$  and  $G^{(4)}$  holomorphic for  $\operatorname{Im} k < 0$  (and continuous for  $\operatorname{Im} k \leq 0$ ), which tend to  $I$  as  $k \rightarrow \infty$ , and which satisfy

$$J_1(0, t, k) G^{(2)}(t, k) = G^{(1)}(t, k) J^{(t)}(t, k), \quad k \in i\mathbb{R}^+, \quad (4.8a)$$

$$J_3(0, t, k) G^{(4)}(t, k) = G^{(3)}(t, k) J^{(t)}(t, k), \quad k \in i\mathbb{R}^-, \quad (4.8b)$$

$$J_4(0, t, k) G^{(4)}(t, k) = G^{(1)}(t, k) J^{(t)}(t, k), \quad k \in \mathbb{R}^+, \quad (4.8c)$$

where  $J^{(t)}(t, k)$  is the jump matrix in (3.15c). Then the equations in (4.8) yield

$$J_2(0, t, k) G^{(2)}(t, k) = G^{(3)}(t, k) J^{(t)}(t, k), \quad k \in \mathbb{R}^-,$$

and equations (4.6) and (4.7) imply that  $M^{(t)}$  satisfies the RH problem defined in (3.15). If the sets  $\{k_j\}$  and  $\{\lambda_j\}$  are empty, this immediately yields the desired result.

We will show that such  $G^{(j)}$  matrices are:

$$\begin{aligned} G^{(1)} &= \begin{bmatrix} \frac{a(k)}{A(k)} & c^+(k) e^{4ik^2(T-t)} \\ 0 & \frac{A(k)}{a(k)} \end{bmatrix}, & G^{(4)} &= \begin{bmatrix} \frac{\overline{A(\bar{k})}}{a(\bar{k})} & 0 \\ \lambda \overline{c^+(\bar{k})} e^{-4i\bar{k}^2(T-t)} & \frac{a(\bar{k})}{A(\bar{k})} \end{bmatrix} \\ G^{(2)} &= \begin{bmatrix} d(k) & \frac{-b(k) e^{-4ik^2 t}}{A(\bar{k})} \\ 0 & \frac{1}{d(k)} \end{bmatrix}, & G^{(3)} &= \begin{bmatrix} \frac{1}{d(\bar{k})} & 0 \\ \frac{-\lambda \overline{b(\bar{k})}}{A(\bar{k})} e^{4i\bar{k}^2 t} & d(\bar{k}) \end{bmatrix}. \end{aligned} \quad (4.9)$$

We first verify (4.8a): The (12) element is proportional to the global relation; the (21) and (22) elements are satisfied identically. The (11) element is satisfied iff

$$d = \frac{a}{A} + \frac{\lambda \bar{B}}{A} c^+ e^{4ik^2 T}. \quad (4.10)$$

Using  $A\bar{A} - \lambda B\bar{B} = 1$ , we find

$$d = \frac{a}{A} A\bar{A} - \lambda b\bar{B} = \frac{a}{A} (1 + \lambda B\bar{B}) - \lambda b\bar{B} = \frac{a}{A} + \frac{\lambda \bar{B}}{A} (aB - bA),$$

which equals the rhs of equation (4.10) in view of the global relation (3.18).

The equation (4.8b) follows from the first one and the symmetry relations,

$$G^{(4)}(k) = \sigma_\lambda \overline{G^{(1)}(\bar{k})} \sigma_\lambda, \quad G^{(3)}(k) = \sigma_\lambda \overline{G^{(2)}(\bar{k})} \sigma_\lambda, \quad J_3(k) = \sigma_\lambda \overline{J_1^{-1}(\bar{k})} \sigma_\lambda,$$

where

$$\sigma_\lambda = \begin{cases} \sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } \lambda = 1 \\ \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \text{if } \lambda = -1 \end{cases}.$$

The third equation (4.8c) can be verified in a way similar to (4.8a). In fact, in this case one has to use all three basic algebraic identities which hold on the real axis, i.e. both the determinant relations,  $|a|^2 - \lambda|b|^2 = 1$  and  $|A|^2 - \lambda|B|^2 = 1$ , and the global relation,  $a(k)B(k) - b(k)A(k) = c^+(k)e^{4ik^2 T}$ .

**Remark 4.2.** In the case  $T = \infty$ , the matrices  $G^{(j)}(t, k)$  are defined and analytic only in the respective quadrants of the complex plane  $k$ . Moreover, the global relation holds only in the first quadrant (see (3.19)). Therefore, one can not use (4.8) to establish the relation  $J_4(0, t, k)G^{(2)}(t, k) = G^{(3)}(t, k)J^{(t)}(t, k)$ ,  $k < 0$ . The latter, however, can be verified independently, with the use of the determinant relations.

We now consider the case that the sets  $\{k_j\}$  and  $\{\lambda_j\}$  are *not* empty.

(a)  $\arg k \in (0, \frac{\pi}{2})$

Let  $M = (M_1, M_2)$ , then equations (4.7) and (4.9) imply

$$M^{(t)}(t, k) = \left( \frac{a(k)}{A(k)} M_1(0, t, k), c^+(k) e^{4ik^2(T-t)} M_1(0, t, k) + \frac{A(k)}{a(k)} M_2(0, t, k) \right).$$

Suppose that  $k_0 \in \{k_j\}_1^{n_1}$  and  $k_0 \notin \{K_j\}_1^{N_1}$ , where  $\{K_j\}_1^{N_1}$  denotes the set of zeros of  $A(k)$  in the first quadrant. Then,  $M^{(t)}(t, k)$  does *not* have a pole at  $k_0$ . Indeed,

$$\text{Res}_{k_0} [M^{(t)}(t, k)]_2 = c^+(k_0) e^{4ik_0^2(T-t)} \text{Res}_{k_0} M_1(0, t, k) + \frac{A(k_0)}{\dot{a}(k_0)} M_2(0, t, k_0).$$

Using

$$\text{Res}_{k_0} M_1(0, t, k) = \frac{M_2(0, t, k_0) e^{4ik_0^2 t}}{\dot{a}(k_0) b(k_0)},$$

we find

$$\operatorname{Res}_{k_0} [M^{(t)}(t, k)]_2 = \frac{M_2(0, t, k_0)}{\dot{a}(k_0)b(k_0)} \left( c^+(k_0)e^{4ik_0^2T} + b(k_0)A(k_0) \right).$$

From the global relation, the term in the parentheses equals  $a(k_0)B(k_0)$ , hence

$$\operatorname{Res}_{k_0} [M^{(t)}(t, k)]_2 = 0.$$

Suppose that  $K_0 \in \{K_j\}_1^{N_1}$  and  $K_0 \notin \{k_j\}_1^{n_1}$ . Then,  $[M^{(t)}(t, k)]_1$  has a pole at  $K_0$ . In order to compute the associated residues we note that

$$\operatorname{Res}_{K_0} [M^{(t)}(t, k)]_1 = \frac{a(K_0)}{\dot{A}(K_0)} M_1(0, t, K_0).$$

Using the definition of the second column of  $M^{(t)}$  evaluated at  $k = K_0$

$$M_1(0, t, K_0) = \frac{e^{4iK_0^2t}[M^{(t)}(t, K_0)]_2}{c^+(K_0)e^{4iK_0^2T}},$$

and the global relation evaluated at  $k = K_0$ ,

$$a(K_0)B(K_0) = c^+(K_0)e^{4iK_0^2T},$$

we find

$$\operatorname{Res}_{K_0} [M^{(t)}(t, k)]_1 = \frac{e^{4iK_0^2t}[M^{(t)}(t, K_0)]_2}{\dot{A}(K_0)B(K_0)},$$

which is the residue condition in (3.15e). (Note that since  $K_0$  is not a common zero for  $a(k)$  and  $A(k)$ , the inequality,  $c^+(K_0) \neq 0$ , holds.)

Suppose now that  $k_0 \equiv K_0$  is a common (simple) zero of the functions  $a(k)$  and  $A(k)$ . Then necessarily

$$c^+(k_0) = 0, \quad (4.11)$$

and the second column of  $M^{(t)}(t, k)$  does not have pole at  $k_0$ . The first column has a pole at  $k_0 \equiv K_0$ , and for the residue condition we have,

$$\operatorname{Res}_{K_0} [M^{(t)}(t, k)]_1 = \frac{\dot{a}(K_0)}{\dot{A}(K_0)} \operatorname{Res}_{K_0} M_1(0, t, k) = \frac{e^{4iK_0^2t}}{\dot{A}(K_0)b(K_0)} M_2(0, t, K_0). \quad (4.12)$$

Using, as before, the definition of the second column of  $M^{(t)}$  evaluated at  $k = K_0$  we obtain the equation,

$$\begin{aligned} [M^{(t)}(t, k)]_2 &= \dot{c}^+(K_0)e^{4iK_0^2(T-t)} \operatorname{Res}_{K_0} M_1(0, t, k) + \frac{\dot{A}(K_0)}{\dot{a}(K_0)} M_2(0, t, K_0) \\ &= M_2(0, t, K_0) \left( \frac{\dot{A}(K_0)}{\dot{a}(K_0)} + \frac{\dot{c}^+(K_0)e^{4iK_0^2T}}{\dot{a}(K_0)b(K_0)} \right) = M_2(0, t, K_0) \frac{B(K_0)}{b(K_0)}, \end{aligned} \quad (4.13)$$

where in the last step we have used the equation

$$\dot{c}^+(K_0)e^{4iK_0^2T} = \dot{a}(K_0)B(K_0) - \dot{A}(K_0)b(K_0),$$

which follows from the global relation and from equation (4.11). In virtue of (4.13), equation (4.12) can be rewritten as

$$\operatorname{Res}_{K_0} [M^{(t)}(t, k)]_1 = \frac{e^{4iK_0^2t}}{\dot{A}(K_0)B(K_0)} [M^{(t)}(t, K_0)]_2,$$

which again reproduces the residue condition in (3.15e).

We note that the last arguments, further simplified by  $\dot{c}^+(K_0)e^{4iK_0^2T} \mapsto 0$ , are precisely the ones we need in the case  $T = \infty$ , when the global relation takes the form (3.19) so that  $\{k_j\}_1^{n_1} = \{K_j\}_1^{N_1}$ .

(b)  $\arg k \in (\frac{\pi}{2}, \pi)$

Equations (4.7) and (4.9) imply

$$M^{(t)}(t, k) = \left( d(k)M_1(0, t, k), -\frac{b(k)}{A(k)}e^{-4ik^2t}M_1(0, t, k) + \frac{M_2(0, t, k)}{d(k)} \right).$$

Suppose that  $\lambda_0 \in \{\lambda_j\}_1^\Lambda$  and  $\lambda_0 \notin \{\bar{K}_j\}_{N_1+1}^N$ , where  $\{K_j\}_{N_1+1}^N$  denotes the set of zeros of  $A(k)$  in the third quadrant. Then,  $M^{(t)}(t, k)$  does *not* have a pole at  $\lambda_0$ . Indeed,

$$\operatorname{Res}_{\lambda_0} [M^{(t)}(t, k)]_2 = \frac{-b(\lambda_0)}{A(\lambda_0)}e^{-4i\lambda_0^2t} \operatorname{Res}_{\lambda_0} M_1(0, t, k) + \frac{M_2(0, t, \lambda_0)}{d(\lambda_0)}.$$

Using

$$\operatorname{Res}_{\lambda_0} M_1(0, t, k) = \frac{\overline{\lambda B(\bar{\lambda}_0)}e^{4i\lambda_0^2t}M_2(0, t, \lambda_0)}{a(\lambda_0)\dot{d}(\lambda_0)}, \quad (4.14)$$

and taking into account that under the assumption on  $\lambda_0$ ,

$$d(\lambda_0) = 0 \quad \implies \quad \frac{\overline{\lambda B(\bar{\lambda}_0)}}{a(\lambda_0)} = \frac{\overline{A(\bar{\lambda}_0)}}{b(\lambda_0)},$$

it follows that  $\operatorname{Res}_{\lambda_0} [M^{(t)}(t, k)]_2 = 0$ .

Suppose that  $K_0 \in \{K_j\}_{N_1+1}^N$  and  $\bar{K}_0 \notin \{\lambda_j\}_1^\Lambda$ . Then,  $[M^{(t)}(t, k)]_2$  has a pole at  $\bar{K}_0$ . In order to compute the associated residues we note that

$$\operatorname{Res}_{\bar{K}_0} [M^{(t)}(t, k)]_2 = \frac{-b(\bar{K}_0)}{\dot{A}(K_0)}e^{-4i\bar{K}_0^2t}M_1(0, t, \bar{K}_0).$$

Using the definition of the first column of  $M^{(t)}$  at  $k = \bar{K}_0$  and recalling that  $d(\bar{K}_0) = -\lambda B(K_0)b(\bar{K}_0)$  (and hence, in particular,  $\overline{B(K_0)}b(\bar{K}_0) \neq 0$ ), we find

$$[M^{(t)}(t, \bar{K}_0)]_1 = -\lambda \overline{B(K_0)}b(\bar{K}_0)M_1(0, t, \bar{K}_0).$$

Thus

$$\operatorname{Res}_{\bar{K}_0} [M^{(t)}(t, k)]_2 = \frac{\lambda e^{-4i\bar{K}_0^2 t} [M^{(t)}(t, \bar{K}_0)]_1}{\overline{\dot{A}(K_0)B(K_0)}},$$

which is the residue condition in (3.15e).

Suppose now that  $\lambda_0 \equiv \bar{K}_0$  is a common (simple) zero of the functions  $d(k)$  and  $\overline{A(\bar{k})}$ . Then necessarily

$$b(\lambda_0) = 0,$$

and for the residue of  $[M^{(t)}(t, k)]_2$  at  $\bar{K}_0$  we have,

$$\begin{aligned} \operatorname{Res}_{\bar{K}_0} [M^{(t)}(t, k)]_2 &= \frac{-\dot{b}(\bar{K}_0)}{\overline{\dot{A}(K_0)}} e^{-4i\bar{K}_0^2 t} \operatorname{Res}_{\bar{K}_0} M_1(0, t, k) + \frac{M_2(0, t, \bar{K}_0)}{\dot{d}(\bar{K}_0)} \\ &= \frac{1}{\overline{\dot{A}(K_0)a(\bar{K}_0)}} M_2(0, t, \bar{K}_0), \end{aligned} \quad (4.15)$$

where we have used the residue condition (4.14) for  $M_1(0, t, k)$  at  $\lambda_0 \equiv \bar{K}_0$  and the equation,

$$\dot{d}(\bar{K}_0) = \overline{\dot{A}(K_0)a(\bar{K}_0)} - \lambda \overline{B(K_0)\dot{b}(\bar{K}_0)}.$$

Using the definition of the first column of  $M^{(t)}$  at  $k = \bar{K}_0$  and the residue equation (4.14) one more time, we conclude that

$$M_2(0, t, \bar{K}_0) = \lambda \frac{a(\bar{K}_0)}{\overline{B(K_0)}} e^{-4i\bar{K}_0^2 t} [M^{(t)}(t, \bar{K}_0)]_1,$$

which together with (4.15) yield again the residue condition in (3.15e).

Similar considerations are valid for  $\arg k \in [\frac{3\pi}{2}, 2\pi]$  and  $\arg k \in [\pi, \frac{3\pi}{2}]$ . Alternatively, one can use the symmetry relations generated by the anti - involution  $k \mapsto \bar{k}$ .  $\square$

Let  $0 < T_* < T$ . Since the solution of the NLS equation for  $0 < t < T_*$  depend only on the boundary data between  $0 < t < T_*$ , the RH problems corresponding to  $T_*$  and  $T$  must be related. This is confirmed by the following proposition.

**Proposition 4.3.** *Let  $A(T_*, k)$ ,  $B(T_*, k)$  be defined by (3.11) with  $T$  replaced by  $T_* < T$ ,  $\tilde{J}_1(x, t, k)$  and  $\tilde{J}_3(x, t, k)$  denote the jump matrices obtained from (2.27) by replacing  $A(k)$  and  $B(k)$  with  $A(T_*, k)$  and  $B(T_*, k)$ , and  $\tilde{J}_2 = \tilde{J}_3 J_4^{-1} \tilde{J}_1$ . Let  $\tilde{M}(x, t, k)$  satisfy a RH problem defined by (4.1) but with jump matrices  $\tilde{J}_1$ ,  $\tilde{J}_2$ ,  $\tilde{J}_3$  and  $J_4$ . Then for  $0 < t < T_*$  the restrictions of  $M(x, t, k)$  and  $\tilde{M}(x, t, k)$  to the four quadrants (cf. Figure 2.2) satisfy*

$$M_1 = \tilde{M}_1, \quad M_4 = \tilde{M}_4, \quad M_2 = \tilde{M}_2 \tilde{J}_1^{-1} J_1, \quad M_3 = \tilde{M}_3 \tilde{J}_3 J_3^{-1}. \quad (4.16)$$

*Proof.* Using equation (4.16) it is straightforward to verify that the jump condition for  $M$ , i.e., equation (4.1) yields similar jump condition for  $\tilde{M}$  with  $J_1, J_2, J_3$  replaced by  $\tilde{J}_1, \tilde{J}_2$  and  $\tilde{J}_3$ . Assuming the solitonless case it remains to show that the functions  $\tilde{J}_1^{-1} J_1$  and  $\tilde{J}_3 J_3^{-1}$  are analytic and bounded for  $\arg k \in (\frac{\pi}{2}, \pi)$  and  $\arg k \in (\pi, \frac{3\pi}{2})$ , respectively, and that both tend

to the identity matrix as  $k \rightarrow \infty$ . We will show this fact for the function  $\tilde{J}_3 J_3^{-1}$ , the proof for  $\tilde{J}_1^{-1} J_1$  follows from symmetry considerations.

