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CONVERGENCE OF GREEDY APPROXIMATION FOR THE TRIGONOMETRIC SYSTEM[1]

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Abstract. We study the following nonlinear method of approximation by trigonometric polynomials in this paper. For a periodic function f we take as an approximant a trigonometric polynomial of the form $G_m(f) := \sum_{k\in\Lambda} \hat{f}(k)e^{i(k,x)}$, where $\Lambda \subset \mathbb{Z}^d$ is a set of cardinality m containing the indices of the m biggest (in absolute value) Fourier coefficients $\hat{f}(k)$ of function f. Note that $G_m(f)$ gives the best m-term approximant in the L_2-norm and, therefore, for each $f \in L_2$, $\|f - G_m(f)\|_2 \to 0$ as $m \to \infty$. It is known from previous results that in the case of $p \neq 2$ the condition $f \in L_p$ does not guarantee the convergence $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$. We study the following question. What conditions (in addition to $f \in L_p$) provide the convergence $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$? In our previous paper [10] in the case $2 < p \leq \infty$ we have found necessary and sufficient conditions on a decreasing sequence $\{A_n\}_{n=1}^\infty$ to guarantee the L_p-convergence of $\{G_m(f)\}$ for all $f \in L_p$, satisfying $a_n(f) \leq A_n$, where $\{a_n(f)\}$ is a decreasing rearrangement of absolute values of the Fourier coefficients of f. In this paper we are looking for necessary and sufficient conditions on a sequence $\{M(m)\}$ such that the conditions $f \in L_p$ and $\|G_{M(m)}(f) - G_m(f)\|_p \to 0$ as $m \to \infty$ imply $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$. We have found these conditions in the case p an even number or $p = \infty$.

1. Introduction

We study in this paper the following nonlinear method of summation of trigonometric Fourier series. Consider a periodic function $f \in L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, $(L_\infty(\mathbb{T}^d) = C(\mathbb{T}^d))$, defined on the d-dimensional torus $\mathbb{T}^d$. Let a number $m \subset n\mathbb{N}$ be given and Λ_m be a set of $k \in \mathbb{Z}^d$ with the properties:

$$\min_{k\in\Lambda_m} |\hat{f}(k)| \geq \max_{k\notin\Lambda_m} |\hat{f}(k)|, \quad |\Lambda_m| = m, \tag{1.1}$$

where

$$\hat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)e^{-i(k,x)}dx$$

is a Fourier coefficient of f. We define

$$G_m(f) := S_{\Lambda_m}(f) := \sum_{k\in\Lambda_m} \hat{f}(k)e^{i(k,x)}$$

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and call it a m-th greedy approximant of f with regard to the trigonometric system $\mathcal{T} := \{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$. Clearly, a m-th greedy approximant may not be unique. In this paper we do not impose any extra restrictions on Λ_m in addition to (1.1). Thus theorems formulated below hold for any choice of Λ_m satisfying (1.1) or in other words for any realization $G_m(f)$ of the greedy approximation.

There has recently been (see surveys [4], [12], [9]) much interest in approximation of functions by m-term approximants with regard to a basis (or minimal system). We will discuss in detail only results concerning the trigonometric system. T.W. Körner answering a question raised by Carleson and Coifman constructed in [6] a function from $L_2(\mathbb{T})$ and then in [7] a continuous function such that $\{G_m(f)\}$ diverges almost everywhere. It has been proved in [11] for $p \neq 2$ and in [3] for $p < 2$ that there exists a $f \in L_p(\mathbb{T})$ such that $\{G_m(f)\}$ does not converge in L_p. It was remarked in [12] that the method from [11] gives a little more: 1) There exists a continuous function f such that $\{G_m(f)\}$ does not converge in $L_p(\mathbb{T})$ for any $p > 2$; 2) There exists a function f that belongs to any $L_p(\mathbb{T})$, $p < 2$, such that $\{G_m(f)\}$ does not converge in measure. Thus the above negative results show that the condition $f \in L_p(\mathbb{T}^d)$, $p \neq 2$, does not guarantee convergence of $\{G_m(f)\}$ in the L_p-norm. The main goal of this paper is to find an additional (to $f \in L_p$) condition on f to guarantee that $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$. Some results in this direction have already been obtained in [10]. In the case $2 < p \leq \infty$ we found in [10] necessary and sufficient conditions on a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ to guarantee the L_p-convergence of $\{G_m(f)\}$ for all $f \in L_p$, satisfying $a_n(f) \leq A_n$, where $\{a_n(f)\}$ is a decreasing rearrangement of absolute values of the Fourier coefficients of f. We will formulate three theorems from [10].