The diagonal elements of  $\tilde{J}_3 J_3^{-1}$  are 1, its (21) element is 0, and its (12) element equals

$$\lambda(\overline{\Gamma(\bar{k})} - \overline{\Gamma(T_*, \bar{k})})e^{-2i\theta} = \frac{B(k)A(T_*, k) - A(k)B(T_*, k)}{d(\bar{k})d(T_*, \bar{k})}e^{-4ik^2 T_*}e^{-2ikx+4ik^2(T_*-t)}, \quad (4.17)$$

where the rhs of this equation follows from the lhs using the definitions of  $\Gamma(k)$ , of  $\Gamma(T_*, k)$ , and the notation

$$d(T_*, k) = a(k)\overline{A(T_*, \bar{k})} - \lambda b(k)\overline{B(T_*, \bar{k})}.$$

The definition of  $A(T_*, k)$  and  $B(T_*, k)$  implies that they have the same properties as  $A(k)$ ,  $B(k)$ , where  $T$  is replaced by  $T_*$  in the second property. Thus since  $d(\bar{k})$  is bounded and analytic for  $k \in D_3$  the same is true for  $d(T_*, \bar{k})$ . Also, the definition of  $A(T_*, k)$  and  $B(T_*, k)$  implies that

$$\begin{aligned} [B(k)A(T_*, k) - B(T_*, k)A(k)]e^{-4ik^2 T_*} \\ = \overline{\Phi_2(T, \bar{k})}\Phi_1(T_*, k) - \Phi_1(T, k)\overline{\Phi_2(T_*, \bar{k})}e^{4ik^2(T-T_*)}. \end{aligned} \quad (4.18)$$

We will show that the r.h.s. of equation (4.18) is bounded and analytic for  $k \in D_1 \cup D_3$ , and that it goes to zero as  $k \rightarrow \infty$ ,  $k \in \overline{D_1} \cup \overline{D_3}$ . This result together with the fact that  $\exp[-2ikx + 4ik^2(T_* - t)]$  is bounded for  $k \in D_3$  imply that the r.h.s. of equation (4.17) is bounded and analytic for  $k \in D_3$ , and it goes to zero as  $k \rightarrow \infty$ ,  $k \in \overline{D_3}$ .

In order to prove that the r.h.s. of (4.18) is bounded and analytic for  $k \in D_1 \cup D_3$  we introduce the notations

$$\begin{aligned} \chi_1(t, k) &= \overline{\Phi_2(T, \bar{k})}\Phi_1(t, k) - \Phi_1(T, k)\overline{\Phi_2(t, \bar{k})}e^{4ik^2(T-t)}, \\ \chi_2(t, k) &= \overline{\Phi_2(T, \bar{k})}\Phi_2(t, k) - \lambda\Phi_1(T, k)\overline{\Phi_1(t, \bar{k})}e^{4ik^2(T-t)}. \end{aligned} \quad (4.19)$$

We will prove that the functions  $\chi_1$  and  $\chi_2$  satisfy the following system of linear integral equations

$$\begin{aligned} \chi_1(t, k) &= - \int_t^T [\tilde{Q}_{11}(\tau, k)\chi_1(\tau, k) + \tilde{Q}_{12}(\tau, k)\chi_2(\tau, k)]e^{4ik^2(\tau-t)} d\tau, \\ \chi_2(t, k) &= 1 - \int_t^T [\tilde{Q}_{22}(\tau, k)\chi_2(\tau, k) + \tilde{Q}_{21}(\tau, k)\chi_1(\tau, k)] d\tau, \end{aligned} \quad (4.20)$$

where  $\tilde{Q}_{ij}$  denote, as usual, the entries of the matrix  $\tilde{Q}(t, k)$ . Indeed, the symmetry properties of  $\tilde{Q}(t, k)$  imply that if the vector  $\Phi(t, k)$  with the two components  $\Phi_1$  and  $\Phi_2$  satisfies equation (3.12), then the vector  $\left(\lambda\overline{\Phi_2(t, \bar{k})}, \overline{\Phi_1(t, \bar{k})}\right)^t e^{-4ik^2 t}$  also satisfies the same equation. Hence the vector  $\chi(t, k)$  with the two components  $\chi_1$  and  $\chi_2$  defined by equations (4.19) satisfies equation (3.12). Furthermore,

$$\chi_1(T, k) = \overline{\Phi_2(\bar{k})}\Phi_1(k) - \Phi_1(k)\overline{\Phi_2(\bar{k})} = 0,$$



$$\chi_2(T, k) = \overline{\Phi_2(\bar{k})}\Phi_2(k) - \lambda\Phi_1(k)\overline{\Phi_1(\bar{k})} = 1.$$

The unique solution of equation (3.12) with the boundary condition  $\{\chi_1(T, k) = 0, \chi_2(T, k) = 1\}$  satisfies equations (4.20).

Equations (4.20) imply that  $\chi_1(t, k)$  is bounded and analytic for  $k \in D_1 \cup D_3$ , and that it goes to zero as  $k \rightarrow \infty$ ,  $k \in \overline{D_1} \cup \overline{D_3}$  uniformly for all  $0 < t < T$ . Since the r.h.s. of equation (4.18) equals  $\chi_1(T_*, k)$ ,  $T_* < T$ , it follows that the r.h.s. of equation (4.18) is also bounded and analytic for  $k \in D_1 \cup D_3$ , and that it goes to zero as  $k \rightarrow \infty$ ,  $k \in \overline{D_1} \cup \overline{D_3}$ .  $\square$

## 5 Construction of Admissible Sets of Functions

For simplicity we consider the special case where  $q_0 = 0$  and a function  $g_0 \in C^\infty([0, T])$  is given such that

$$\frac{d^\ell g_0}{dt^\ell}(0) = 0 \quad \text{for } \ell = 0, 1, 2, \dots \quad (5.1)$$

We will show that there exists a unique  $g_1 \in C^\infty([0, T])$  such that  $\{g_0, g_1\}$  is an admissible set of functions with respect to  $q_0$ .

Since  $q_0 = 0$ , we have  $a(k) = 1$  and  $b(k) = 0$  for all  $k \in \mathbb{C}$ . It follows from (3.9)–(3.11) that in this case the global relation (3.18) is reduced to

$$\Phi_1(T, k) = -c^+(k), \quad (5.2)$$

where the function  $\Phi_1$  is determined by

$$\Phi_{1,t} + 4ik^2\Phi_1 = -i\lambda|g_0(t)|^2\Phi_1 + (2kg_0(t) + ig_1(t))\Phi_2, \quad (5.3a)$$

$$\Phi_{2,t} = \lambda(2k\overline{g_0(t)} - \overline{ig_1(t)})\Phi_1 + i\lambda|g_0(t)|^2\Phi_2, \quad (5.3b)$$

$$\Phi_1(0, k) = 0, \quad \Phi_2(0, k) = 1. \quad (5.3c)$$

Substituting

$$\Phi_1 = \exp\left(-i\lambda \int_0^t |g_0(s)|^2 ds\right)\tilde{\Phi}_1 \quad \text{and} \quad \Phi_2 = \exp\left(i\lambda \int_0^t |g_0(s)|^2 ds\right)\tilde{\Phi}_2 \quad (5.4)$$

into (5.3), we find

$$\tilde{\Phi}_{1,t} + 4ik^2\tilde{\Phi}_1 = [2kf_0(t) + if_1(t)]\tilde{\Phi}_2, \quad (5.5a)$$

$$\tilde{\Phi}_{2,t} = \lambda[2k\overline{f_0(t)} - \overline{if_1(t)}]\tilde{\Phi}_1, \quad (5.5b)$$

$$\tilde{\Phi}_1(0, k) = 0, \quad \tilde{\Phi}_2(0, k) = 1, \quad (5.5c)$$

where

$$f_0(t) = g_0(t) \exp\left(2i\lambda \int_0^t |g_0(s)|^2 ds\right), \quad (5.6a)$$

$$f_1(t) = g_1(t) \exp\left(2i\lambda \int_0^t |g_0(s)|^2 ds\right). \quad (5.6b)$$

We can rewrite (5.5) into a system of integral equations:

$$\tilde{\Phi}_1(t, k) = \int_0^t e^{-4ik^2(t-t')} [2kf_0(t') + if_1(t')] \tilde{\Phi}_2(t', k) dt', \quad (5.7a)$$

$$\tilde{\Phi}_2(t, k) = 1 + \int_0^t \lambda [2k\overline{f_0(t')} - \overline{if_1(t')}] \tilde{\Phi}_1(t', k) dt'. \quad (5.7b)$$

It follows easily from (5.7) that  $\tilde{\Phi}_1(t, k)$  (and hence  $\Phi_1(t, k)$ ) is bounded and analytic for  $k$  in the second quadrant and it decays at the order of  $(1/k)$  as  $k \rightarrow \infty$ . Therefore the only condition imposed by the global relation (5.2) is for  $k$  in the first quadrant.

Now we let

$$\phi(t, k) = e^{4ik^2t} \tilde{\Phi}_1(t, k) \quad \text{and} \quad \psi(t, k) = e^{4ik^2t} \tilde{\Phi}_2(t, k), \quad (5.8)$$

and transform (5.7) into

$$\phi(t, k) = \int_0^t [2kf_0(t') + if_1(t')] \psi(t', k) dt', \quad (5.9a)$$

$$\psi(t, k) = e^{4ik^2t} + \int_0^t e^{4ik^2(t-t')} \lambda [2k\overline{f_0(t')} - \overline{if_1(t')}] \phi(t', k) dt'. \quad (5.9b)$$

From (5.2), (5.4) and (5.8), we can write the global relation as

$$\phi(T, k) = e^{4ik^2T} c_1(k). \quad (5.10)$$

where  $c_1(k)$  is bounded and analytic in the first quadrant, and it decays at the order of  $(1/k)$  as  $k \rightarrow \infty$ . So the analysis of the global relation involves exactly the equations (5.9) and (5.10), where  $f_0 \in C^\infty([0, T])$  (which vanishes to all orders at 0) is given and  $f_1$  (equivalently  $g_1$ ) is the unknown to be constructed.

We define the Sobolev space

$$H_{0*}^m = \{v \in H^m(0, T) : v(0) = v'(0) = \dots = v^{(m-1)}(0) = 0\}$$

and assume at first that

$$g_1 \text{ (equivalently } f_1) \in H_{0*}^1(0, T) = \{v \in H^1(0, T) : v(0) = 0\}. \quad (5.11)$$

Note that the Poincaré inequality

$$\max_{\alpha \leq t \leq \beta} |g(t)| \leq |g(\alpha)| + \sqrt{T} \|g'\|_{L_2(\alpha, \beta)} \quad \forall g \in H^1(\alpha, \beta) \quad (5.12)$$

implies that

$$\max_{0 \leq t \leq T} |g(t)| \leq \sqrt{T} \|g'\|_{L_2(0, T)} \quad \forall g \in H_{0*}^1(0, T). \quad (5.13)$$

By eliminating  $\phi$  from (5.9) we have

$$\psi = e^{4ik^2t} + \mathbf{F}(k, g_1)\psi, \quad (5.14)$$

where the operator  $\mathbf{F}(k, g_1)$  is given by

$$\mathbf{F}(k, g_1)\psi = \int_0^t e^{4ik^2(t-t')} \lambda [2k\overline{f_0(t')} - i\overline{f_1(t')}] \left[ \int_0^{t'} [2kf_0(s) + if_1(s)]\psi(s, k) ds \right] dt', \quad (5.15)$$

and  $f_1$  is given by (5.6b).

Let  $\mathcal{Q} = \{k \in \mathbb{C} : \operatorname{Re} k > 0, \operatorname{Im} k > 0\}$  be the first quadrant of the complex plane and  $\mathbb{R}_* = (-\infty, -1) \cup (1, \infty)$ . The proofs of the following lemmas on the solution of (5.14) can be found in Appendix C.

**Lemma 5.1.** *Under the condition (5.11), the integral equation (5.14) has a unique solution in  $C([0, T])$  for each  $k \in \bar{\mathcal{Q}}$  (the closure of  $\mathcal{Q}$ ),  $\psi(t, k)$  is bounded on  $[0, T] \times \bar{\mathcal{Q}}$  and the map  $k \mapsto \psi(\cdot, k)$  is analytic in  $\mathcal{Q}$  and continuous on  $\bar{\mathcal{Q}}$ . Moreover, the map  $g_1 \mapsto \psi$  from  $H_{0*}^1(0, T)$  into  $C([0, T] \times \bar{\mathcal{Q}})$  is locally Lipschitz continuous.*

**Remark 5.2.** More precisely, a map  $\mathcal{M}$  from the normed linear space  $X$  to the normed linear space  $Y$  is locally Lipschitz continuous if

$$\|\mathcal{M}(x_1) - \mathcal{M}(x_2)\|_Y \leq B(\|x_1\|_X, \|x_2\|_X)\|x_1 - x_2\|_X \quad \forall x_1, x_2 \in X, \quad (5.16)$$

where the function  $B(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}^+$  is continuous.

**Lemma 5.3.** *Under condition (5.11), we have the following asymptotic expansion for  $\psi$  :*

$$\psi(t, k) = e^{4ik^2t} \left( \chi_0(t) + \frac{\chi_1(t)}{k} + \frac{\chi_2(t)}{k^2} + \frac{\chi_3(t)}{k^3} + \frac{\chi_4(t)}{k^4} \right) + \psi_4(t, k) \quad (5.17)$$

for  $k \in \bar{\mathcal{Q}}$  and  $|k| > 1$ , where

$$\chi_0 = e^{-i\lambda \int_0^t |g_0(s)|^2 ds}, \quad \chi_1, \chi_2 \in H^2(0, T) \cap H_{0*}^1(0, T), \quad \chi_3, \chi_4 \in H_{0*}^1(0, T), \quad (5.18)$$

$$\psi_4(t, k) = O\left(\frac{1}{k^3}\right), \quad (5.19)$$

and

$$\text{the map } t \rightarrow \psi_4(t, \sqrt{\xi}) \text{ belongs to } C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi)). \quad (5.20)$$

Here  $k = \sqrt{\xi}$  is the inverse of  $\xi = k^2$  for  $k \in \bar{\mathcal{Q}}$ .

We will denote by  $\tilde{\psi}_4$  the map in  $C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$  defined by

$$[\tilde{\psi}_4(t)](\xi) = \psi_4(t, \sqrt{\xi}). \quad (5.21)$$

Given  $g_1 \in H_{0*}^1(0, T)$ , we define the maps  $E_1, E_2 : H_{0*}^1(0, T) \rightarrow H^2(0, T)$  by

$$E_1(g_1) = \chi_1 \quad \text{and} \quad E_2(g_1) = \chi_2,$$

the maps  $E_3, E_4 : H_{0*}^1(0, T) \rightarrow H^1(0, T)$  by

$$E_3(g_1) = \chi_3 \quad \text{and} \quad E_4(g_1) = \chi_4,$$

and the map  $E : H_{0*}^1(0, T) \rightarrow C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$  by

$$E(g_1) = \tilde{\psi}_4,$$

where  $\chi_j$  and  $\tilde{\psi}_4$  are the functions that appear in the asymptotic expansion (5.17) and (5.21).

**Lemma 5.4.** *The maps  $E_j$  ( $1 \leq j \leq 4$ ) and  $E$  are locally Lipschitz continuous.*

We now examine the global relation (5.10) and note immediately that it implies

$$\int_{\mathcal{L}} e^{-4ik^2t} \phi(T, k) 8k dk = 0 \quad \forall t < T, \quad (5.22)$$

where  $\mathcal{L}$  is the positively oriented boundary of  $\mathcal{Q}$  and the integral is taken in the sense of Cauchy principal value.

On the other hand from (5.9a) we have

$$\phi(T, k) = \int_0^T [2kf_0(t') + if_1(t')] \psi(t', k) dt'. \quad (5.23)$$

The asymptotic expansion (5.17) and (5.23) imply that

$$\phi(T, k) = 2k \int_0^T e^{4ik^2t'} f_0(t') \chi_0(t') dt' + R(T, k), \quad (5.24)$$

where the function  $R(T, k)$  is analytic in  $\mathcal{Q}$ , continuous on  $\bar{\mathcal{Q}}$  and decays at the order of  $1/k^2$  as  $k \rightarrow \infty$ . Let  $\alpha_0(T) = f_0(T) \chi_0(T)$ . We can rewrite (5.24) as

$$\phi(T, k) - \frac{\alpha_0(T)}{2i} \frac{e^{4ik^2T}}{(k+i)} = 2k \int_0^T e^{4ik^2t'} f_0(t') \chi_0(t') dt' - \frac{\alpha_0(T)}{2i} \frac{e^{4ik^2T}}{(k+i)} + R(T, k),$$

which shows that

$$\text{the function } \xi \mapsto \phi(T, \sqrt{\xi}) - \frac{\alpha_0(T)}{2i} \frac{e^{4i\xi T}}{(\sqrt{\xi} + i)} \text{ belongs to } L_2(\mathbb{R}). \quad (5.25)$$

Under condition (5.22), we have

$$\int_{\mathcal{L}} e^{-4ik^2t} \left[ \phi(T, k) - \frac{\alpha_0(T)}{2i} \frac{e^{4ik^2T}}{(k+i)} \right] 8k dk = 0 \quad \forall t < T.$$

Let

$$\phi_1(T, k) = e^{-4ik^2T} \left[ \phi(T, k) - \frac{\alpha_0(T)}{2i} \frac{e^{4ik^2T}}{(k+i)} \right].$$

Then (5.25) and the Paley-Wiener Theorem imply that the function  $\xi \mapsto \phi_1(T, \sqrt{\xi})$  belongs to the Hardy space  $H^2(\mathbb{C}_+)$ . Further regularization of  $\phi(T, k)$  yields the global relation (5.10).