For $f \in L_1(\mathbb{T}^d)$ let $\{\hat{f}(k(l))\}_{l=1}^{\infty}$ denote the decreasing rearrangement of $\{\hat{f}(k)\}_{k\in\mathbb{Z}^d}$, i.e.

$$|\hat{f}(k(1))| \geq |\hat{f}(k(2))| \geq \dots \quad . \tag{1.2}$$

Denote $a_n(f) := |\hat{f}(k(n))|$.

Theorem 1 [10]. *Let* $2 < p < \infty$ *and let a decreasing sequence* $\{A_n\}_{n=1}^{\infty}$ *satisfy the condition:*

$$A_n = o(n^{1/p-1}) \quad as \quad n \to \infty. \tag{1.3}$$

Then for any $f \in L_p(\mathbb{T}^d)$ *with the property* $a_n(f) \leq A_n$, $n = 1, 2, \dots,$ *we have*

$$\lim_{m\to\infty} \|f - G_m(f)\|_p = 0. \tag{1.4}$$

We also proved in [10] that for any decreasing sequence $\{A_n\}$, satisfying

$$\limsup_{n\to\infty} A_n n^{1-1/p} > 0$$

there exists a function $f \in L_p$ such that $a_n(f) \leq A_n$, $n = 1, \dots,$ with divergent in the L_p sequence of greedy approximants $\{G_m(f)\}$.

Theorem 2 [10]. *Let a decreasing sequence* $\{A_n\}_{n=1}^\infty$ *satisfy the condition* $(\mathcal{A}_\infty)$*:*

$$\sum_{M<n\le e^M} A_n = o(1) \quad as \quad M\to\infty. \tag{1.5}$$

Then for any $f\in C(\mathbb{T})$ *with the property* $a_n(f)\le A_n$, $n=1,2,\dots$, *we have*

$$\lim_{m\to\infty}\|f-G_m(f)\|_\infty = 0. \tag{1.6}$$

The following theorem shows that the condition $(\mathcal{A}_\infty)$ in Theorem 2 is sharp.

Theorem 3 [10]. *Assume that a decreasing sequence* $\{A_n\}_{n=1}^\infty$ *does not satisfy the condition* $(\mathcal{A}_\infty)$*. Then there exists a function* $f\in C(\mathbb{T})$ *with the property* $a_n(f)\le A_n$, $n=1,2,\dots$, *and such that we have*

$$\limsup_{m\to\infty}\|f-G_m(f)\|_\infty > 0$$

for some realization $G_m(f)$*.*

In this paper we concentrate on imposing extra conditions in the following form. We assume that for some sequence $\{M(m)\}$, $M(m)>m$, we have

$$\|G_{M(m)}(f)-G_m(f)\|_p\to 0 \quad \text{as} \quad m\to\infty. \tag{1.7}$$

This exrta assumption on f is in a style of A.S. Belov [2]. He studied convergence of Fourier series in L_p with $p=1,\infty$ and imposed extra conditions on f in the form $\|S_{2n}(f)-S_n(f)\|_p = o(1)$. In the case p is an even number or $p=\infty$ we find necessary and sufficient conditions on the growth of the sequence $\{M(m)\}$ to provide convergence $\|f-G_m(f)\|_p\to 0$ as $m\to\infty$. We prove the following theorem in Section 3 (see Theorem 3.2).

Theorem 4. *Let* $p=2q$, $q\in\mathbb{N}$, *be an even integer,* $\delta>0$*. Assume that* $f\in L_p(\mathbb{T})$ *and there exists a sequence of positive integer* $M(m)>m^{1+\delta}$ *such that*

$$\|G_{M(m)}(f)-G_m(f)\|_p\to 0 \quad as \quad m\to\infty.$$

Then we have

$$\|f-G_m(f)\|_p\to 0 \quad as \quad m\to\infty.$$

In Section 4 we prove that the condition $M(m)>m^{1+\delta}$ cannot be replaced by a condition $M(m)>m^{1+o(1)}$. The following theorem is a direct corollary of Theorem 4.1.

Theorem 5. *For any* $p\in(2,\infty)$ *there exists a function* $f\in L_p(\mathbb{T})$ *with divergent in the* $L_p(\mathbb{T})$ *sequence* $\{G_m(f)\}$ *of greedy approximations with the following property. For any sequence* $\{M(m)\}$ *such that* $m\le M(m)\le m^{1+o(1)}$ *we have*

$$\|G_{M(m)}(f)-G_m(f)\|_p\to 0 \quad (m\to 0).$$

In Section 5 we discuss the case $p=\infty$. We prove there necessary and sufficient conditions for convergence of greedy approximations in the uniform norm. For a mapping $\alpha: W\to W$ we denote α_k its k-fold iteration: $\alpha_k := \alpha\circ\alpha_{k-1}$.

Theorem 6. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then the following conditions are equivalent:*
a) for some $k \in \mathbb{N}$ and for any sufficiently large $m \in \mathbb{N}$ we have $\alpha_k(m) > e^m$;
b) if $f \in C(\mathbb{T})$ and

$$\left\|G_{\alpha(m)}(f) - G_m(f)\right\|_\infty \to 0 \quad (m \to \infty)$$

then

$$\|f - G_m(f)\|_\infty \to 0 \quad (m \to \infty).$$

The proof of necessary condition is based on the above Theorem 3 from [10]. In the proof of sufficient condition we use the following special inequality (see Theorem 2.1 in Section 2).

By $\Sigma_m(\mathcal{T})$ we denote the set of all trigonometric polynomials with at most m nonzero coefficients.

Theorem 7. *For any $h \in \Sigma_m(\mathcal{T})$ and any $g \in L_\infty$ one has*

$$\|h + g\|_\infty \geq K^{-2}\|h\|_\infty - e^{C(K)m}\|\{\hat{g}(k)\}\|_{\ell_\infty}, \quad K > 1. \tag{1.8}$$

We note that in the proof of the above inequality we use a deep result on the uniform approximation property of the space $C(X)$ (see [5]). Section 2 contains some other inequalities in the style of (1.8).

Greedy approximations are close to thresholding approximations (thresholding greedy approximations). Thresholding approximations are defined as follows

$$T_\varepsilon(f) := S_{\Lambda(\varepsilon)}(f) := \sum_{k:|\hat{f}(k)|\geq\varepsilon} \hat{f}(k)e^{i(k,x)}, \quad \varepsilon > 0.$$

Clearly, for any $\varepsilon > 0$ there exists an $m(\varepsilon)$ such that $T_\varepsilon(f) = G_{m(\varepsilon)}(f)$. Therefore, convergence of $\{G_m(f)\}$ as $m \to \infty$ implies convergence of $\{T_\varepsilon(f)\}$ as $\varepsilon \to 0$. In Sections 3–5 we obtain results on convergence of $\{T_\varepsilon(f)\}$, $\varepsilon \to 0$, that are similar to the above mentioned results on convergence of $\{G_m(f)\}$.

We use the same notations in both cases $d = 1$ and $d > 1$. We point out that in Sections 2,3 we consider the general case $d \geq 1$ and in Sections 4,5 we confine ourselves to the case $d = 1$. The reason for that is that we prove necessary conditions in Section 4 and in a part of Section 5, where, clearly, we consider the case $d = 1$ without loss of generality. We note that sufficient conditions in Theorems 5.1 and 5.2 also hold for $d > 1$ (the proof is the same with natural modifications).

2. Some inequalities

In this section we prove some inequalities that will be used in the paper. The general style of these inequalities is the following. A function that has a sparse representation with regard to the trigonometric system cannot be approximated in L_p by functions with small

Fourier coefficients. We begin our discussion with some concepts that are useful in proving such inequalities.

The following new characteristic of a Banach space L_p plays an important role in such inequalities. We introduce some more notations. Let Λ be a finite subset of $\mathbb{Z}^d$. By $|\Lambda|$ we denote its cardinality and by $\mathcal{T}(\Lambda)$ the span of $\{e^{i(k,x)}\}_{k\in\Lambda}$. It is clear that

$$\Sigma_m(\mathcal{T}) = \cup_{\Lambda:|\Lambda|\le m}\mathcal{T}(\Lambda).$$

For $f \in L_p$, $F \in L_{p'}$, $1 \le p \le \infty$, $p' = p/(p-1)$, we write

$$\langle F, f\rangle := \int_{\mathbb{T}^d} F\bar{f}d\mu, \quad d\mu := (2\pi)^{-d}dx.$$