In view of (5.24) and Jordan's lemma, equation (5.22) holds automatically for  $-\infty < t \leq 0$ . Therefore, the global relation is equivalent to

$$\int_{\mathcal{L}} e^{-4ik^2t} \phi(T, k) 8k dk = 0 \quad \text{for } 0 < t < T. \quad (5.26)$$

We now substitute (5.23) into (5.26) and use (5.6), (C.8) and the Fourier inversion formula to obtain the following equation:

$$g_1(t) = \left(\frac{4i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} [2kg_0(t') + ig_1(t')] [\psi(t', k) - e^{4ik^2t'} \chi_0(t')] dt' \right] dk + g_*(t) \quad \text{for } 0 < t < T, \quad (5.27)$$

where

$$\varpi(t) = \lambda \int_0^t |g_0(s)|^2 ds, \quad (5.28)$$

$$g_*(t) = \left(\frac{8i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k^2 \left[ \int_0^T e^{4ik^2t'} e^{i\varpi(t')} g_0(t') dt' \right] dk. \quad (5.29)$$

Note that  $g_* \in H_{0*}^1(0, T)$  and (5.27) is a nonlinear integral equation on  $H_{0*}^1(0, T)$  for the unknown  $g_1$  (since  $\psi(t, k)$  and  $\chi_0(t)$  also depend on  $g_1$ ). Below we will first show by a contraction mapping argument that it has a unique solution in  $H_{0*}^1(0, T)$  if  $T$  is sufficiently small.

Let  $\mathcal{L}_0$  be the part of  $\mathcal{L}$  that is inside the unit circle, and  $\mathcal{L}_\infty$  be the part of  $\mathcal{L}$  that is outside. We define

$$\mathbf{L}_0(g_1) = \left(\frac{4i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}_0} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} [2kg_0(t') + ig_1(t')] [\psi(t', k) - e^{4ik^2t'} \chi_0(t')] dt' \right] dk, \quad (5.30)$$

$$\mathbf{L}_\infty(g_1) = \left(\frac{4i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} [2kg_0(t') + ig_1(t')] [\psi(t', k) - e^{4ik^2t'} \chi_0(t')] dt' \right] dk. \quad (5.31)$$

The integral equation (5.27) can then be written concisely as

$$g_1 = \mathbf{L}_0(g_1) + \mathbf{L}_\infty(g_1) + g_*. \quad (5.32)$$

**Remark 5.5.** In the following analysis of the nonlinear operators  $\mathbf{L}_0$  and  $\mathbf{L}_\infty$ , we present estimates that are applicable in a more general setting (cf. (5.54) below). For example, we do not take advantage of the fact that  $g_0(0) = 0$ .

The estimate for the nonlinear map  $\mathbf{L}_0$  is straightforward. From Lemma 5.1, (C.8), (5.13), (5.28) and (5.30), we have

$$\|\mathbf{L}_0(g_1)\|_{C^1([0, T])} \leq T \cdot \mathcal{B}_{01}(\|g_1\|_{H^1(0, T)}), \quad (5.33)$$

$$\|\mathbf{L}_0(g_1) - \mathbf{L}_0(g_2)\|_{C^1([0, T])} \leq T \cdot \mathcal{B}_{02}(\|g_1\|_{H^1(0, T)}, \|g_2\|_{H^1(0, T)}) \|g_1 - g_2\|_{H^1(0, T)}, \quad (5.34)$$

for all  $g_1, g_2 \in H_{0*}^1(0, T)$ , where  $\mathcal{B}_*(\cdot)$  (resp.  $\mathcal{B}_*(\cdot, \cdot)$ ) from now on denote continuous functions from  $\mathbb{R}^+ \cup \{0\}$  (resp.  $(\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})$ ) into  $\mathbb{R}^+$ .

In order to estimate the nonlinear map  $\mathbf{L}_\infty$  we substitute the expansion (5.17) into (5.31) and write

$$\mathbf{L}_\infty = \sum_{j=1}^4 \mathbf{L}_{\infty,j}, \quad (5.35)$$

where

$$\mathbf{L}_{\infty,1}(g_1) = \frac{8i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} g_0(t') \sum_{j=1}^4 e^{4ik^2t'} \frac{\chi_j(t')}{k^{j-1}} dt' \right] dk, \quad (5.36)$$

$$\mathbf{L}_{\infty,2}(g_1) = -\frac{4}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} g_1(t') \sum_{j=1}^4 e^{4ik^2t'} \frac{\chi_j(t')}{k^j} dt' \right] dk, \quad (5.37)$$

$$\mathbf{L}_{\infty,3}(g_1) = \frac{8i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k^2 \left[ \int_0^T e^{2i\varpi(t')} g_0(t') \psi_4(t', k) dt' \right] dk, \quad (5.38)$$

$$\mathbf{L}_{\infty,4}(g_1) = -\frac{4}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} g_1(t') \psi_4(t', k) dt' \right] dk. \quad (5.39)$$

Using integration by parts and Jordan's lemma, We can rewrite (5.36) as

$$\begin{aligned} \mathbf{L}_{\infty,1}(g_1) &= \frac{2}{\pi} e^{-i\varpi(t)} \int_{\mathcal{C}_1} e^{4ik^2(T-t)} e^{2i\varpi(T)} g_0(T) \sum_{j=1}^2 \frac{\chi_j(T)}{k^j} dk \\ &\quad + \frac{8i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} \left[ \int_0^T e^{2i\varpi(t')} g_0(t') \sum_{j=3}^4 e^{4ik^2t'} \frac{\chi_j(t')}{k^{j-2}} dt' \right] dk \\ &\quad - \frac{2}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} \left[ \int_0^T e^{4ik^2t'} \sum_{j=1}^2 [e^{2i\varpi} g_0 \chi_j]'(t') k^{-j} dt' \right] dk, \end{aligned} \quad (5.40)$$

where  $\mathcal{C}_1$  is the part of the unit circle in the first quadrant connecting 1 to  $i$ . From the Plancherel theorem, Lemma 5.4, (5.40) and the Poincaré inequality (5.12), we obtain the following estimate for  $\mathbf{L}_{\infty,1}$ :

$$\|\mathbf{L}_{\infty,1}(g_1)\|_{H^1(0,T)} \leq \|g_0\|_{H^1(0,T)} \sum_{j=1}^4 \|\chi_j\|_{H^1(0,T)} \leq \mathcal{B}_{1,1}(\|g_1\|_{H^1(0,T)}), \quad (5.41)$$

$$\begin{aligned} \|\mathbf{L}_{\infty,1}(g_1) - \mathbf{L}_{\infty,1}(g_2)\|_{H^1(0,T)} &\leq \|g_0\|_{H^1(0,T)} \mathcal{B}_{1,2}(\|g_1\|_{L_2(0,T)}, \|g_2\|_{L_2(0,T)}) \\ &\quad \times \|g_1 - g_2\|_{H^1(0,T)}, \end{aligned} \quad (5.42)$$

for all  $g_1, g_2 \in H_{0*}^1(0, T)$ .

We can similarly rewrite (5.37) using integration by parts and Jordan's lemma to obtain

$$\begin{aligned} \mathbf{L}_{\infty,2}(g_1) &= \frac{i}{\pi} e^{-i\varpi(t)} \int_{iR}^i e^{4ik^2(T-t)} e^{2i\varpi(T)} g_1(T) \frac{\chi_1(T)}{k^2} dk \\ &\quad + \frac{i}{\pi} e^{-i\varpi(t)} \int_1^R e^{4ik^2(T-t)} e^{2i\varpi(T)} g_1(T) \frac{\chi_1(T)}{k^2} dk \end{aligned}$$

$$\begin{aligned}
& + \frac{i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{C}_R} e^{4ik^2(T-t)} e^{2i\varpi(T)} g_1(T) \frac{\chi_1(T)}{k^2} dk \\
& - \frac{i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}, |k| > R} e^{-4ik^2t} \left[ \int_0^T e^{4ik^2t'} [e^{2i\varpi} g_1 \chi_1]' k^{-2} dt' \right] dk \\
& - \frac{i}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}, 1 < |k| < R} e^{-4ik^2t} \left[ \int_0^T e^{4ik^2t'} [e^{2i\varpi} g_1 \chi_1]' k^{-2} dt' \right] dk \\
& - \frac{4}{\pi} e^{-i\varpi(t)} \int_{\mathcal{L}_\infty} e^{-4ik^2t} k \left[ \int_0^T e^{2i\varpi(t')} g_1(t') \sum_{j=2}^4 e^{4ik^2t'} \frac{\chi_j(t')}{k^j} dt' \right] dk, \tag{5.43}
\end{aligned}$$

where  $\mathcal{C}_R$  is the part of the circle of radius  $R$  in the first quadrant connecting  $R$  to  $iR$ . From (5.43) and Lemma 5.4 we find

$$\begin{aligned}
\|\mathbf{L}_{\infty,2}(g_1)\|_{H^1(0,T)} & \leq C \left( \frac{1}{R} + R\sqrt{T} \right) \|g_1\|_{H^1(0,T)} \sum_{j=1}^4 \|\chi_j\|_{H^1(0,T)} \\
& \leq \left( \frac{1}{R} + R\sqrt{T} \right) B_{2,1}(\|g_1\|_{H^1(0,T)}), \tag{5.44}
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{L}_{\infty,2}(g_1) - \mathbf{L}_{\infty,2}(g_2)\|_{H^1(0,T)} & \leq \left( \frac{1}{R} + R\sqrt{T} \right) B_{2,2}(\|g_1\|_{H^1(0,T)}, \|g_2\|_{H^1(0,T)}) \\
& \quad \times \|g_1 - g_2\|_{H^1(0,T)}, \tag{5.45}
\end{aligned}$$

for all  $g_1, g_2 \in H_{0*}^1(0, T)$ , where we have used the Plancherel theorem, the Cauchy-Schwarz inequality and the Poincaré inequality (5.12).

Using (5.38), the change of variable  $\xi = k^2$ , the Plancherel theorem and Lemma 5.4, we can derive the following estimates for  $\mathbf{L}_{\infty,3}$ :

$$\begin{aligned}
\|\mathbf{L}_{\infty,3}(g_1)\|_{H^1(0,T)} & \leq CT \|g_0\|_{C[0,T]} \max_{0 \leq t \leq T} \|\psi_4(t, \sqrt{\xi})\|_{L_2(\mathbb{R}_*, |\xi|^3 d\xi)} \\
& \leq T\mathcal{B}_{3,1}(\|g_1\|_{H^1(0,T)}), \tag{5.46}
\end{aligned}$$

$$\|\mathbf{L}_{\infty,3}(g_1) - \mathbf{L}_{\infty,3}(g_2)\|_{H^1(0,T)} \leq T\mathcal{B}_{3,2}(\|g_1\|_{H^1(0,T)}, \|g_2\|_{H^1(0,T)}) \|g_1 - g_2\|_{H^1(0,T)}, \tag{5.47}$$

for all  $g_1, g_2 \in H_{0*}^1(0, T)$ .

Similarly, we have the following estimates for  $\mathbf{L}_{\infty,4}$ :

$$\begin{aligned}
\|\mathbf{L}_{\infty,4}(g_1)\|_{H^1(0,T)} & \leq CT \|g_1\|_{H^1(0,T)} \max_{0 \leq t \leq T} \|\psi_4(t, \sqrt{\xi})\|_{L_2(\mathbb{R}_*, |\xi|^3 d\xi)} \\
& \leq T\mathcal{B}_{4,1}(\|g_1\|_{H^1(0,T)}), \tag{5.48}
\end{aligned}$$

$$\|\mathbf{L}_{\infty,4}(g_1) - \mathbf{L}_{\infty,4}(g_2)\|_{H^1(0,T)} \leq T\mathcal{B}_{4,2}(\|g_1\|_{H^1(0,T)}, \|g_2\|_{H^1(0,T)}) \|g_1 - g_2\|_{H^1(0,T)}, \tag{5.49}$$

for all  $g_1, g_2 \in H_{0*}^1(0, T)$ .

It follows from (5.33)–(5.35), (5.41), (5.42) and (5.44)–(5.49) that  $\mathbf{L}_0(\cdot) + \mathbf{L}_\infty(\cdot)$  is a contraction map from  $H_{0*}^1(0, T_*)$  into itself, provided  $T > 0$  is sufficiently small. We have therefore established the following lemma on the existence and uniqueness of a local solution for (5.32).

**Lemma 5.6.** For  $T_1 > 0$  sufficiently small, there exists a unique solution of (5.27) (with  $T = T_1$ ) in  $H_{0*}^1(0, T_1)$ .

We can apply the same technique to establish a unique solution of (5.27) in the space  $H_{0*}^2(0, T_2)$  for  $T_2 > 0$  sufficiently small. Furthermore, it can be checked that the magnitudes of  $T_2$  and  $\|g_1\|_{H^2(0, T_2)}$  are controlled by the magnitude of  $\|g_1\|_{H^1(0, T_1)}$ . Similarly, there is a unique solution of (5.27) in the space  $H_{0*}^3(0, T_3)$  for  $T_3 > 0$  sufficiently small, where the magnitudes of  $T_3$  and  $\|g_1\|_{H^3(0, T_3)}$  are controlled by  $\|g_1\|_{H^2(0, T_2)}$ , and so on.

Therefore, we have the following generalization of Lemma 5.6.

**Lemma 5.7.** Given any positive integer  $m$ , the integral equation (5.27) has a unique solution in  $H_{0*}^m(0, T_m)$ , where  $T_m \geq \mathcal{B}_{m,1}(\|g_1\|_{H^1(0, T_1)})$ ,  $\|g_1\|_{H^m(0, T_m)} \leq \mathcal{B}_{m,2}(\|g_1\|_{H^1(0, T_1)})$ , and  $\mathcal{B}_{m,j} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function for  $j = 1, 2$ .

Next we consider the question of extending the solution  $g_1$  of (5.27) in Lemma 5.7.

From the results of § 4, we see that  $g_0 = q(0, t)$  and  $g_1 = q_x(0, t)$ , where  $q$  is a solution of (2.1) with  $q_0 = 0$ , and, for  $m$  sufficiently large,  $q$  has high order of regularity and decay. Hence, the global relation is valid for any  $t \leq T_m$ , i.e., we have

$$\phi(t, k) = e^{4ik^2t} c^+(t, k). \quad (5.50)$$

where  $c^+(t, \cdot)$  is analytic and bounded on the first quadrant and  $c^+(t, k) = O(1/k)$  as  $k \rightarrow \infty$ .

For  $t > T_m$ , we can therefore rewrite (5.9) as

$$\phi(t, k) = e^{4ik^2T_m} c^+(T_m, k) + \int_{T_m}^t [2kf_0(t') + if_1(t')] \psi(t', k) dt', \quad (5.51a)$$

$$\psi(t, k) = e^{4ik^2t} c_*^+(k) + \int_{T_m}^t e^{4ik^2(t-t')} \lambda [2k\overline{f_0(t')} - \overline{if_1(t')}] \phi(t', k) dt', \quad (5.51b)$$

where  $c_*^+(k)$  is also analytic and bounded on the first quadrant. It is not difficult to see that the solution of (5.51) obtained by the Neumann series has the property that

$$\int_{\mathcal{L}} e^{-4ik^2s} \phi(t, k) 8k dk = 0 \quad \text{for } 0 < s < T_m < t. \quad (5.52)$$

Therefore, for  $T > T_m$ , the global relation (5.26) is equivalent to

$$\int_{\mathcal{L}} e^{-4ik^2t} \phi(T, k) 8k dk = 0 \quad \text{for } T_m < t < T. \quad (5.53)$$

So the problem of extending the solution from  $(0, T_m)$  to  $(0, T)$  is reduced to solving the integral equation (5.27) for  $T_m < t < T$ , which can be written in the form



$$\begin{aligned}
g_1(t) &= G(t) + \left(\frac{8i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} \left[ \int_{T_m}^T e^{4ik^2t'} e^{2i\varpi(t')} g_0(t') \left( \sum_{j=1}^2 [\chi_j(t') - \chi_j(T_m)] k^{2-j} \right) dt' \right] dk \\
&+ \left(\frac{8i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k^2 \left[ \int_{T_m}^T e^{2i\varpi(t')} g_0(t') [\psi_4(t', k) + e^{4ik^2t'} \sum_{j=3}^4 \frac{\chi_j(t')}{k^j}] dt' \right] dk \\
&- \left(\frac{4}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} \left[ \int_{T_m}^T e^{4ik^2t'} e^{2i\varpi(t')} [g_1(t') \chi_1(t') - g_1(T_m) \chi_1(T_m)] dt' \right] dk \\
&- \left(\frac{4}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k \left[ \int_{T_m}^T e^{2i\varpi(t')} g_1(t') \left( \psi_4(t', k) + e^{4ik^2t'} \sum_{j=2}^4 \frac{\chi_j(t')}{k^j} \right) dt' \right] dk, \quad (5.54)
\end{aligned}$$

where

$$\begin{aligned}
G(t) &= g_*(t) + \left(\frac{8i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} \left[ \int_0^T e^{4ik^2t'} e^{2i\varpi(t')} g_0(t') \left( \sum_{j=1}^2 \tilde{\chi}_j(t') k^{2-j} \right) dt' \right] dk \\
&+ \left(\frac{8i}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k^2 \left[ \int_0^{T_m} e^{2i\varpi(t')} g_0(t') [\psi_4(t', k) + e^{4ik^2t'} \sum_{j=3}^4 \frac{\chi_j(t')}{k^j}] dt' \right] dk \\
&- \left(\frac{4}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} \left[ \int_0^T e^{4ik^2t'} e^{2i\varpi(t')} \tilde{g}_1(t') \tilde{\chi}_1(t') dt' \right] dk \quad (5.55) \\
&- \left(\frac{4}{\pi}\right) e^{-i\varpi(t)} \int_{\mathcal{L}} e^{-4ik^2t} k \left[ \int_0^{T_m} e^{2i\varpi(t')} g_1(t') \left( \psi_4(t', k) + e^{4ik^2t'} \sum_{j=2}^4 \frac{\chi_j(t')}{k^j} \right) dt' \right] dk,
\end{aligned}$$

and  $\tilde{\chi}_1$  (resp.  $\tilde{\chi}_2$  and  $\tilde{g}_1$ ) are extensions of  $\chi_1$  (resp.  $\chi_2$  and  $g_1$ ) from  $(0, T_m)$  to  $(0, T)$  that takes the constant value  $\chi_1(T_m)$  (resp.  $\chi_2(T_m)$  and  $g_1(T_m)$ ) on  $(T_m, T)$ .