Definition 2.1. *Let Λ be a finite subset of $\mathbb{Z}^d$ and $1 \le p \le \infty$. We call a set $\Lambda' := \Lambda'(p,\gamma)$, $\gamma \in (0,1]$ a (p,γ)-dual to Λ if for any $f \in \mathcal{T}(\Lambda)$ there exists $F \in \mathcal{T}(\Lambda')$ such that $\|F\|_{p'} = 1$ and $\langle F, f\rangle \ge \gamma\|f\|_p$.*

Denote by $D(\Lambda, p, \gamma)$ the set of all (p,γ)-dual sets Λ'. The following function is important for us

$$v(m,p,\gamma) := \sup_{\Lambda:|\Lambda|=m}\ \inf_{\Lambda'\in D(\Lambda,p,\gamma)} |\Lambda'|.$$

We note that in a particular case $p = 2q$, $q \in \mathbb{N}$ we have

$$v(m,p,1) \le m^{p-1}. \tag{2.1}$$

This follows immediately from the form of the norming functional F for $f \in L_p$:

$$F = f^{q-1}(\bar{f})^q\|f\|_p^{1-p}. \tag{2.2}$$

We will use the quantity $v(m,p,\gamma)$ in greedy approximation. We first prove a lemma.

Lemma 2.1. *Let $2 \le p \le \infty$. For any $h \in \Sigma_m(\mathcal{T})$ and any $g \in L_p$ one has*

$$\|h+g\|_p \ge \gamma\|h\|_p - v(m,p,\gamma)^{1-1/p}\|\{\hat{g}(k)\}\|_{\ell_\infty}.$$

Proof. Let $h \in \mathcal{T}(\Lambda)$ with $|\Lambda| = m$ and let $\Lambda' \in D(\Lambda,p,\gamma)$. Then using the Definition 2.1 we find $F(h,\gamma) \in \mathcal{T}(\Lambda')$ such that

$$\|F(h,\gamma)\|_{p'} = 1 \quad \text{and} \quad \langle F(h,\gamma), h\rangle \ge \gamma\|h\|_p.$$

We have

$$\langle F(h,\gamma), h\rangle = \langle F(h,\gamma), h+g\rangle - \langle F(h,\gamma), g\rangle \le \|h+g\|_p + |\langle F(h,\gamma), g\rangle|.$$

Next,

$$|\langle F(h,\gamma), g\rangle| \le \|\{\hat{F}(h,\gamma)(k)\}\|_{\ell_1}\|\{\hat{g}(k)\}\|_{\ell_\infty}.$$

Using $F(h,\gamma) \in \mathcal{T}(\Lambda')$ and the Hausdorf-Young theorem [14,Chap.12,Section 2] we obtain

$$\|\{\hat{F}(h,\gamma)(k)\}\|_{\ell_1} \le |\Lambda'|^{1-1/p}\|\{\hat{F}(h,\gamma)(k)\}\|_{\ell_p} \le |\Lambda'|^{1-1/p}\|F(h,\gamma)\|_{p'} = |\Lambda'|^{1-1/p}.$$

It remains to combine the above inequalities and use the definition of $v(m,p,\gamma)$.

Definition 2.2. *Let X be a finite dimensional subspace of L_p, $1 \le p \le \infty$. We call a subspace $Y \subset L_{p'}$ a (p,γ)-dual to X, $\gamma \in (0,1]$, if for any $f \in X$ there exists $F \in Y$ such that $\|F\|_{p'} = 1$ and $\langle F, f\rangle \ge \gamma \|f\|_p$.*

Similarly to the above we denote by $D(X,p,\gamma)$ the set of all (p,γ)-dual subspaces Y. Consider the following function

$$w(m,p,\gamma) := \sup_{X:\dim X = m} \inf_{Y \in D(X,p,\gamma)} \dim Y.$$

We begin our discussion by a particular case $p = 2q$, $q \in \mathbb{N}$. Let X be given and $e_1, \dots, e_m$ form a basis of X. Using the Hölder inequality for n functions $f_1, \dots, f_n \in L_n$

$$\int |f_1 \cdots f_n| d\mu \le \|f_1\|_n \cdots \|f_n\|_n$$

with $f_i = |e_j|^{p'}$, $n = p-1$ we get that any function of the form

$$\prod_{i=1}^m |e_i|^{k_i}, \quad k_i \in \mathbb{N}, \quad \sum_{i=1}^m k_i = p-1,$$

belongs to $L_{p'}$. It now follows from (2.2) that

$$w(m,p,1) \le m^{p-1}, \quad p = 2q, \quad q \in \mathbb{N}. \tag{2.3}$$

There is a general theory of uniform approximation property (UAP) that provides some estimates for $w(m,p,\gamma)$. We begin with some definitions from this theory. For a given subspace X of L_p, $\dim X = m$, and a constant $K > 1$ let $k_p(X,K)$ be the smallest k such that there is an operator $I_X : L_p \to L_p$ with $I_X(f) = f$ for $f \in X$, $\|I_X\|_{L_p \to L_p} \le K$, and $\operatorname{rank} I_X \le k$. Denote

$$k_p(m,K) := \sup_{X:\dim X = m} k_p(X,K).$$

Let us discuss how $k_p(m,K)$ can be used in estimating $w(m,p,\gamma)$. Consider I_X^* the dual to I_X operator. Then $\|I_X^*\|_{L_{p'} \to L_{p'}} \le K$ and $\operatorname{rank} I_X^* \le k_p(m,K)$. Let $f \in X$, $\dim X = m$, and let F_f be the norming functional for f. Define

$$F := I_X^*(F_f)/\|I_X^*(F_f)\|_{p'}.$$

Then $(f \in X)$

$$\langle f, I_X^*(F_f)\rangle = \langle I_X(f), F_f\rangle = \langle f, F_f\rangle = \|f\|_p$$

and

$$\|I_X^*(F_f)\|_{p'} \le K$$

imply

$$\langle f, F\rangle \ge K^{-1}\|f\|_p.$$

Therefore

$$w(m,p,K^{-1}) \le k_p(m,K). \tag{2.4}$$

We note that the behavior of functions $w(m,p,\gamma)$ and $k_p(m,K)$ may be very different. J. Bourgain [1] proved that for any $p \in (1,\infty)$, $p \neq 2$ the function $k_p(m,K)$ grows faster than any polynomial in m. The estimate (2.3) shows that in the particular case $p = 2q$, $q \in \mathbb{N}$ the growth of $w(m,p,\gamma)$ is at most polynomial. This means that we cannot expect to obtain accurate estimates for $w(m,p,K^{-1})$ using the inequality (2.4). We give one more application of the UAP in the style of Lemma 2.1.

Lemma 2.2. *Let $2 \le p \le \infty$. For any $h \in \Sigma_m(\mathcal{T})$ and any $g \in L_p$ one has*

$$\|h+g\|_p \ge K^{-1}\|h\|_p - k_p(m,K)^{1/2}\|g\|_2; \tag{2.5}$$

$$\|h+g\|_p \ge K^{-2}\|h\|_p - k_p(m,K)\|\{\hat{g}(k)\}\|_{\ell_\infty}. \tag{2.6}$$

Proof. Let $h \in \mathcal{T}(\Lambda)$, $|\Lambda| = m$. Take $X = \mathcal{T}(\Lambda)$ and consider the operator I_X provided by the UAP. Let $\psi_1,\dots,\psi_M$ form an orthonormal basis for the range Y of the operator I_X. Then $M \le k_p(m,K)$. Let

$$I_X(e^{i(k,x)}) = \sum_{j=1}^{M} c_j^k \psi_j.$$

Then the property $\|I_X\|_{L_p\to L_p} \le K$ implies

$$(\sum_{j=1}^{M} |c_j^k|^2)^{1/2} = \|I_X(e^{i(k,x)})\|_2 \le \|I_X(e^{i(k,x)})\|_p \le K.$$

Consider along with the operator I_X a new one

$$A := (2\pi)^{-d}\int_{\mathbb{T}^d} T_t I_X T_{-t}dt$$

where T_t is a shifting operator: $T_t(f) = f(\cdot + t)$. Then

$$A(e^{i(k,x)}) = \sum_{j=1}^{M} c_j^k (2\pi)^{-d}\int_{\mathbb{T}^d} e^{-i(k,t)}\psi_j(x+t)dt = (\sum_{j=1}^{M} c_j^k \hat{\psi}_j(k))e^{i(k,x)}.$$