Let  $\alpha = g_1(T_m)$ . Note that the function  $G$  is known and it belongs to the affine subspace  $H_{\alpha*}^1(T_m, T)$  of  $H^1(T_m, T)$  defined by

$$H_{\alpha*}^1(T_m, T) = \{v \in H^1(T_m, T) : v(T_m) = \alpha\}.$$

Equation (5.54) is an integral equation for  $g_1 \in H_{\alpha*}^1$  and it can be analyzed in the same way as (5.27). (cf. Remark 5.5). We can therefore extend the solution  $g_1$  from  $(0, T_m)$  to  $(0, T_m + \Delta T_1)$  provided  $\Delta T_1$  is small enough, where the magnitudes of  $\Delta T_1 > 0$  and  $\|g_1\|_{H^1(0, T_m + \Delta T_1)}$  are controlled by  $\|g_1\|_{H^1(0, T_m)}$ . Similarly we can extend  $g_1$  to a solution in  $H_{0*}^m(0, T_m + \Delta_m)$ , where the magnitudes of  $\Delta T_m$  and  $\|g_1\|_{H^m(0, T_m + \Delta_m)}$  are both controlled by  $\|g_1\|_{H^1(0, T_m)}$ .

Hence the extension procedure can be repeated until a solution on  $H_{0*}^m(0, T)$  is reached provided there is an *a priori* bound for  $\|g_1\|_{H^1(0, T)}$ . It turns out that in the case where  $\lambda = 1$ , such an *a priori* bound exists for any given  $g_0$ , and it also exists in the case where  $\lambda = -1$  if  $\|g_0\|_{L_2(0, T)}$  is sufficiently small (see Appendix D for details).

We have therefore established the following theorem.

**Theorem 5.8.** *Given any  $g_0 \in C^\infty([0, T])$  such that  $g_0$  vanishes to all orders at  $t = 0$ , there exists  $g_1 \in C^\infty([0, T])$ , also vanishing to all orders at  $t = 0$ , such that  $\{g_0, g_1\}$  form an admissible pair for (2.1) with  $\lambda = 1$  and initial value  $q_0 = 0$ . This is also true for  $\lambda = -1$  if  $\|g_0\|_{L_2(0, T)}$  is sufficiently small.*

## 6 Linearizable Boundary Conditions

It was shown in § 4 that  $q(x, t)$  can be expressed in terms of the solution of a  $2 \times 2$  RH problem, which is uniquely defined in terms of the spectral functions  $a(k)$ ,  $b(k)$ ,  $A(k)$ ,  $B(k)$ . The functions  $a(k)$  and  $b(k)$  are defined in terms of  $q_0(x)$  through the solution of a linear Volterra integral equation, see Definition 3.1. However, the functions  $A(k)$  and  $B(k)$  are in general defined in terms of the initial and boundary conditions through the solution of a nonlinear Volterra integral equation, see Definitions 3.1, 3.3 and § 5. In what follows we present a general methodology which identifies a *particular class of boundary value problems* for which it is possible to compute  $A(k)$  and  $B(k)$  using only the algebraic manipulation of the global relation. We will refer to this class of boundary value problems as *linearizable*.

Recall that  $A(k)$  and  $B(k)$  are defined in terms of  $\mu_2(t, k)$ . Let  $M(t, k) = \mu_2(t, k)e^{-2ik^2t\sigma_3}$ , i.e.,

$$M(t, k) = \begin{bmatrix} \overline{M_2(t, \bar{k})} & M_1(t, k) \\ \lambda \overline{M_1(t, \bar{k})} & M_2(t, k) \end{bmatrix}, \quad M_1 = \Phi_1 e^{2ik^2t}, \quad M_2 = \Phi_2 e^{2ik^2t}.$$

Then  $M(t, k)$  satisfies

$$M_t + 2ik^2\sigma_3 M = \tilde{Q}(t, k)M, \quad M(0, k) = I. \quad (6.1)$$

The function  $M(t, -k)$  satisfies a similar equation where  $\tilde{Q}(t, k)$  is replaced by  $\tilde{Q}(t, -k)$ . Suppose that there exists a  $t$ -independent, nonsingular matrix  $N(k)$  such that

$$(2ik^2\sigma_3 - \tilde{Q}(t, -k))N(k) = N(k)(2ik^2\sigma_3 - \tilde{Q}(t, k)). \quad (6.2)$$

Then

$$M(t, -k) = N(k)M(t, k)N(k)^{-1}. \quad (6.3)$$

This equation evaluated at  $t = 0$  defines a relation between the spectral functions at  $k$  and the spectral functions at  $-k$ .

We note that a necessary condition for the existence of  $N(k)$  is that the determinant of the matrix  $2ik^2\sigma_3 - \tilde{Q}(t, k)$  depends on  $k$  in the form of  $k^2$ . This condition implies

$$q(0, t)\bar{q}_x(0, t) - \bar{q}(0, t)q_x(0, t) = 0. \quad (6.4)$$

If this condition is satisfied, equation (6.2) yield

$$(2kq - iq_x)N_3 = -\lambda(2k\bar{q} - i\bar{q}_x)N_2, \quad (6.5a)$$

$$(2kq + iq_x)N_1 + (2kq - iq_x)N_4 = -2(2ik^2 + i\lambda|q|^2)N_2, \quad (6.5b)$$

where we have used the notations

$$N_1 = N_{11}, \quad N_2 = N_{12}, \quad N_3 = N_{21}, \quad N_4 = N_{22}. \quad (6.6)$$

We now discuss in detail three particular cases of (6.5).

**(a)**  $q(0, t) = 0$

In this case  $\tilde{Q}(t, k)$  is a function of  $t$  and  $k^2$ , thus there is no need to introduce  $N(k)$ , i.e.  $N(k) = I$ . Then the second column of equation (6.3) evaluated at  $t = T$  yields

$$A(k) = A(-k), \quad B(k) = B(-k), \quad k \in \mathbb{C}. \quad (6.7)$$

**(b)**  $q_x(0, t) = 0$

Equation (6.5) implies that  $N(k)$  does not depend on  $q(0, t)$  provided that  $N_2 = N_3 = 0$  and  $N_4 = -N_1$ . Then the second column of equation (6.3) evaluated at  $t = T$  yields

$$A(k) = A(-k), \quad B(k) = -B(-k), \quad k \in \mathbb{C}. \quad (6.8)$$

**(c)**  $q_x(0, t) - \rho q(0, t) = 0$ ,  $\rho$  positive constant

Equations (6.5) imply that  $N(k)$  does not depend separately on  $q(0, t)$  and on  $q_x(0, t)$  provided that  $N_2 = N_3 = 0$  and

$$(2k - i\rho)N_4 + (2k + i\rho)N_1 = 0.$$

Then the second column of equation (6.3) evaluated at  $t = T$  yields

$$A(k) = A(-k), \quad B(k) = -\frac{2k + i\rho}{2k - i\rho}B(-k), \quad k \in \mathbb{C}. \quad (6.9)$$

Using the transformations (6.7)–(6.9), together with the global relation it is possible to express  $A(k)$  and  $B(k)$  in terms of  $a(k)$  and  $b(k)$ .

For convenience we assume  $T = \infty$ . It can be shown that a similar analysis is valid if  $T < \infty$ . If  $T = \infty$ , the global relation becomes

$$a(k)B(k) - b(k)A(k) = 0, \quad \arg k \in [0, \pi/2]. \quad (6.10)$$

We note again that since  $T = \infty$ ,  $A(k)$  and  $B(k)$  are *not* entire functions but are defined for

$$\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2].$$

**(a)**  $q(0, t) = 0$

Letting  $k \mapsto -k$  in the definition of  $\overline{d(k)}$  and using the symmetry relation (6.7) we find

$$A(k)\overline{a(-\bar{k})} - \lambda B(k)\overline{b(-\bar{k})} = \overline{d(-\bar{k})}, \quad \arg k \in [0, \pi/2]. \quad (6.11)$$

This equation and the global relation (6.10) are *two algebraic* equations for  $A(k)$  and  $B(k)$ . Their solution yields

$$A(k) = \frac{a(k)\overline{d(-\bar{k})}}{\Delta_0(k)}, \quad B(k) = \frac{b(k)\overline{d(-\bar{k})}}{\Delta_0(k)}, \quad \arg k \in [0, \pi/2], \quad (6.12)$$

where

$$\Delta_0(k) := a(k)\overline{a(-\bar{k})} - \lambda b(k)\overline{b(-\bar{k})}. \quad (6.13)$$

The function  $\overline{d(\bar{k})}$  can be computed explicitly in terms of  $a(k)$  and  $b(k)$ . However, it does not affect the solution of the RH problem of Theorem 4.1. Indeed, this RH problem is defined in terms of  $\gamma(k) = b(k)/\bar{a}(k)$ ,  $k \in \mathbb{R}$  and of  $\Gamma(k)$  which involves  $a(k)$ ,  $b(k)$  and  $A(k)/B(k)$ ,

$$\Gamma(k) = \frac{\lambda(\overline{B(\bar{k})/A(\bar{k})})}{a(k)\left(a(k) - \lambda b(k)\overline{(B(\bar{k})/A(\bar{k}))}\right)} = \frac{\overline{\lambda b(-\bar{k})}}{a(k)\Delta_0(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \quad (6.14)$$

The function  $\Delta_0(k)$  is an analytic function in the upper half  $k$  plane, and it satisfies the symmetry equation,

$$\Delta_0(k) = \overline{\Delta_0(-\bar{k})}. \quad (6.15)$$

It can be shown that the zero set of  $\Delta_0(k)$  is the union

$$\{\lambda_j\}_{j=1}^\Lambda \cup \{-\bar{\lambda}_j\}_{j=1}^\Lambda. \quad (6.16)$$

Indeed, the global relation (6.10) implies that the zero sets of the functions  $a(k)$  and  $A(k)$  coincide in the first quadrant. It also implies that if the zeros of  $a(k)$  are simple the zeros of  $A(k)$  have the same property. This and equation (6.11) imply that the zero sets of the functions  $\overline{d(-\bar{k})}$  and  $\Delta_0(k)$  coincide in the first quadrant as well. Equation (6.15) implies that the zero set of  $\Delta_0(k)$  is the set given in (6.16).

Since the zeros  $\lambda_j$  of  $d(k)$  coincide with the second quadrant zeros of  $\Delta_0(k)$ , equations (6.12) and (6.7) imply the following modification of the residue conditions in (2.29):

$$\operatorname{Res}_{k_j} [M(x, t, k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)} [M(x, t, k_j)]_2, \quad j = 1, \dots, n_1, \quad (6.17a)$$

$$\operatorname{Res}_{\bar{k}_j} [M(x, t, k)]_2 = \frac{\lambda}{\overline{\dot{a}(k_j)b(k_j)}} e^{-2i\theta(\bar{k}_j)} [M(x, t, \bar{k}_j)]_1, \quad j = 1, \dots, n_1, \quad (6.17b)$$

$$\operatorname{Res}_{\lambda_j} [M(x, t, k)]_1 = \frac{\lambda b(-\bar{\lambda}_j)}{a(\lambda_j)\dot{\Delta}_0(\lambda_j)} e^{2i\theta(\lambda_j)} [M(x, t, \lambda_j)]_2, \quad j = 1, \dots, \Lambda, \quad (6.17c)$$

$$\operatorname{Res}_{\bar{\lambda}_j} [M(x, t, k)]_2 = \frac{b(-\bar{\lambda}_j)}{a(\lambda_j)\overline{\dot{\Delta}_0(\lambda_j)}} e^{-2i\theta(\bar{\lambda}_j)} [M(x, t, \bar{\lambda}_j)]_1, \quad j = 1, \dots, \Lambda, \quad (6.17d)$$

where

$$\theta(k_j) = k_j x + 2k_j^2 t. \quad (6.18)$$

(b)  $q_x(0, t) = 0$

Equations (6.12) are valid but  $\Delta_0(k)$  is replaced by

$$\Delta_1(k) = a(k)\overline{a(-\bar{k})} + \lambda b(k)\overline{b(-\bar{k})}. \quad (6.19)$$

The zeros  $\lambda_j$  are now the second quadrant zeros of  $\Delta_1(k)$ , and equation (6.14) should be replaced by

$$\Gamma(k) = -\frac{\overline{\lambda b(-\bar{k})}}{a(k)\Delta_1(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \quad (6.20)$$

The modified residue conditions are given by the equations

$$\operatorname{Res}_{k_j} [M(x, t, k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)} [M(x, t, k_j)]_2, \quad j = 1, \dots, n_1, \quad (6.21a)$$

$$\operatorname{Res}_{\bar{k}_j} [M(x, t, k)]_2 = \frac{\lambda}{\dot{a}(k_j)b(k_j)} e^{-2i\theta(\bar{k}_j)} [M(x, t, \bar{k}_j)]_1, \quad j = 1, \dots, n_1, \quad (6.21b)$$

$$\operatorname{Res}_{\lambda_j} [M(x, t, k)]_1 = -\frac{\lambda \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_1(\lambda_j)} e^{2i\theta(\lambda_j)} [M(x, t, \lambda_j)]_2, \quad j = 1, \dots, \Lambda, \quad (6.21c)$$

$$\operatorname{Res}_{\bar{\lambda}_j} [M(x, t, k)]_2 = -\frac{b(-\bar{\lambda}_j)}{a(\lambda_j) \dot{\Delta}_1(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)} [M(x, t, \bar{\lambda}_j)]_1, \quad j = 1, \dots, \Lambda. \quad (6.21d)$$

(c)  $q_x(0, t) - \rho q(0, t) = 0$ ,  $\rho$  constant

In this case,  $\Delta_0(k)$  is replaced by

$$\Delta_\rho(k) = a(k) \overline{a(-\bar{k})} + \lambda \frac{2k - i\rho}{2k + i\rho} b(k) \overline{b(-\bar{k})}. \quad (6.22)$$

The zeros  $\lambda_j$  are now the second quadrant zeros of  $\Delta_\rho(k)$ , and

$$\Gamma(k) = -\frac{\lambda \frac{2k - i\rho}{2k + i\rho} \overline{b(-\bar{k})}}{a(k) \Delta_\rho(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \quad (6.23)$$

The modified residue conditions are given by the equations

$$\operatorname{Res}_{k_j} [M(x, t, k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)} [M(x, t, k_j)]_2, \quad j = 1, \dots, n_1, \quad (6.24a)$$

$$\operatorname{Res}_{\bar{k}_j} [M(x, t, k)]_2 = \frac{\lambda}{\dot{a}(k_j)b(k_j)} e^{-2i\theta(\bar{k}_j)} [M(x, t, \bar{k}_j)]_1, \quad j = 1, \dots, n_1, \quad (6.24b)$$

$$\operatorname{Res}_{\lambda_j} [M(x, t, k)]_1 = -\frac{\lambda \frac{2\lambda_j - i\rho}{2\lambda_j + i\rho} \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_\rho(\lambda_j)} e^{2i\theta(\lambda_j)} [M(x, t, \lambda_j)]_2, \quad j = 1, \dots, \Lambda, \quad (6.24c)$$

$$\operatorname{Res}_{\bar{\lambda}_j} [M(x, t, k)]_2 = -\frac{\frac{2\bar{\lambda}_j + i\rho}{2\lambda_j - i\rho} \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_\rho(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)} [M(x, t, \bar{\lambda}_j)]_1, \quad j = 1, \dots, \Lambda. \quad (6.24d)$$

Theorem 4.1 and the above results imply the following.

**Theorem 6.1.** *Let  $q(x, t)$  satisfy the NLS equation (2.1), the initial condition*

$$q(x, 0) = q_0 \in S(\mathbb{R}^+), \quad 0 < x < \infty,$$

*and any of the following boundary conditions*

$$(a) \quad q(0, t) = 0, \quad t > 0,$$

$$(b) \quad q_x(0, t) = 0, \quad t > 0,$$

*or*

$$(c) \quad q_x(0, t) - \rho q(0, t) = 0, \quad \rho > 0, \quad t > 0.$$

Assume that the initial and boundary conditions are compatible at  $x = t = 0$ . Furthermore, assume if  $\lambda = -1$ :

- (i)  $a(k)$ , which is defined in Definition 3.1, has a finite number of simple zeros for  $\text{Im } k > 0$ .
- (ii)  $\Delta_0(k)$  in case (a), or  $\Delta_1(k)$  in case (b), or  $\Delta_\rho(k)$  in case (c), have a finite number of simple zeros in the second quadrant which do not coincide with the possible zeros of  $a(k)$  ( $\Delta_0, \Delta_1, \Delta_\rho$  are defined in equations (6.13), (6.19), (6.22)).

The solution  $q(x, t)$  can be constructed through equation (4.3), where  $M$  satisfies the RH problem defined in Theorem 4.1, with  $\Gamma(k)$  given by equation (6.14) in case (a), by equation (6.20) in case (b), and by equation (6.23) in case (c). The relevant residue conditions are given by equation (6.17) in case (a), by equation (6.21) in case (b), and by equation (6.24) in case (c).

**Remark 6.2.** Linearizable boundary value problems have been studied via techniques based on an appropriate continuation of the boundary problem to the problem on the line in [11]–[16]. The solutions are given via the Riemann-Hilbert problems corresponding to the extended initial value problems. These continuations are described by explicit conditions on the scattering data associated with the initial value problem on the line (see [11], [13]–[16]). In the case of the first two boundary problems studied here, these conditions can be easily translated to the even or odd continuation of the initial data  $q_0(x)$  (see [11]). For the third boundary problem, the spectral data can also be computed in terms of the initial data  $q_0(x)$ ,  $x > 0$ , see [17]. We emphasize that the method of continuing the problem to the full line can not be implemented for odd order problems, such as the KdV and the modified KdV. On the other hand the method developed here works for such integrable PDEs [18]. Theorem 6.1 presents the solution of the third boundary problem with the same level of efficiency as the one for the full axis problem. Indeed, the relevant Riemann-Hilbert problem is formulated in terms of the spectral data,  $a(k)$ ,  $b(k)$ , which are calculated *directly* via the given initial data  $q_0(x)$ ,  $x > 0$ . The only difference, which does not affect the effectiveness of the solution, is that the Riemann-Hilbert problem is now formulated on a cross and not on the real line. It also worth noticing that in this case (as well as in the other linearizable cases) the Riemann-Hilbert problem can be deformed back to the real line and then in fact coincides with the Riemann-Hilbert problem of [17] (see [19]).