Denote

$$\lambda_k := \sum_{j=1}^{M} c_j^k \hat{\psi}_j(k).$$

We have

$$\sum_k |\lambda_k|^2 \le \sum_k (\sum_{j=1}^M |c_j^k|^2)(\sum_{j=1}^M |\hat\psi_j(k)|^2) \le K^2 M. \tag{2.7}$$

Also $\lambda_k = 1$ for $k \in \Lambda$. For the operator A we have

$$\|A\|_{L_p\to L_p} \le K \quad \text{and} \quad \|A\|_{L_2\to L_\infty} \le KM^{1/2}.$$

Therefore

$$\|A(h+g)\|_p \le K\|h+g\|_p$$

and

$$\|A(h+g)\|_p \ge \|h\|_p - KM^{1/2}\|g\|_2.$$

This proves inequality (2.5).

Consider the operator $B := A^2$. Then

$$B(h) = h, \quad h \in \mathcal{T}(\Lambda); \quad B(e^{i(k,x)}) = \lambda_k^2 e^{i(k,x)}, \quad k \in \mathbb{Z}^d; \quad \|B\|_{L_p\to L_p} \le K^2$$

and, by (2.7),

$$\|B(f)\|_\infty \le \sum_k |\lambda_k|^2 \|\{\hat f(k)\}\|_{\ell_\infty} \le K^2 M \|\{\hat f(k)\}\|_{\ell_\infty}.$$

Now, on the one hand

$$\|B(h+g)\|_p \le K^2\|h+g\|_p$$

and on the other hand

$$\|B(h+g)\|_p = \|h + B(g)\|_p \ge \|h\|_p - K^2 M\|\{\hat g(k)\}\|_{\ell_\infty}.$$

This proves inequality (2.6).

Theorem 2.1. *For any $h \in \Sigma_m(\mathcal{T})$ and any $g \in L_\infty$ one has*

$$\|h+g\|_\infty \ge K^{-1}\|h\|_\infty - e^{C(K)m/2}\|g\|_2;$$

$$\|h+g\|_\infty \ge K^{-2}\|h\|_\infty - e^{C(K)m}\|\{\hat g(k)\}\|_{\ell_\infty}.$$

Proof. This theorem is a direct corollary of Lemma 2.2 and the following known (see [5]) estimate

$$k_\infty(m,K) \le e^{C(K)m}.$$

As we already mentioned $k_p(m,K)$ increases faster than any polynomial. We will improve inequality (2.5) in the case $p < \infty$ by using other arguments.

Lemma 2.3. *Let $2 \le p < \infty$. For any $h \in \Sigma_m(\mathcal{T})$ and any $g \in L_p$ one has*

$$\|h+g\|_p^p \ge \|h\|_p^p - pm^{(p-2)/4}\|h\|_p^{p-1}\|g\|_2.$$

Proof. Since the function $f(x) = |x|^p$ is convex, we have $f(x-y) \ge f(x) - yf'(x)$. Therefore,

$$|h+g|^p \ge |h|^p - p|h|^{p-1}|g|. \tag{2.8}$$

Taking the integral of (2.8) over $\mathbb{T}^d$ with respect to the measure μ with $d\mu := (2\pi)^{-d}dx$ we get

$$\int_{\mathbb{T}^d} |h+g|^p d\mu \ge \int_{\mathbb{T}^d} |h|^p d\mu - p\int_{\mathbb{T}^d} |h|^{p-1}|g|d\mu. \tag{2.9}$$

Next, by Cauchy's inequality,

$$\int_{\mathbb{T}^d} |h|^{p-1}|g|d\mu \le \left(\int_{\mathbb{T}^d} |h|^{2p-2}d\mu \int_{\mathbb{T}^d} |g|^2 d\mu\right)^{1/2}$$

$$\le \|g\|_2 \left(\int_{\mathbb{T}^d} |h|^p \|h\|_\infty^{p-2} d\mu\right)^{1/2} = \|g\|_2 \|h\|_p^{p/2} \|h\|_\infty^{(p-2)/2}. \tag{2.10}$$

Using Cauchy's inequality again, we obtain

$$\|h\|_\infty \le m^{1/2}\|h\|_2 \le m^{1/2}\|h\|_p. \tag{2.11}$$

Combining (2.9)—(2.11) we complete the proof of Lemma 2.3.

We will mention some known inequalities in a style of inequalities in Lemmas 2.1–2.3.