**Remark 6.3.** Linearizable boundary value problems have infinitely many conserved quantities [20, 21].

## 7 Conclusions

We have introduced a rigorous methodology for solving boundary value problems for nonlinear integrable evolution equations. This involves the following steps: (1) Assume that there exists a smooth, global solution  $q(x, t)$ , and perform the simultaneous spectral analysis of the associated Lax pair. This yields a representation of  $q(x, t)$  in terms of the solution  $M(x, t, k)$  of a matrix RH problem. This RH problem is uniquely defined in terms of certain spectral

functions  $a(k)$ ,  $b(k)$ ,  $A(k)$ ,  $B(k)$ , which satisfy a simple global relation. (2) Motivated from the results of (1), *postulate* the global relation and *define* the spectral functions:  $a(k)$ ,  $b(k)$  are defined in terms of the initial conditions  $q_0(x)$ , and  $A(k)$ ,  $B(k)$  are defined in terms of an admissible set of functions  $g_0(t)$ ,  $g_1(t)$ , where a set is called admissible if  $A(k)$ ,  $B(k)$  satisfy the postulated global relation. (3) Motivated from the results of (1), define  $M(x, t, k)$  as the solution of a matrix RH problem, uniquely defined in terms of  $a(k)$ ,  $b(k)$ ,  $A(k)$ ,  $B(k)$ . Prove that this RH problem has a unique, global solution. Define  $q(x, t)$  in terms of  $M(x, t, k)$  and prove that  $q(x, t)$  solves the nonlinear PDE, and it satisfies  $q(x, 0) = q_0(x)$ ,  $q(0, t) = g_0(t)$ ,  $q_x(0, t) = g_1(t)$ . (4) Investigate the existence of the admissible set. For example, show that given  $q_0(x)$  and  $g_0(t)$ , there exists a unique  $g_1(t)$ . This involves the investigation of a nonlinear Volterra integral equation.

We have also introduced a methodology for analyzing a particular class of boundary value problems, which we call linearizable. This class is distinctive in the sense that  $A(k)$ ,  $B(k)$  can be computed directly in terms of  $a(k)$ ,  $b(k)$  using the algebraic manipulation of the global relation, without the need to analyze the nonlinear Volterra integral equation. Thus for linearizable boundary conditions, boundary value problems can be solved as effectively as initial value problems.

We conclude with some remarks.

**Remark 7.1.** It was realized by the first author [22] that for the solution of initial boundary value problems of integrable nonlinear evolution equations, one needs to perform, in addition to the spectral analysis of the  $x$  part of the Lax pairs, the spectral analysis of its  $t$ -part. For the NLS equation this was done in [9]. However, the importance of performing the *simultaneous* spectral analysis, as well as the key role played by the *global relation* was not understood at that time.

**Remark 7.2.** A rigorous characterization of the properties of the spectral functions associated with the NLS on the half-line is given in [7].

**Remark 7.3.** Under the *assumption* of existence of solutions, a rigorous determination of the long time behavior of the solution of the NLS equation on the half-line is given in [9], using the Deift-Zhou approach [23]. In particular, it is shown in [9] that the long time asymptotics is dominated by the solitonic part of the solution. These results together with the results presented here imply that for the linearizable class of boundary conditions, the long time asymptotics is explicitly determined in terms of the initial and boundary conditions. The asymptotic results are summarized in Appendix B (the poles generated by the zeros of  $a(k)$ ,  $\arg k \in (\frac{\pi}{2}, \pi)$  were missed in [9] but are included here).

**Remark 7.4.** In recent years there have been important developments in the analysis of boundary value problems of nonlinear PDEs using PDE techniques [28], [29]. It is remarkable that some of these techniques yield *global* results. It is satisfying that there exists now a rigorous theory using the integrability machinery, so that it is possible to make comparisons between these different approaches. Although at the moment the PDE results are proven in less restrictive functional spaces, the advantage of our method is that it yields rigorous asymptotic results. We reiterate that this is a consequence of our representation of the solution in terms of the RH problem whose jump matrices depend on the  $x$  and  $t$  in a simple oscillatory way which, in turn, allows to apply the Deift-Zhou method.

**Acknowledgements** The authors are grateful to A. Boutet de Monvel and V. Kotlyarov for pointing out the missing part of the discrete spectral data in [9]. The work of A.S.F. was partially supported by EPSRC. The work of A.R.I. was partially supported by EPSRC and by NSF under grants DMS-9801608 and DMS-0099812. The work of L.Y.S. was partially supported by EPSRC. Part of this work was carried out when A.R.I. and L.Y.S. visited the Mathematics Department of Imperial College. A.R.I. and L.Y.S. are grateful to the staff of the department for their hospitality during the visits.

## Appendices

### A The Inverse Problems

#### A.1 The $x$ - inverse problem

Consider the functions  $q_0(x)$ ,  $\varphi(x, k)$ ,  $a(k)$ ,  $b(k)$  introduced in Definition 3.1. Let the vector function,  $\psi(x, k) = (\psi_1, \psi_2)^t$  be defined as the unique solution of

$$\begin{aligned}\psi_{1,x} &= q_0(x)\psi_2, \\ \psi_{2,x} - 2ik\psi_2 &= \lambda\bar{q}_0(x)\psi_1, \quad 0 < x < \infty, \quad k \in \mathbb{C}, \\ \psi(0, k) &= (1, 0)^t.\end{aligned}$$

Note that the vector  $\psi$  satisfies the linear Volterra equations,

$$\psi_1(x, k) = 1 + \int_0^x q_0(y)\psi_2(y, k)dy, \quad k \in \mathbb{C}, \quad (\text{A.1a})$$

$$\psi_2(x, k) = \lambda \int_0^x e^{2ik(x-y)}\bar{q}_0(y)\psi_1(y, k)dy, \quad k \in \mathbb{C}. \quad (\text{A.1b})$$

Denote,

$$\varphi^*(x, k) \equiv \left( \bar{\varphi}_2(x, \bar{k}), \lambda\bar{\varphi}_1(x, \bar{k}) \right)^t \quad \text{and} \quad \psi^*(x, k) \equiv \left( \bar{\psi}_2(x, \bar{k}), \lambda\bar{\psi}_1(x, \bar{k}) \right)^t.$$

Define  $\mu_3(x, k)$  and  $\mu_2(x, k)$  by

$$\mu_3(x, k) = \left( \varphi^*(x, k), \varphi(x, k) \right) \quad \text{and} \quad \mu_2(x, k) = \left( \psi(x, k), \lambda\psi^*(x, k) \right).$$

They satisfy the matrix equation,

$$\mu_x + ik[\sigma_3, \mu] = \begin{bmatrix} 0 & q_0 \\ \lambda\bar{q}_0 & 0 \end{bmatrix} \mu. \quad (\text{A.2})$$

This in turn implies that the above vectors are simply related,

$$\begin{aligned}\left( \varphi^*(x, k), \varphi(x, k) \right) &= \left( \psi(x, k), \lambda\psi^*(x, k) \right) e^{-ikx\hat{\sigma}_3} s(k) \\ &= \left( \psi(x, k), \lambda\psi^*(x, k) \right) \begin{bmatrix} \bar{a}(k) & b(k)e^{-2ikx} \\ \lambda\bar{b}(k)e^{2ikx} & a(k) \end{bmatrix}, \quad k \in \mathbb{R}. \quad (\text{A.3})\end{aligned}$$



Let

$$M_-^{(x)} = \left( \varphi^*, \frac{\lambda \psi^*}{\bar{a}(k)} \right), \quad \text{Im } k \leq 0, \quad (\text{A.4a})$$

$$M_+^{(x)} = \left( \frac{\psi}{a(k)}, \varphi \right), \quad \text{Im } k \geq 0. \quad (\text{A.4b})$$

Equation (A.3) can be rewritten as

$$M_-^{(x)}(x, k) = M_+^{(x)}(x, k) J^{(x)}(x, k), \quad k \in \mathbb{R}, \quad (\text{A.5})$$

where  $J^{(x)}(x, k)$  is the jump matrix defined by (3.6c). Furthermore,  $M^{(x)}$  satisfies the RH problem defined in (3.6). Indeed, we only need to prove the residue conditions at the possible simple zeros,  $\{k_j\}_1^n$ , of  $a(k)$ . To this end we note that in virtue of (A.3) the equation,

$$\varphi = b(k)e^{-2ikx}\psi + a(k)\lambda\psi^*, \quad (\text{A.6})$$

holds. The function  $\psi$ , and hence the function  $\psi^*$  are entire functions of  $k$ . Therefore, we can evaluate (A.6) at  $k = k_j$ . This yields the relation,

$$\varphi(x, k_j) = \psi(x, k_j)b(k_j)e^{-2ik_jx},$$

or, taking into account the definition (A.4) of the function  $M^{(x)}(x, k)$ ,

$$\text{Res}_{k_j} [M^{(x)}(x, k_j)]_1 = \frac{e^{2ik_jx}}{\bar{a}(k_j)b(k_j)} [M^{(x)}(x, k_j)]_2.$$

The residue condition at  $k = \bar{k}_j$  is derived similarly.

A substitution of the asymptotic expansion,

$$M^{(x)}(x, k) = I + \frac{m_1(x)}{k} + O\left(\frac{1}{k^2}\right), \quad k \rightarrow \infty,$$

into equation (A.2) yields

$$q_0(x) = 2i \left( m_1(x) \right)_{12} = 2i \lim_{k \rightarrow \infty} \left( k M^{(x)}(x, k) \right)_{12}. \quad (\text{A.7})$$

Our next task is to show that this relation defines the map,

$$\mathbb{Q} : \{a(k), b(k)\} \mapsto \{q_0(x)\},$$

which is inverse to the spectral map,

$$\mathbb{S} : \{q_0(x)\} \mapsto \{a(k), b(k)\}.$$

In more detail this problem is formulated as follows. Given  $\{a(k), b(k)\}$ , construct the jump matrix  $J^{(x)}(x, k)$  according to equation (3.6c) and define the RH problem by (3.6). Let  $q_0(x)$

be the function defined by (A.7) in terms of the solution  $M^{(x)}(x, k)$  of this RH problem. Denote by  $\{a_0(k), b_0(k)\}$  the spectral data corresponding to  $q_0(x)$ . We have to show that

$$a_0(k) = a(k) \quad \text{and} \quad b_0(k) = b(k). \quad (\text{A.8})$$

Using the standard arguments of the dressing method [10] it is straightforward to prove that  $M^{(x)}(x, k)$  satisfies equation (A.2) with the potential  $q_0(x)$  defined by (A.7). This means in particular that the matrix solution  $\mu_3(x, k)$ ,  $k \in \mathbb{R}$  corresponding to the potential  $q_0(x)$  is given by the equation,

$$\mu_3(x, k) = M_+^{(x)}(x, k) e^{-ikx\hat{\sigma}_3} C_+(k), \quad k \in \mathbb{R}, \quad (\text{A.9})$$

for an appropriate matrix  $C_+(k)$ . This matrix does not depend on  $x$  and hence can be evaluated by letting  $x \rightarrow \infty$  in (A.9).

It follows from the theory of the inverse scattering problem for the Dirac equation (A.2) (see for e.g. [24]; or from the direct use of the nonlinear steepest descent method [23, 26]) that

$$M_+^{(x)}(x, k) = \begin{bmatrix} 1 & 0 \\ -\frac{\lambda\bar{b}(k)}{a(k)} e^{2ikx} & 1 \end{bmatrix} + o(1), \quad x \rightarrow \infty, \quad k \in \mathbb{R} \quad (\text{A.10})$$

(under the usual assumptions on the Riemann-Hilbert data  $\{a(k), b(k)\}$ ). Since  $\mu_3 \rightarrow I$  as  $x \rightarrow \infty$ , it follows that

$$C_+(k) = \begin{bmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{bmatrix}. \quad (\text{A.11})$$

Equations (A.9) and (A.11) imply that the scattering data,

$$s_0(k) = \begin{bmatrix} \bar{a}_0(k) & b_0(k) \\ \lambda\bar{b}_0(k) & a_0(k) \end{bmatrix} = \mu_3(0, k),$$

corresponding to the potential  $q_0(x)$  defined in (A.8) are given by the equation,

$$s_0(k) = M_+^{(x)}(0, k) \begin{bmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{bmatrix}.$$

If  $x = 0$  (in fact, for all  $x \leq 0$ ) the above Riemann-Hilbert problem can be solved explicitly. Indeed,

$$J^{(x)}(0, k) = \begin{bmatrix} 1 & -\frac{b(k)}{\bar{a}(k)} \\ \frac{\lambda\bar{b}(k)}{a(k)} & \frac{1}{|a|^2} \end{bmatrix} = \begin{bmatrix} a(k) & -b(k) \\ 0 & \frac{1}{a(k)} \end{bmatrix} \begin{bmatrix} \bar{a}(k) & 0 \\ \lambda\bar{b}(k) & \frac{1}{\bar{a}(k)} \end{bmatrix}.$$

This implies,

$$M_+^{(x)}(0, k) = \begin{bmatrix} \frac{1}{a(k)} & b(k) \\ 0 & a(k) \end{bmatrix}$$

(note that the residue conditions are satisfied), and hence

$$s_0(k) = \begin{bmatrix} \frac{1}{a(k)} & b(k) \\ 0 & a(k) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{bmatrix} = \begin{bmatrix} \bar{a}(k) & b(k) \\ \lambda\bar{b}(k) & a(k) \end{bmatrix} = s(k),$$

i.e. equation (A.8) follows.

## A.2 The $t$ - inverse problem

Consider the functions  $g_0(t)$ ,  $g_1(t)$ ,  $\Phi(x, k)$ ,  $A(k)$ ,  $B(k)$  introduced in Definition 3.3. Let the vector function,  $\Psi(x, k) = (\Psi_1, \Psi_2)^t$  be defined as the unique solution of

$$\begin{aligned}\Psi_{1_t} &= \tilde{Q}_{11}\Psi_1 + \tilde{Q}_{12}\Psi_2, \\ \Psi_{2_t} - 4ik^2\Psi_2 &= \tilde{Q}_{21}\Psi_1 + \tilde{Q}_{22}\Psi_2, \quad 0 < t < T, \quad k \in \mathbb{C}, \\ \Psi(T, k) &= (1, 0)^t,\end{aligned}$$

where (cf. (3.9))

$$\tilde{Q}(t, k) = 2k \begin{bmatrix} 0 & g_0(t) \\ \lambda\bar{g}_0(t) & 0 \end{bmatrix} - i \begin{bmatrix} 0 & g_1(t) \\ \lambda\bar{g}_1(t) & 0 \end{bmatrix} \sigma_3 - i\lambda|g_0(t)|^2\sigma_3, \quad \lambda = \pm 1. \quad (\text{A.12})$$

Note that the vector  $\Psi$  satisfies the linear Volterra equations,

$$\Psi_1(t, k) = 1 + \int_T^t (\tilde{Q}_{11}\Psi_1 + \tilde{Q}_{12}\Psi_2)(\tau, k) d\tau, \quad (\text{A.13a})$$

$$\Psi_2(t, k) = \int_T^t e^{4ik^2(t-\tau)} (\tilde{Q}_{21}\Psi_1 + \tilde{Q}_{22}\Psi_2)(\tau, k) d\tau. \quad (\text{A.13b})$$

Denote, as before,

$$\Phi^*(t, k) = \left( \bar{\Phi}_2(t, \bar{k}), \lambda\bar{\Phi}_1(t, \bar{k}) \right)^t \quad \text{and} \quad \Psi^*(t, k) = \left( \bar{\Psi}_2(t, \bar{k}), \lambda\bar{\Psi}_1(t, \bar{k}) \right)^t.$$

Define

$$\mu_1(t, k) = \left( \Psi(t, k), \lambda\Psi^*(t, k) \right) \quad \text{and} \quad \mu_2(t, k) = \left( \Phi^*(t, k), \Phi(t, k) \right).$$

They satisfy the matrix equation,

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(t, k)\mu. \quad (\text{A.14})$$

This in turn implies (cf. (2.12)) that

$$\begin{aligned}\left( \Phi^*(t, k), \Phi(t, k) \right) &= \left( \Psi(t, k), \lambda\Psi^*(t, k) \right) e^{-2ik^2t\hat{\sigma}_3} S(k) \\ &= \left( \Psi(t, k), \lambda\Psi^*(t, k) \right) \begin{bmatrix} \bar{A}(k) & B(k)e^{-4ik^2t} \\ \lambda\bar{B}(k)e^{4ik^2t} & A(k) \end{bmatrix}, \quad k \in \mathbb{R} \cup i\mathbb{R}.\end{aligned} \quad (\text{A.15})$$

**Remark A.1.** We recall that the function  $\Phi(t, k)$ , as a function of  $k$ , is analytic and bounded in the second and fourth quadrants, while the function  $\Psi(t, k)$  is analytic for all  $k$  and bounded in the first and third quadrants. Also, if  $T < \infty$  all of the above functions are entire functions of  $k$ . This means in particular that in this case equation (A.15) is valid for all complex values of  $k$ .

Let

$$M_-^{(t)} = \left( \Phi^*, \frac{\lambda \Psi^*}{A(k)} \right), \quad \arg k \in [\pi/2, \pi] \cup [3\pi/2, 2\pi], \quad (\text{A.16a})$$

$$M_+^{(t)} = \left( \frac{\Psi}{A(k)}, \Phi \right), \quad \arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]. \quad (\text{A.16b})$$

Equation (A.15) can be rewritten as

$$M_-^{(t)}(t, k) = M_+^{(t)}(t, k) J^{(t)}(t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad (\text{A.17})$$

where  $J^{(t)}(t, k)$  is the jump matrix defined in (3.15c). Furthermore,  $M^{(t)}$  satisfies the RH problem defined in (3.15). Indeed, as in the  $x$ -case, we only need to prove the residue conditions at the possible simple zeros,  $\{K_j\}_1^N$ , of  $A(k)$ . The proof is the same as in the case of the function  $M^{(x)}(x, k)$ .