Lemma 2.4 [10]. *Let $2 \le p < \infty$ and $h \in L_p$, $\|h\|_p \ne 0$. Then for any $g \in L_p$ we have*

$$\|h\|_p \le \|h+g\|_p + (\|h\|_{2p-2}/\|h\|_p)^{p-1}\|g\|_2.$$

Lemma 2.5 [10]. *Let $h \in \Sigma_m(\mathcal{T})$, $\|h\|_\infty = 1$. Then for any function g such that $\|g\|_2 \le \frac{1}{4}(4\pi m)^{-m/2}$ we have*

$$\|h+g\|_\infty \ge 1/4.$$

We proceed to estimating $v(m,p,\gamma)$ for $p \in [2,\infty)$. In the special case of even p we have by (2.1) that

$$v(m,p,1) \le m^{p-1}.$$

Lemma 2.6. *Let $2 \le p < \infty$. Denote $\alpha := p/2 - [p/2]$. Then we have*

$$v(m, p, \gamma) \le m^{c(\alpha,\gamma)m^{1/2}+p-1}.$$

Proof. In the case p an even number the statement follows from (2.1). We will assume that p is not an even number. Let $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| = m$ be given. Take any nonzero $h \in \mathcal{T}(\Lambda)$ and assume for convenience that $\|h\|_p = 1$. We wil l construct a γ-norming functional $F(h, \gamma)$ ($\langle F, h\rangle \ge \gamma \|h\|_p$). We use the formula for the norming functional of h

$$F = \|h\|_p^{1-p} \bar{h}|h|^{p-2} = \bar{h}(|h|^2)^{p/2-1} = \bar{h}(|h|^2)^{[p/2]-1}(|h|^2)^{\alpha}.$$

By (2.11), we have

$$\|h\|_\infty \le m^{1/2}.$$

The idea is to replace $(|h|^2)^\alpha$ by an algebraic polynomial on $|h|^2$. We approximate the function x^α on the interval $[0, m]$. We use the Telyakovskii's result [13]: there exists an algebraic polynomial of degree n such that

$$|y^\alpha - P_n(y)| \le C_1(\alpha)(y^{1/2}/n)^\alpha, \quad y \in [0, 1]. \tag{2.12}$$

Substituting $y = x/m$ into (2.12) we get

$$|x^\alpha - m^\alpha P_n(x/m)| \le C_1(\alpha) x^{\alpha/2} m^{\alpha/2} n^{-\alpha}.$$

We take $\theta = \frac{1-\gamma}{1+\gamma} \in (0, 1)$ and choose $n(m) \le C_2(\alpha, \gamma)m^{1/2}$ with $C_2(\alpha, \gamma)$ big enough to have

$$C_1(\alpha)x^{\alpha/2}m^{\alpha/2}n^{-\alpha} \le \theta x^{\alpha/2}.$$

Denote

$$F_m := m^\alpha P_{n(m)}(|h|^2/m)\bar{h}(|h|^2)^{[p/2]-1}.$$

Then ($x = |h|^2$)

$$|F - F_m| \le \theta |h|^{2[p/2]-1+\alpha}.$$

Therefore,

$$\|F - F_m\|_{p'} \le \theta \| |h|^{2[p/2]-1+\alpha} \|_{p'}. \tag{2.13}$$

Using $2[p/2] = p - 2\alpha$ we get

$$\| |h|^{p-1-\alpha} \|_{p'} \le \| |h|^{p-\alpha-1} \|_{(p-\alpha)'} = \|h\|_{p-\alpha}^{p-\alpha-1} \le \|h\|_p^{p-\alpha-1} = 1. \tag{2.14}$$

Combining (2.13) and (2.14) we get

$$\|F - F_m\|_{p'} \le \theta.$$

This implies that

$$\|F_m\|_{p'} \le 1 + \theta$$

and

$$\langle F_m, h\rangle = \langle F, h\rangle + \langle F_m - F, h\rangle \ge \|h\|_p - \theta\|h\|_p = (1-\theta)\|h\|_p.$$

Thus $F(h, \gamma) := F_m/\|F_m\|_{p'}$ is a γ-norming functional for h. It remains to note that the dimension of a subspace $\mathcal{T}(\Lambda')$ containing all $P_{n(m)}(|h|^2/m)\bar{h}(|h|^2)^{[p/2]-1}$ when h runs over $\mathcal{T}(\Lambda)$ does not exceed $m^{c(\alpha,\gamma)m^{1/2}+p-1}$.