The substitution of the asymptotic expansion,

$$M^{(t)}(t, k) = I + \frac{m_1(t)}{k} + \frac{m_2(t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty,$$

into equation (A.14) leads to the relations,

$$g_0(t) = 2i \left( m_1(t) \right)_{12} = 2i \lim_{k \rightarrow \infty} \left( k M^{(t)}(t, k) \right)_{12}, \quad (\text{A.18})$$

$$\begin{aligned} g_1(t) &= 4 \left( m_2(t) \right)_{12} + 2i g_0(t) \left( m_1(t) \right)_{22} \\ &= \lim_{k \rightarrow \infty} \left\{ 4(k^2 M^{(t)}(t, k))_{12} + 2i g_0(t) k (M^{(t)}(t, k))_{22} \right\}. \end{aligned} \quad (\text{A.19})$$

We will show that these relations define the map,

$$\tilde{\mathbb{Q}} : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t)\},$$

which is inverse to the spectral map,

$$\tilde{\mathbb{S}} : \{g_0(t), g_1(t)\} \mapsto \{A(k), B(k)\}.$$

Similar to the  $x$ -case, we have to prove that

$$A_0(k) = A(k) \quad \text{and} \quad B_0(k) = B(k), \quad (\text{A.20})$$

where the left hand side is the spectral data corresponding to  $g_0(t)$  and  $g_1(t)$ . We follow precisely the same procedure as the one used for  $x$ -problem: Using arguments of the dressing method [10] it follows that if  $M^{(t)}(t, k)$  is the solution of the Riemann-Hilbert problem then it satisfies equation (A.14) with potentials  $g_0(t)$  and  $g_1(t)$  defined by (A.18) and (A.19). This means, in particular, that the matrix solution  $\mu_1(t, k)$ ,  $k \in \mathbb{C}$  (we assume that  $T < \infty$ ) corresponding to the potentials  $g_0(t)$  and  $g_1(t)$  is given by the equation,

$$\mu_1(t, k) = M_+^{(t)}(t, k) e^{-2ik^2 t \hat{\sigma}_3} D_+(k), \quad k \in \mathbb{C}, \quad (\text{A.21})$$

for an appropriate matrix  $D_+(k)$ . This matrix does not depend on  $t$  and hence can be evaluated by letting  $t = T$  in (A.21).

Observe that for all  $t$  the jump matrix  $J^{(t)}(t, k)$  can be factorized as

$$J^{(t)}(t, k) = \begin{bmatrix} 1 & 0 \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2 t} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{B(k)}{A(k)} e^{-4ik^2 t} \\ 0 & 1 \end{bmatrix}. \quad (\text{A.22})$$

Recall that  $A(k)$  and  $B(k)$  are entire functions satisfying the asymptotic relations,

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2 T}}{k}\right), \quad B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2 T}}{k}\right), \quad k \rightarrow \infty.$$

Hence,

$$\frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2 T} \rightarrow 0, \quad k \rightarrow \infty, \quad \arg k \in [0, \pi/2] \cup [\pi, 3\pi/2], \quad (\text{A.23})$$

and

$$\frac{B(k)}{\bar{A}(\bar{k})} e^{-4ik^2 T} \rightarrow 0, \quad k \rightarrow \infty, \quad \arg k \in [\pi/2, \pi] \cup [3\pi/2, 2\pi]. \quad (\text{A.24})$$

Also, taking into account that

$$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1, \quad k \in \mathbb{C},$$

it follows that if  $K_j$  is a zero of  $A(k)$  then

$$\operatorname{Res}_{K_j} \begin{bmatrix} 1 \\ -\frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2 T} \end{bmatrix} = -\frac{\lambda \bar{B}(\bar{K}_j)}{\dot{A}(K_j)} e^{4iK_j^2 T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\dot{A}(K_j)B(K_j)} e^{4iK_j^2 T} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Similarly, at  $k = \bar{K}_j$ ,

$$\operatorname{Res}_{\bar{K}_j} \begin{bmatrix} -\frac{B(k)}{A(k)} e^{-4ik^2 T} \\ 1 \end{bmatrix} = \frac{1}{\overline{\dot{A}(K_j)B(K_j)}} e^{-4i\bar{K}_j^2 T} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

These equations, together with (A.22) and the estimates (A.23), (A.24) imply that for  $t = T$  the RH problem defined in (3.15) can be solved explicitly:

$$M_+^{(t)}(T, k) = \begin{bmatrix} 1 & 0 \\ -\frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2 T} & 1 \end{bmatrix}. \quad (\text{A.25})$$

Thus

$$D_+(k) = \begin{bmatrix} 1 & 0 \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} & 1 \end{bmatrix}. \quad (\text{A.26})$$

**Remark A.2.** In the case  $T = \infty$ , the factorization (A.22) does not provide the exact solution for the  $t$ -RH problem. However, the methodology of the nonlinear steepest descent method [23] can be applied. The factorization (A.22) can be used to deform the jump contour  $\mathcal{L} = \mathbb{R} \cup i\mathbb{R}$  to the hyperbola  $(\operatorname{Re} k)(\operatorname{Im} k) = \delta > 0$  (cf. [25, 26]). Since  $\operatorname{Re}(ik^2) > 0$  in the first and third quadrants, the jump matrix of the deformed RH problem tends exponentially fast to the identity matrix as  $t \rightarrow \infty$ . The possible error terms coming from the zeros of  $A(k)$  are exponentially small. This implied that instead of the exact equation (A.25), the following asymptotic relation (cf. (A.10)) is valid,

$$M_+^{(t)}(t, k) = \begin{bmatrix} 1 & 0 \\ -\frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2 t} & 1 \end{bmatrix} + o(1), \quad t \rightarrow \infty, \quad k \in \mathbb{R} \cup i\mathbb{R}. \quad (\text{A.27})$$

Indeed, the  $t$ -Riemann - Hilbert problem is a particular case of the oscillatory Riemann-Hilbert problem corresponding to the NLS equation on the whole axis. The asymptotics (A.27) is the leading term of the known asymptotics of the solution of the NLS Riemann-Hilbert problem (see e.g. [25, 26] and the earlier works [27, 28])<sup>2</sup>. Equation (A.27) implies that the formula (A.26) for the matrix  $D_+(k)$  is valid for  $T = \infty$  as well.

Equations (A.21) and (A.26) imply that the scattering data,

$$S_0(k) = \begin{bmatrix} \bar{A}_0(\bar{k}) & B_0(k) \\ \lambda \bar{B}_0(\bar{k}) & A_0(k) \end{bmatrix} = \mu_1(0, k),$$

corresponding to the potentials  $g_0(t)$  and  $g_1(t)$  defined in (A.18) and (A.19), are given by the equation,

$$S_0(k) = M_+^{(t)}(0, k) \begin{bmatrix} 1 & 0 \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} & 1 \end{bmatrix}.$$

If  $t = 0$  (in fact, for all  $t \leq 0$ ) the factorization,

$$\begin{bmatrix} 1 & -\frac{B(k)}{A(k)} \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} & \frac{1}{A(k)A(k)} \end{bmatrix} = \begin{bmatrix} A(k) & -B(k) \\ 0 & \frac{1}{A(k)} \end{bmatrix} \begin{bmatrix} \bar{A}(\bar{k}) & 0 \\ \lambda \bar{B}(\bar{k}) & \frac{1}{A(k)} \end{bmatrix},$$

yields a (unique) solution to the RH problem defined in (3.15). This implies,

$$M_+^{(t)}(0, k) = \begin{bmatrix} \frac{1}{A(k)} & B(k) \\ 0 & A(k) \end{bmatrix},$$

and hence

$$S_0(k) = \begin{bmatrix} \frac{1}{A(k)} & B(k) \\ 0 & A(k) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} & 1 \end{bmatrix} = \begin{bmatrix} \bar{A}(\bar{k}) & B(k) \\ \lambda \bar{B}(\bar{k}) & A(k) \end{bmatrix} = S(k),$$

i.e. equation (A.20) follows.

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<sup>2</sup>In general, the error term in (A.27) is not exponentially small; the deformation process includes a certain rational approximation of the function  $\bar{B}(\bar{k})$  which produces an additional error (cf. again [23, 26]).

## B Long Time Asymptotics

The formulation presented in this paper is very convenient for computing the long time asymptotics of the solution  $q(x, t)$  in the case  $T = \infty$ . Indeed the function  $q(x, t)$  is given in terms of the solution  $M(x, t, k)$  of the Riemann-Hilbert problem formulated in Theorem 4.1. The corresponding jump matrix,  $J(x, t, k)$ , depends on the parameters  $x, t$  according to the *explicit* formula,

$$J(x, t, k) = e^{-ikx\sigma_3 - 2ikt^2\sigma_3} J(0, 0, k) e^{ikx\sigma_3 + 2ikt^2\sigma_3},$$

which is perfectly suited for the application of the nonlinear steepest descent method of [23] (see also [25, 26] and earlier works [27, 28]). Moreover, a similar Riemann-Hilbert problem has already been analyzed via the steepest descent method in [9]. In fact, there exists the following correspondence between the Riemann-Hilbert problem considered here and the one of [9]:

$$\begin{aligned} \hat{Z}_p^{[9]}(x, t, k) &= M(x, t, k), \\ b^{[9]}(k) &= \gamma(k), \\ c^{[9]}(k) &= \Gamma(k), \\ \{k_j^{[9]}\}_{j=1}^{N^{[9]}} &= \{\lambda_j\}_{j=1}^{\Lambda}. \end{aligned}$$

Let

$$N = n_1 + \Lambda;$$

define  $c_j$ ,  $j = 1, \dots, N$ , by

$$c_j = \frac{\overline{\lambda B(\lambda_j)}}{a(\lambda_j) \dot{d}(\lambda_j)}, \quad j = 1, \dots, \Lambda, \quad c_{\Lambda+j} = \frac{1}{\dot{a}(k_j) b(k_j)}, \quad j = 1, \dots, n_1. \quad (\text{B.1})$$

Then,

$$\{c_j^{[9]}\}_{j=1}^{N^{[9]}} = \{c_j\}_{j=1}^{\Lambda}.$$

The zeros  $k_j$  of the function  $a(k)$  ( $\equiv s_2^{+[9]}(k)$ ) were missed in [9] (see [7]). Nevertheless, if we just make the extensions,

$$\{c_j^{[9]}\}_{j=1}^{N^{[9]}} \mapsto \{c_j\}_{j=1}^N, \quad (\text{B.2a})$$

$$\{k_j^{[9]}\}_{j=1}^{N^{[9]}} \mapsto \{\kappa_j\}_{j=1}^N, \quad (\text{B.2b})$$

$$\kappa_j = \lambda_j, \quad j = 1, \dots, \Lambda, \quad (\text{B.2c})$$

$$\kappa_{\Lambda+j} = k_j, \quad j = 1, \dots, n_1, \quad (\text{B.2d})$$

then all the asymptotic considerations of the work [9] can be repeated word for word, and we arrive at the following result.

**Theorem B.1.** *Suppose that the conditions of Theorem 4.1 are satisfied. Then the solution  $q(x, t)$  of the NLS equation on the half - line corresponding to the initial-boundary data  $q_0(x)$ ,  $g_0(t)$  and  $g_1(t)$  exhibits the following large  $t$  behavior.*

- (i) If the set  $\{\kappa_j\}_{j=1}^\Lambda = \{\lambda_j\}_{j=1}^\Lambda$  is empty then the asymptotics has a quasilinear dispersive character, i.e. it is described by the Zakharov - Manakov type formulae,

$$q(x, t) = t^{-1/2} \alpha \left( -\frac{x}{4t} \right) \exp \left\{ \frac{ix^2}{4t} - 2i\lambda\alpha^2 \left( -\frac{x}{4t} \right) \log t + i\phi \left( -\frac{x}{4t} \right) \right\} + o(t^{-1/2}), \quad (\text{B.3})$$

$$t \rightarrow \infty, \quad \frac{x}{4t} = O(1),$$

with the amplitude  $\alpha$  and the phase  $\phi$  given by the equations (cf. [29])

$$\alpha^2(k) = -\frac{\lambda}{4\pi} \log \left( 1 - \lambda |\gamma(k) - \lambda \overline{\Gamma(k)}|^2 \right), \quad (\text{B.4})$$

$$\begin{aligned} \phi(k) = & -6\lambda\alpha^2(k) \log 2 + \frac{\pi(2-\lambda)}{4} + \arg \left( \gamma(k) - \lambda \overline{\Gamma(k)} \right) + \arg \Gamma(2i\lambda\alpha^2(k)) \\ & - 4\lambda \int_{-\infty}^k \log |\mu - k| d\alpha^2(\mu), \end{aligned} \quad (\text{B.5})$$

where  $\Gamma(z)$  denotes Euler's gamma-function.

- (ii) If  $\lambda = -1$  and the set  $\{\kappa_j\}_{j=1}^\Lambda = \{\lambda_j\}_{j=1}^\Lambda$  is not empty then solitons, which are moving away from the boundary, are generated. This means that there are  $\Lambda$  directions on the  $(x, t)$ -plane, namely

$$t \rightarrow \infty, \quad -\frac{x}{4t} = \xi_j + O\left(\frac{1}{t}\right), \quad j \in \{1, \dots, \Lambda\}, \quad (\text{B.6})$$

along which the asymptotics is given by the one-soliton formula,

$$q(x, t) = -\frac{2\eta_j \exp \left( -2i\xi_j x - 4i(\xi_j^2 - \eta_j^2)t - i\phi_j \right)}{\cosh \left( 2\eta_j(x + 4\xi_j t) - \Delta_j \right)} + O(t^{-1/2}), \quad (\text{B.7})$$

where

$$\eta_j = \text{Im}(\kappa_j), \quad \xi_j = \text{Re}(\kappa_j),$$

and the parameters  $\phi_j$  and  $\Delta_j$  are described by the following equations:

$$\begin{aligned} \phi_j = & -\frac{\pi}{2} + \arg c_j + \sum_{l=1, l \neq j}^N \left( 1 - \text{sign}(\xi_l - \xi_j) \right) \arg \left( \frac{\lambda_j - \kappa_l}{\lambda_j - \bar{\kappa}_l} \right) \\ & + \frac{1}{\pi} \int_{-\infty}^{-x/4t} \frac{\log \left( 1 - \lambda |\gamma(k) - \lambda \overline{\Gamma(k)}|^2 \right)}{(\mu - \xi_j)^2 + \eta_j^2} (\mu - \xi_j) d\mu, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \Delta_j = & -\log 2\eta_j + \log |c_j| + \sum_{l=1, l \neq j}^N \left( 1 - \text{sign}(\xi_l - \xi_j) \right) \log \left| \frac{\lambda_j - \kappa_l}{\lambda_j - \bar{\kappa}_l} \right| \\ & - \frac{\eta_j}{\pi} \int_{-\infty}^{-x/4t} \frac{\log \left( 1 - \lambda |\gamma(k) - \lambda \overline{\Gamma(k)}|^2 \right)}{(\mu - \xi_j)^2 + \eta_j^2} d\mu. \end{aligned} \quad (\text{B.9})$$



Away from the rays (B.6) the asymptotics has again dispersive character, and it can be described by formulae (B.3)–(B.5), evaluated at  $\lambda = -1$ , and with the term,

$$\phi_{\text{solitons}} = 2 \sum_{j=1}^N \arg(\kappa_j - k) \operatorname{sign}(\xi_j - k),$$

added to the right hand side of (B.5).

**Remark B.2.** The zeros  $k_j, j = 1, \dots, n_1$ , of the function  $a(k)$  lying in the first quadrant, although they participate in the residue conditions of the Riemann-Hilbert problem, they do not generate solitons (there are exactly  $\Lambda$  but not  $N = n_1 + \Lambda$  soliton rays indicated in (B.6). They, however, do participate in formulae (B.8) and (B.9) describing the parameters of the soliton (B.7) (the summations in the right hand sides of these formulae run from 1 to  $N = \Lambda + n_1$ ). A qualitative explanation of the absence in the asymptotics of the solitons corresponding to  $k_j$  is quite simple: these solitons move to the left, and hence after a finite time disappear from the first quadrant.

**Remark B.3.** In the cases of the linearizable boundary conditions all the parameters in the above formulae can be expressed in terms of the spectral functions  $a(k)$  and  $b(k)$ , i.e. in terms of the initial data only. Indeed we have,

$$c_j = \frac{\overline{\lambda b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_0(\lambda_j)}, \quad j = 1, \dots, \Lambda,$$

or

$$c_j = -\frac{\overline{\lambda b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_1(\lambda_j)}, \quad j = 1, \dots, \Lambda,$$

or

$$c_j = -\frac{\lambda \frac{2\lambda_j - i\rho}{2\lambda_j + i\rho} \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j) \dot{\Delta}_\rho(\lambda_j)}, \quad j = 1, \dots, \Lambda.$$

Also,

$$\Gamma(k) = \frac{\overline{\lambda b(-\bar{k})}}{a(k) \Delta_0(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+,$$

or

$$\Gamma(k) = -\frac{\overline{\lambda b(-\bar{k})}}{a(k) \Delta_1(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+,$$

or

$$\Gamma(k) = -\frac{\lambda \frac{2k - i\rho}{2k + i\rho} \overline{b(-\bar{k})}}{a(k) \Delta_\rho(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+.$$

## C Proofs of Lemmas 5.1, 5.3 and 5.4

### Proof of Lemma 5.1

For  $|k| > 1$ , we can write

$$\mathbf{F}(k, g_1)\psi = \sum_{j=1}^3 \mathbf{F}_j(k, g_1)\psi, \quad (\text{C.1})$$

where

$$\begin{aligned} \mathbf{F}_1(k, g_1)(e^{4ik^2t}\chi) &= \left(\frac{\lambda}{i}\right) e^{4ik^2t} \int_0^t |f_0(t')|^2 \chi(t') dt' + \left(\frac{\lambda}{2k}\right) e^{4ik^2t} \int_0^t \overline{f_0(t')} f_1(t') \chi(t') dt' \\ &\quad - \left(\frac{\lambda}{2k}\right) e^{4ik^2t} \int_0^t \overline{f_1(t')} f_0(t') \chi(t') dt' - \left(\frac{i\lambda}{4k^2}\right) e^{4ik^2t} \int_0^t |f_1(t')|^2 \chi(t') dt', \end{aligned} \quad (\text{C.2a})$$

$$\begin{aligned} \mathbf{F}_2(k, g_1)(e^{4ik^2t}\chi) &= -\left(\frac{\lambda}{i}\right) \overline{f_0(t)} \int_0^t e^{4ik^2t'} f_0(t') \chi(t') dt' - \left(\frac{\lambda}{2k}\right) \overline{f_0(t)} \int_0^t e^{4ik^2t'} f_1(t') \chi(t') dt' \\ &\quad + \left(\frac{\lambda}{2k}\right) \overline{f_1(t)} \int_0^t e^{4ik^2t'} f_0(t') \chi(t') dt' + \left(\frac{i\lambda}{4k^2}\right) \overline{f_1(t)} \int_0^t e^{4ik^2t'} f_1(t') \chi(t') dt', \end{aligned} \quad (\text{C.2b})$$

and

$$\begin{aligned} \mathbf{F}_3(k, g_1)(e^{4ik^2t}\chi) &= \left(\frac{\lambda}{i}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2s} f_0(s) \chi(s) ds \right] dt' \\ &\quad + \left(\frac{\lambda}{2k}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2s} f_1(s) \chi(s) ds \right] dt' \\ &\quad - \left(\frac{\lambda}{2k}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2s} f_0(s) \chi(s) ds \right] dt' \\ &\quad - \left(\frac{i\lambda}{4k^2}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2s} f_1(s) \chi(s) ds \right] dt'. \end{aligned} \quad (\text{C.2c})$$

Indeed, using (5.9), (5.11), (5.14), (5.15) and integration by parts, we find

$$\begin{aligned} \psi(t, k) &= e^{4ik^2t} - \left(\frac{\lambda}{2ik}\right) \overline{f_0(t)} \phi(t, k) + \left(\frac{\lambda}{2ik}\right) \int_0^t e^{4ik^2(t-t')} [\overline{f_0'(t')} \phi(t', k) + \overline{f_0(t')} \phi_t(t', k)] dt' \\ &\quad - i\lambda \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \phi(t', k) dt' \\ &= e^{4ik^2t} - \left(\frac{\lambda}{2ik}\right) \overline{f_0(t)} \phi(t, k) + \left(\frac{\lambda}{2ik}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \phi(t', k) dt' \\ &\quad + \left(\frac{\lambda}{2ik}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_0(t')} [2k f_0(t') + i f_1(t')] \psi(t', k) dt' + \left(\frac{\lambda}{4k^2}\right) \overline{f_1(t)} \phi(t, k) \\ &\quad - \left(\frac{\lambda}{4k^2}\right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \phi(t', k) dt' \end{aligned}$$

$$- \left( \frac{\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1(t')} [2kf_0(t') + if_1(t')] \psi(t', k) dt' \quad (\text{C.3})$$

Using (5.9a) we can further eliminate  $\phi(t, k)$  from (C.3):

$$\begin{aligned} \psi(t, k) &= e^{4ik^2t} - \left( \frac{\lambda}{2i} \right) \overline{f_0(t)} \left[ \int_0^t 2f_0(t') \psi(t', k) dt' + \frac{i}{k} \int_0^t f_1(t') \psi(t', k) dt \right] \\ &+ \left( \frac{\lambda}{2i} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} 2f_0(s) \psi(s, k) ds + \frac{i}{k} \int_0^{t'} f_1(s) \psi(s, k) ds \right] dt' \\ &+ \left( \frac{\lambda}{i} \right) \int_0^t e^{4ik^2(t-t')} |f_0(t')|^2 \psi(t', k) dt' \\ &+ \left( \frac{\lambda}{2k} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0(t')} f_1(t') \psi(t', k) dt' \\ &+ \left( \frac{\lambda}{4k} \right) \overline{f_1(t)} \left[ \int_0^t 2f_0(t') \psi(t', k) dt' + \frac{i}{k} \int_0^t f_1(t') \psi(t', k) dt \right] \\ &- \left( \frac{\lambda}{4k} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} 2f_0(s) \psi(s, k) ds + \frac{i}{k} \int_0^{t'} f_1(s) \psi(s, k) ds \right] dt' \\ &- \left( \frac{\lambda}{2k} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1(t')} f_0(t') \psi(t', k) dt' \\ &- \left( \frac{i\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} |f_1(t')|^2 \psi(t', k) dt', \end{aligned}$$

which is equivalent to (C.1) and (C.2).

Let  $v \in C([0, T])$ . From (5.15), (C.1), (C.2) and the embedding  $H^1(0, T) \hookrightarrow C([0, T])$ , we find

$$|[\mathbf{F}(k, g_1)v](t)| \leq \mathcal{B}_1(\|g_1\|_{H^1(0, T)}) \int_0^t |\mathcal{K}_1(g_1)(t')| \max_{0 \leq s \leq t} |v(s)| dt' \quad \forall k \in \mathcal{Q}, \quad (\text{C.4})$$

where  $\mathcal{B}_1(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous,  $\mathcal{K}_1(g_1) \in L_2(0, T)$  and there exists a continuous function  $\mathcal{B}_2(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$\|\mathcal{K}_1(g_1)\|_{L_2(0, T)} \leq \mathcal{B}_2(\|g_1\|_{H^1(0, T)}). \quad (\text{C.5})$$

It follows immediately from (C.4) that the Neumann series  $\sum_{j=0}^{\infty} [\mathbf{F}(k, g_1)]^j$  is convergent in the space of bounded operators on  $C([0, T])$ , uniformly with respect to  $k \in \mathcal{Q}$ . Therefore, the operator  $\mathbf{I} - \mathbf{F}(k, g_1)$  is invertible on  $C([0, T])$  and

$$\text{the norm of } [\mathbf{I} - \mathbf{F}(k, g_1)]^{-1} \text{ is uniformly bounded for } k \in \bar{\mathcal{Q}}. \quad (\text{C.6})$$

Moreover, the map  $k \mapsto [\mathbf{I} - \mathbf{F}(k, g_1)]^{-1}$  is analytic in  $\mathcal{Q}$  and continuous on  $\bar{\mathcal{Q}}$ . These properties of  $\mathbf{I} - \mathbf{F}(k, g_1)$  imply the analytic properties of  $\psi(t, k)$ .

Finally we observe that, from (5.6b), (5.15), (C.1) and (C.2), the dependence of the operator  $\mathbf{F}(k, g_1)$  on  $g_1$  is locally Lipschitz continuous. The local Lipschitz continuity of the map  $g_1 \mapsto \psi$  then follows immediately.  $\square$

### Proof of Lemma 5.3

The following calculations are based on (C.1), (C.2) and integration by parts.

We define  $\chi_0$  to be the solution of

$$\chi_0(t) = 1 + \left(\frac{\lambda}{i}\right) \int_0^t |f_0(t')|^2 \chi_0(t') dt', \quad (\text{C.7})$$

i.e.,

$$\chi_0(t) = e^{-i\lambda \int_0^t |g_0(s)|^2 ds}. \quad (\text{C.8})$$

Then we have

$$e^{4ik^2t} + \mathbf{F}(k, g_1)(e^{4ik^2t} \chi_0) = e^{4ik^2t} \left( \chi_0(t) + \frac{\omega_1(t)}{k} \right) + R_1(\chi_0)(t, k), \quad (\text{C.9})$$

where

$$\omega_1 = \mathcal{G}_1 \chi_0, \quad (\text{C.10})$$

$$(\mathcal{G}_1 \chi)(t) = \left(\frac{\lambda}{2}\right) \int_0^t [\overline{f_0(t')} f_1(t') - \overline{f_1(t')} f_0(t')] \chi(t') dt', \quad (\text{C.11})$$

and

$$R_1(\chi)(t, k) = -\left(\frac{i\lambda}{4k^2}\right) e^{4ik^2t} \int_0^t |f_1(t')|^2 \chi(t') dt' + \sum_{j=2}^3 \mathbf{F}_j(k, g_1)(e^{4ik^2t} \chi). \quad (\text{C.12})$$

Let  $\chi_1$  be defined by the Volterra integral equation

$$\chi_1(t) = \omega_1(t) + \left(\frac{\lambda}{i}\right) \int_0^t |f_0(t')|^2 \chi_1(t') dt', \quad (\text{C.13})$$

then we have  $\chi_1(0) = 0$  and

$$\begin{aligned} & e^{4ik^2t} + \mathbf{F}(k, g_1) \left( e^{4ik^2t} \left( \chi_0 + \frac{\chi_1}{k} \right) \right) \\ &= e^{4ik^2t} \left( \chi_0(t) + \frac{\chi_1(t)}{k} + \frac{\omega_2(t)}{k^2} \right) + R_2(\chi_0) + \frac{R_1(\chi_1)}{k}, \end{aligned} \quad (\text{C.14})$$

where

$$\omega_2 = \mathcal{G}_2 \chi_0 + \mathcal{G}_1 \chi_1 \quad (\text{C.15})$$

$$\begin{aligned} (\mathcal{G}_2 \chi)(t) &= -\left(\frac{i\lambda}{4}\right) \int_0^t |f_1(t')|^2 \chi(t') dt' + \left(\frac{\lambda}{4}\right) |f_0(t)|^2 \chi(t) \\ &\quad - \left(\frac{\lambda}{4}\right) \int_0^t \overline{f_0'(t')} f_0(t') \chi(t') dt', \end{aligned} \quad (\text{C.16})$$

and

$$\begin{aligned}
R_2(\chi)(t, k) = & - \left( \frac{\lambda}{4k^2} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} [f_0 \chi]'(t') dt' \\
& - \left( \frac{\lambda}{2k} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} f_1(t') \chi(t') dt' + \left( \frac{\lambda}{2k} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} f_0(t') \chi(t') dt' \\
& + \left( \frac{i\lambda}{4k^2} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} f_1(t') \chi(t') dt' \\
& + \left( \frac{\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_0 \chi]'(s) ds \right] dt' \\
& + \left( \frac{\lambda}{2k} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} f_1(s) \chi(s) ds \right] dt' \\
& - \left( \frac{\lambda}{2k} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} f_0(s) \chi(s) ds \right] dt' \\
& - \left( \frac{i\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} f_1(s) \chi(s) ds \right] dt'. \tag{C.17}
\end{aligned}$$

The function  $\chi_2$  is then defined by the Volterra integral equation

$$\chi_2(t) = \omega_2(t) + \left( \frac{\lambda}{i} \right) \int_0^t |f_0(t')|^2 \chi_2(t') dt', \tag{C.18}$$

and we have  $\chi_2(0) = 0$  and

$$\begin{aligned}
& e^{4ik^2 t} + \mathbf{F}(k, g_1) \left( e^{4ik^2 t} \sum_{\ell=0}^2 \frac{\chi_\ell}{k^\ell} \right) \\
& = e^{4ik^2 t} \left( \sum_{\ell=0}^2 \frac{\chi_\ell(t)}{k^\ell} + \frac{\omega_3(t)}{k^3} \right) + R_3(\chi_0) + \frac{R_2(\chi_1)}{k} + \frac{R_1(\chi_2)}{k^2}, \tag{C.19}
\end{aligned}$$

where

$$\omega_3 = \mathcal{G}_3 \chi_0 + \mathcal{G}_2 \chi_1 + \mathcal{G}_1 \chi_2 \tag{C.20}$$

$$\begin{aligned}
\mathcal{G}_3 \chi(t) = & - \left( \frac{\lambda}{8i} \right) \overline{f_0(t)} f_1(t) \chi(t) + \left( \frac{\lambda}{8i} \right) \overline{f_1(t)} f_0(t) \chi(t) \\
& + \left( \frac{\lambda}{8i} \right) \int_0^t \overline{f_0'(t')} f_1(t') \chi(t') dt' \\
& - \left( \frac{\lambda}{8i} \right) \int_0^t \overline{f_1'(t')} f_0(t') \chi(t') dt', \tag{C.21}
\end{aligned}$$

and

$$\begin{aligned}
R_3(\chi) = & - \left( \frac{\lambda}{4k^2} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} [f_0 \chi]'(t') dt' \\
& + \left( \frac{\lambda}{8ik^3} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} [f_1 \chi]'(t') dt' - \left( \frac{\lambda}{8ik^3} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} [f_0 \chi]'(t') dt' \\
& + \left( \frac{i\lambda}{4k^2} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} f_1(t') \chi(t') dt' \\
& + \left( \frac{\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_0 \chi]'(s) ds \right] dt' \\
& - \left( \frac{\lambda}{8ik^3} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_1 \chi]'(s) ds \right] dt' \\
& + \left( \frac{\lambda}{8ik^3} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_0 \chi]'(s) ds \right] dt' \\
& - \left( \frac{i\lambda}{4k^2} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} f_1(s) \chi(s) ds \right] dt'. \tag{C.22}
\end{aligned}$$

Next we define  $\chi_3$  by the Volterra integral equation

$$\chi_3(t) = \omega_3(t) + \left( \frac{\lambda}{i} \right) \int_0^t |f_0(t')|^2 \chi_3(t') dt'. \tag{C.23}$$

It follows that  $\chi_3(0) = 0$  and

$$\begin{aligned}
& e^{4ik^2 t} + \mathbf{F}(k, g_1) \left( e^{4ik^2 t} \sum_{\ell=0}^3 \frac{\chi_\ell}{k^\ell} \right) \\
& = e^{4ik^2 t} \left( \sum_{\ell=0}^3 \frac{\chi_\ell(t)}{k^\ell} + \frac{\omega_4(t)}{k^4} \right) + R_4(\chi_0) + \frac{R_3(\chi_1)}{k} + \frac{R_2(\chi_2)}{k^2} + \frac{R_1(\chi_3)}{k^3}, \tag{C.24}
\end{aligned}$$

where

$$\omega_4(t) = \mathcal{G}_4 \chi_0 + \mathcal{G}_3 \chi_1 + \mathcal{G}_2 \chi_2 + \mathcal{G}_1 \chi_3, \tag{C.25}$$

$$\begin{aligned}
(\mathcal{G}_4 \chi)(t) = & - \left( \frac{\lambda}{16i} \right) \overline{f_0(t)} [f_0 \chi]'(t) + \left( \frac{\lambda}{16} \right) |f_1(t)|^2 \chi(t) \\
& + \left( \frac{\lambda}{16i} \right) \int_0^t \overline{f_0'(t')} [f_0 \chi]'(t') dt' - \left( \frac{\lambda}{16} \right) \int_0^t \overline{f_1'(t')} f_1(t') \chi(t') dt', \tag{C.26}
\end{aligned}$$

and

$$\begin{aligned}
R_4(\chi)(t, k) = & \left( \frac{\lambda}{16ik^4} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} [f_0 \chi]''(t') dt' \\
& + \left( \frac{\lambda}{8ik^3} \right) \overline{f_0(t)} \int_0^t e^{4ik^2 t'} [f_1 \chi]'(t') dt' - \left( \frac{\lambda}{8ik^3} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} [f_0 \chi]'(t') dt'
\end{aligned}$$

$$\begin{aligned}
& - \left( \frac{\lambda}{16k^4} \right) \overline{f_1(t)} \int_0^t e^{4ik^2 t'} [f_1 \chi]'(t') dt' \\
& - \left( \frac{\lambda}{16ik^4} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_0 \chi]''(s) ds \right] dt' \\
& - \left( \frac{\lambda}{8ik^3} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_0'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_1 \chi]'(s) ds \right] dt' \\
& + \left( \frac{\lambda}{8ik^3} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_0 \chi]'(s) ds \right] dt' \\
& + \left( \frac{\lambda}{16k^4} \right) \int_0^t e^{4ik^2(t-t')} \overline{f_1'(t')} \left[ \int_0^{t'} e^{4ik^2 s} [f_1 \chi]'(s) ds \right] dt'. \tag{C.27}
\end{aligned}$$

Finally, we define  $\chi_4$  by the Volterra integral equation

$$\chi_4(t) = \omega_4(t) + \left( \frac{\lambda}{i} \right) \int_0^t |f_0(t')|^2 \chi_4(t') dt'. \tag{C.28}$$

Then we have  $\chi_4(0) = 0$  and

$$\begin{aligned}
& e^{4ik^2 t} + \mathbf{F}(k, g_1) \left( e^{4ik^2 t} \sum_{\ell=0}^4 \frac{\chi_\ell}{k^\ell} \right) \\
& = e^{4ik^2 t} \left( \sum_{\ell=0}^4 \frac{\chi_\ell(t)}{k^\ell} \right) + \tau(t, k), \tag{C.29}
\end{aligned}$$

where

$$\tau(t, k) = \frac{e^{4ik^2 t}}{k^5} (\mathcal{G}_1 \chi_4)(t) + \frac{R_1(\chi_4)}{k^4} + \frac{R_1(\chi_3)}{k^3} + \frac{R_2(\chi_2)}{k^2} + \frac{R_3(\chi_1)}{k} + R_4(\chi_0). \tag{C.30}$$

Using the assumptions on  $f_0$  and  $f_1$ , (C.8), (C.10), (C.11), (C.13), (C.15), (C.16), (C.18), (C.20), (C.21), (C.23), (C.25), (C.26) and (C.28), we can easily establish successively  $\chi_0 \in C^\infty([0, T])$ ,  $\omega_1 \in H^2(0, T)$ ,  $\chi_1 \in H^2(0, T)$ ,  $\omega_2 \in H^2(0, T)$ ,  $\chi_2 \in H^2(0, T)$ ,  $\omega_3 \in H^1(0, T)$ ,  $\chi_3 \in H^1(0, T)$ ,  $\omega_4 \in H^1(0, T)$  and  $\chi_4 \in H^1(0, T)$ .

Let

$$\psi_4(t, k) = \psi(t, k) - e^{4ik^2 t} \sum_{\ell=0}^4 \frac{\chi_\ell(t)}{k^\ell}.$$

Combining (C.29) and (5.14), we find

$$\tau + \mathbf{F}(k, g_1) \psi_4 = \psi_4. \tag{C.31}$$

From (C.11), (C.12), (C.17), (C.22), (C.27) and (C.30), we immediately have

$$\tau(t, k) = O\left(\frac{1}{k^3}\right). \tag{C.32}$$

The estimate (5.19) then follows from (C.6), (C.31) and (C.32).

Observe that

$$t \mapsto |\xi|^{3/2} \tau(t, \sqrt{\xi}) \text{ is a continuous map from } [0, t] \text{ into } L_2(\mathbb{R}_*), \quad (\text{C.33})$$

where  $\mathbb{R}_* = (-\infty, -1) \cup (1, \infty)$ . Indeed, the map

$$t \mapsto |\xi|^{3/2} \frac{e^{4i\xi t}}{\xi^{5/2}} (\mathcal{G}_1 \chi_4)(t)$$

belongs to  $C([0, T], L_2(\mathbb{R}_*))$  because  $\mathcal{G}_1 \chi_4 \in C([0, T])$  (cf. (C.11)), and the map

$$t \mapsto |\xi|^{3/2} \left[ \frac{R_1(\chi_4)(t, \sqrt{\xi})}{\xi^2} + \frac{R_1(\chi_3)(t, \sqrt{\xi})}{\xi^{3/2}} + \frac{R_2(\chi_2)(t, \sqrt{\xi})}{\xi} + \frac{R_3(\chi_1)(t, \sqrt{\xi})}{\xi^{1/2}} + R_4(\chi_0)(t, \sqrt{\xi}) \right]$$

belongs to  $C([0, T], L_2(\mathbb{R}_*))$  because of the (negative) powers of  $k$  that appear in (C.12), (C.17), (C.22) and (C.27), and because

$$t \mapsto \int_0^t e^{4i\xi t'} v(t') dt'$$

defines a continuous map from  $[0, T]$  into  $L_2(\mathbb{R})$  for any  $v \in L_2(0, T)$ .

For  $\xi \in \mathbb{R}_*$ , we obtain from (C.31)

$$\tau(t, \sqrt{\xi}) + \mathbf{F}(\sqrt{\xi}, g_1) \psi_4(t, \sqrt{\xi}) = \psi_4(t, \sqrt{\xi}) \quad (\text{C.34})$$

which in view of (C.33) can be considered as an integral equation on  $C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$ . Using (C.1) and (C.2a)–(C.2c), we have the following analog of (C.4):

$$\begin{aligned} & \left\| [\mathbf{F}(\sqrt{\xi}, g_1)v](t) \right\|_{L_2(\mathbb{R}_*, |\xi|^3 d\xi)} \\ & \leq \mathcal{B}_2(\|g_1\|_{H^1(0, T)}) \int_0^t |\mathcal{K}_2(g_1)(t')| \max_{0 \leq s \leq t} \|v(s)\|_{L_2(\mathbb{R}_*, |\xi|^3 d\xi)} dt', \end{aligned} \quad (\text{C.35})$$

where  $\mathcal{B}_2(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous and  $\mathcal{K}_2(g_1) \in L_2(0, T)$  satisfy an estimate similar to (C.5). It follows from (C.35) that the operator  $\mathbf{I} - \mathbf{F}(\sqrt{\xi}, g_1)$  is invertible on the space  $C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$ , and (5.20) follows.  $\square$

## Proof of Lemma 5.4

We use the standard notation  $L(X, Y)$  to denote the space of bounded linear operators from the normed linear space  $X$  to the normed linear space  $Y$ , which is simplified to  $L(X)$  in the case  $Y = X$ .

The operators  $\mathcal{G}_1, \dots, \mathcal{G}_4$  defined by (C.11), (C.16), (C.21) and (C.26) depend on the function  $g_1$ . Henceforth we will denote them as  $\mathcal{G}_1(g_1), \dots, \mathcal{G}_4(g_1)$ .

It is easy to see from (C.11) that the map  $g_1 \mapsto \mathcal{G}_1(g_1)$  from  $H^1(0, T)$  into the space  $L(H^1(0, T), H^2(0, T))$  is bounded and linear. It then follows from (C.10) and (C.13) that  $E_1 : H_{0*}^1(0, T) \rightarrow H^2(0, T)$  is also bounded and linear, and thus locally Lipschitz continuous.



Similarly, we see from (C.16) that the map  $g_1 \mapsto \mathcal{G}_2(g_1)$  from  $H^1(0, T)$  into the space  $L(H^2(0, T), H^2(0, T))$  is locally Lipschitz continuous, and then (C.15) and (C.18) imply that  $E_2 : H_{0*}^1(0, T) \longrightarrow H^2(0, T)$  is also locally Lipschitz continuous.

From (C.21) we obtain the local Lipschitz continuity of the map  $g_1 \mapsto \mathcal{G}_3(g_1)$  from  $H^1(0, T)$  into  $L(H^1(0, T), H^1(0, T))$ , and then the local Lipschitz continuity of the map  $E_3 : H_{0*}^1(0, T) \longrightarrow H^1(0, T)$  follows from (C.20) and (C.23).

Finally, we see from (C.26) that the map  $g_1 \mapsto \mathcal{G}_4(g_1)$  from  $H^1(0, T)$  into the space  $L(H^2(0, T), H^1(0, T))$  is locally Lipschitz continuous. Combining (C.25) and (C.28), we then obtain the local Lipschitz continuity of  $E_4 : H_{0*}^1(0, T) \longrightarrow H^1(0, T)$ .

To see the local Lipschitz continuity of the map  $E : H_{0*}^1(0, T) \longrightarrow C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$ , we first observe that

$$\begin{aligned} & \text{the map } g_1 \mapsto \mathbf{F}(\sqrt{\xi}, g_1) \text{ from } H_{0*}^1(0, T) \text{ into } L(C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))) \\ & \text{is locally Lipschitz continuous,} \end{aligned} \tag{C.36}$$

by (C.2). Furthermore, the function  $\tau(t, \sqrt{\xi})$ , which depends on  $g_1$ , can be represented more precisely as the continuous function  $\tilde{t}(g_1)$  from  $[0, T]$  into  $L_2(\mathbb{R}_*, |\xi|^3 d\xi)$  given by the formula

$$[\tilde{t}(g_1)(t)](\xi) = \tau(t, \sqrt{\xi}),$$

and the integral equation (C.34) can be written as

$$\tilde{t}(g_1) + \mathbf{F}(\sqrt{\xi}, g_1)\tilde{\psi}_4 = \tilde{\psi}_4. \tag{C.37}$$

Since the map  $g_1 \mapsto \tilde{t}(g_1)$  from  $H_{0*}^1(0, T)$  into  $C([0, T], L_2(\mathbb{R}_*, |\xi|^3 d\xi))$  is locally Lipschitz continuous by (C.11), (C.12), (C.17), (C.22), (C.27) and (C.30), the local Lipschitz continuity of  $E$  follows from (C.36) and (C.37).  $\square$

## D A Priori Bounds

### A Priori Bound for $\|q_x(0, \cdot)\|_{L_2(0, T)}$

Let  $q$  be a smooth solution of (2.1) for  $0 \leq t \leq T$  with sufficient decay as  $x \rightarrow \infty$  and let  $q(x, 0) = 0$ . Multiplying (2.1) by  $\bar{q}$  and integrating over  $\mathbb{R}^+$  we obtain

$$i(q_t, q) + (q_{xx}, q) - 2\lambda(|q|^2 q, q) = 0, \tag{D.1}$$

where  $(u, v) = \int_0^\infty u \bar{v} dx$ . The imaginary part of (D.1) is equivalent to

$$\frac{d}{dt}(q, q) - 2 \operatorname{Im} [q_x(0, t)q(0, t)] = 0 \quad \text{for } 0 < t < T. \tag{D.2}$$

Integration of (D.2) over  $(0, T)$  yields the following estimate

$$\|q(\cdot, t)\|_{L_2(0, \infty)}^2 \leq 2\|q_x(0, \cdot)\|_{L_2(0, T)}\|q(0, \cdot)\|_{L_2(0, T)} \quad \text{for } 0 \leq t \leq T. \tag{D.3}$$

We now multiply (2.1) by  $\bar{q}_t$  and integrate the resulting equation over  $\mathbb{R}^+$  to arrive at

$$i(q_t, q_t) + (q_{xx}, q_t) - 2\lambda(|q|^2 q, q_t) = 0. \tag{D.4}$$

The real part of (D.4) gives

$$-2 \operatorname{Re} [q_x(0, t)q_t(0, t)] - \frac{d}{dt}(q_x, q_x) - \lambda \frac{d}{dt}(|q|^2, |q|^2) = 0 \quad \text{for } 0 < t < T. \quad (\text{D.5})$$

Integration of (D.5) over  $(0, T)$  then yields

$$\|q_x(\cdot, t)\|_{L_2(0, \infty)}^2 + \lambda \|q(\cdot, t)\|_{L_4(0, \infty)}^4 \leq 2 \|q_x(0, \cdot)\|_{L_2(0, T)} \|q_t(0, \cdot)\|_{L_2(0, T)} \quad (\text{D.6})$$

for  $0 \leq t \leq T$ .

Multiplying (2.1) by  $\bar{q}_x$  and integrating over  $\mathbb{R}^+$  gives

$$i(q_t, q_x) + (q_{xx}, q_x) - 2\lambda(|q|^2 q, q_x) = 0. \quad (\text{D.7})$$

The real part of (D.7) can be written as

$$i \frac{d}{dt}(q, q_x) + iq(0, t)q_t(0, t) - |q_x(0, t)|^2 + \lambda |q(0, t)|^4 = 0 \quad \text{for } 0 < t < T. \quad (\text{D.8})$$

Integration of (D.8) over  $(0, T)$  then yields the following estimate:

$$\begin{aligned} \|q_x(0, \cdot)\|_{L_2(0, T)}^2 &\leq \|q(0, \cdot)\|_{L_4(0, T)}^4 + \|q(0, \cdot)\|_{L_2(0, T)} \|q_t(0, \cdot)\|_{L_2(0, T)} \\ &\quad + \|q(\cdot, T)\|_{L_2(0, \infty)} \|q_x(\cdot, T)\|_{L_2(0, \infty)}. \end{aligned} \quad (\text{D.9})$$

In the case where  $\lambda = 1$ , it follows immediately from (D.3), (D.6) and (D.9) that

$$\|q_x(0, \cdot)\|_{L_2(0, T)} \leq \mathcal{B}_1(\|q(0, \cdot)\|_{H^1(0, T)}), \quad (\text{D.10})$$

where  $\mathcal{B}_\#(\cdot)$  is a (generic) continuous map from  $\mathbb{R}^+ \cup \{0\}$  into  $\mathbb{R}^+$  satisfying  $\mathcal{B}_\#(0) = 0$ .

On the other hand, the Sobolev embedding  $H^1(0, \infty) \hookrightarrow L_4(0, \infty)$  implies that

$$\|u\|_{L_4(0, \infty)}^4 \leq \|u\|_{L_2(0, \infty)}^3 \|u_x\|_{L_2(0, \infty)}. \quad (\text{D.11})$$

Therefore, in the case where  $\lambda = -1$ , we conclude from (D.3), (D.6), (D.9) and (D.11) that (D.10) remains valid provided  $\|q(0, \cdot)\|_{L_2(0, T)}$  is sufficiently small.

We note that the estimates (D.3), (D.6) and (D.10) (and (D.11) when  $\lambda = -1$ ) also imply

$$\max_{0 \leq t \leq T} \|q(\cdot, t)\|_{H^1(0, \infty)} \leq \mathcal{B}_2(\|q(0, \cdot)\|_{H^1(0, T)}). \quad (\text{D.12})$$

## A Priori Bound for $\|q_{xt}(0, \cdot)\|_{L_2(0, T)}$

Let  $v = q_t$ . The following equation for  $v$  is derived by differentiating (2.1) in  $t$ :

$$iv_t + v_{xx} - 4\lambda|q|^2 v - 2\lambda q^2 \bar{v} = 0. \quad (\text{D.13})$$

Multiplying (D.13) by  $\bar{v}$  and integrating over  $\mathbb{R}^+$  we find

$$i(v_t, v) + (v_{xx}, v) - 4\lambda(|q|^2 v, v) - 2\lambda(q^2 \bar{v}, v) = 0 \quad \text{for } 0 < t < T. \quad (\text{D.14})$$

The imaginary part of (D.14) then gives

$$\frac{d}{dt}(v, v) - 2 \operatorname{Im} [v_x(0, t)v(0, t)] - 4\lambda \operatorname{Im} (q^2 \bar{v}, v) = 0. \quad (\text{D.15})$$

Note that we have the Sobolev inequality

$$\|u\|_{L_\infty(0, \infty)}^2 \leq \|u\|_{L_2(0, \infty)} \|u_x\|_{L_2(0, \infty)}. \quad (\text{D.16})$$

Integrating (D.15) in  $t$ , we obtain from (D.12) and (D.16) that

$$\begin{aligned} \|v(\cdot, t)\|_{L_2(0, \infty)}^2 &\leq 2\|v_x(0, \cdot)\|_{L_2(0, T)} \|q_t(0, \cdot)\|_{L_2(0, T)} \\ &+ \mathcal{B}_3(\|q(0, \cdot)\|_{H^1(0, T)}) \int_0^t \|v(\cdot, s)\|_{L_2(0, \infty)}^2 ds \quad \text{for } 0 \leq t \leq T. \end{aligned} \quad (\text{D.17})$$

Gronwall's inequality and (D.17) imply the following estimate:

$$\|v(\cdot, t)\|_{L_2(0, \infty)}^2 \leq \mathcal{B}_4(\|q(0, \cdot)\|_{H^1(0, T)}) \|v_x(0, \cdot)\|_{L_2(0, T)} \quad \text{for } 0 \leq t \leq T. \quad (\text{D.18})$$

We now multiply (D.13) by  $\bar{v}_t$  and integrate the resulting equation over  $\mathbb{R}^+$  to obtain

$$i(v_t, v_t) + (v_{xx}, v_t) - 4\lambda(|q|^2 v, v_t) - 2\lambda(q^2 \bar{v}, v_t) = 0. \quad (\text{D.19})$$

The real part of (D.19) yields the estimate

$$\begin{aligned} \frac{d}{dt} [(v_x, v_x) + 4\lambda(|q|^2, |v|^2) + 2\lambda \operatorname{Re} (q^2, v^2)] \\ \leq 2|v_x(0, t)v_t(0, t)| + 12 \int_0^\infty |q(x, t)| |v(x, t)|^3 dx. \end{aligned} \quad (\text{D.20})$$

We have, by (D.16),

$$\begin{aligned} \int_0^\infty |q(x, t)| |v(x, t)|^3 dx &\leq \|v(\cdot, t)\|_{L_\infty(0, \infty)}^2 \|q(\cdot, t)\|_{L_2(0, \infty)} \|v(\cdot, t)\|_{L_2(0, \infty)} \\ &\leq \|q(\cdot, t)\|_{L_2(0, \infty)} \|v(\cdot, t)\|_{L_2(0, \infty)}^2 \|v_x(\cdot, t)\|_{L_2(0, \infty)}. \end{aligned} \quad (\text{D.21})$$

Integrating (D.20) we find, by (D.12), (D.16), (D.18) and (D.21),

$$\begin{aligned} \|v_x(\cdot, t)\|_{L_2(0, \infty)}^2 &\leq \mathcal{B}_5(\|q(0, \cdot)\|_{H^2(0, T)}) \left[ \|v_x(0, \cdot)\|_{L_2(0, T)} + \|v_x(0, \cdot)\|_{L_2(0, T)}^2 \right] \\ &+ \int_0^t \|v_x(\cdot, s)\|_{L_2(0, \infty)}^2 ds. \end{aligned} \quad (\text{D.22})$$

Gronwall's inequality and (D.22) imply

$$\|v_x(\cdot, t)\|_{L_2(0, \infty)}^2 \leq \mathcal{B}_6(\|q(0, \cdot)\|_{H^2(0, T)}) \left[ \|v_x(0, \cdot)\|_{L_2(0, T)} + \|v_x(0, \cdot)\|_{L_2(0, T)}^2 \right] \quad (\text{D.23})$$

for  $0 \leq t \leq T$ .

Finally we multiply (D.13) by  $\bar{v}_x$  and integrate over  $\mathbb{R}^+$  to obtain

$$i(v_t, v_x) + (v_{xx}, v_x) - 4\lambda(|q|^2 v, v_x) - 2\lambda(q^2 \bar{v}, v_x) = 0. \quad (\text{D.24})$$

Taking the real part of (D.24) we find the estimate

$$|v_x(0, t)|^2 \leq 12 \int_0^\infty |q(x, t)|^2 |v(x, t)| |v_x(x, t)| dx + i \left[ q_t(0, t) \overline{q_{tt}(0, t)} + \frac{d}{dt}(v, v_x) \right]. \quad (\text{D.25})$$

Integrating (D.25) over  $(0, T)$  we have, by (D.18) and (D.23),

$$\begin{aligned} \|v_x(0, \cdot)\|_{L_2(0, T)}^2 &\leq \mathcal{B}_7(\|q(0, \cdot)\|_{H^2(0, T)}) \left[ \|v_x(0, \cdot)\|_{L_2(0, T)} + \|v_x(0, \cdot)\|_{L_2(0, T)}^{3/2} \right] \\ &\quad + \|q(0, \cdot)\|_{H^2(0, T)}^2. \end{aligned} \quad (\text{D.26})$$

It follows from (D.26) that

$$\|q_{xt}(0, \cdot)\|_{L_2(0, T)} = \|v_x(0, \cdot)\|_{L_2(0, T)} \leq \mathcal{B}_8(\|q(0, \cdot)\|_{H^2(0, T)}), \quad (\text{D.27})$$

which holds for arbitrary  $q(0, t)$  when  $\lambda = 1$  and for  $\|q(0, \cdot)\|_{L_2(0, T)}$  sufficiently small when  $\lambda = -1$ .

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