3. Sufficient conditions in the case $p \in (2, \infty)$

We will prove now several statements which give sufficient conditions for convergence of greedy approximation in L_p, $2 < p < \infty$.

Theorem 3.1. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer. For $f \in L_p(\mathbb{T}^d)$ assume that two sequences Λ_m and Y_m of sets of frequences satisfy the following conditions*

$$|\Lambda_m| \le m^a, \quad a > 0, \tag{3.1}$$

$$\sup_{k \notin Y_m} |\hat{f}(k)| = o(m^{a(1-p)}), \tag{3.2}$$

$$\|S_{\Lambda_m}(f) - S_{Y_m}(f)\|_p \to 0 \quad as \quad m \to \infty.$$

Then we have

$$\|S_{\Lambda_m}(f) - f\|_p \to 0 \quad as \quad m \to \infty.$$

Proof. We use the M. Riesz theorem [8, Chap. 4, Section 3] that for all $1 < p < \infty$ we have the convergence $\|f - S_N(f)\|_p \to 0$ as $N \to \infty$, where

$$S_N(f) := \sum_{k \in K(N)} \hat{f}(k) e^{i(k,x)}, \quad K(N) := \{k : \max_j |k_j| \le N^{1/d}\}.$$

Let

$$\varepsilon_m := \sup_{k \notin Y_m} |\hat{f}(k)|, \quad N = [m^{ap}].$$

We estimate

$$\|S_N(f) - S_{Y_m}(f)\|_p \le \tag{3.3}$$

$$\le \|\sum_{k:|k|\le N; k \notin Y_m} \hat{f}(k) e^{i(k,x)}\|_p + \|\sum_{k:|k|>N; k \in Y_m} \hat{f}(k) e^{i(k,x)}\|_p =: \|\Sigma_1\|_p + \|\Sigma_2\|_p.$$

We have by the Paley theorem [14, Chap. 12,Section 5] that

$$\|\Sigma_1\|_p = O(\varepsilon_m N^{1-1/p}) = o(1).$$

For the second sum we have

$$\Sigma_2 = f - S_N(f) - g \quad \text{with} \quad g := \sum_{k:|k|>N; k \notin Y_m} \hat{f}(k) e^{i(k,x)}. \tag{3.4}$$

Let us rewrite

$$\Sigma_2 = (Id - S_N)(S_{Y_m}(f)) = \tag{3.5}$$

$$= (Id - S_N)(S_{\Lambda_m}(f)) + (Id - S_N)(S_{Y_m}(f) - S_{\Lambda_m}(f)) =: h_1 + h_2.$$

By the theorem's assumption and the M. Riesz theorem we get $\|h_2\|_p = o(1)$ and, therefore, we get from (3.4) and (3.5) that $\|h_1 + g\|_p = o(1)$. We note that h_1 is a polynomial with at most m terms and g is a function with small Fourier coefficients. We have the following lemma for this situation.

Lemma 3.1. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer. Assume that h is an m-term trigonometric polynomial and g is such that $|\hat{g}(k)| \le \varepsilon$ for all k. Then*

$$\|h\|_p \le \|h + g\|_p + m^{p-1}\varepsilon.$$

Proof. This lemma follows from Lemma 2.1 and the estimate (2.1).

Applying Lemma 3.1 we get for h_1 that $\|h_1\|_p = o(1)$ and, therefore, $\|\Sigma_2\|_p = o(1)$. This implies in turn (see (3.3)) that

$$\|S_N(f) - S_{Y_m}(f)\|_p = o(1).$$

Thus we get $\|f - S_{\Lambda_m}(f)\|_p \to 0$ as $m \to \infty$. The proof of Theorem 3.1 is complete.

We now formulate a straightforward corollary of Theorem 3.1. Let us note first that convergence of $\{G_m(f)\}$ in L_p is equivalent to

$$\|G_m(f) - G_n(f)\|_p \to 0 \quad \text{as} \quad m, n \to \infty.$$

Corollary 3.1. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer. For $f \in L_p(\mathbb{T}^d)$ assume that there exists a sequence $\{\varepsilon_m\}$, $\varepsilon_m = o(m^{1-p})$, such that*

$$\|G_m(f) - T_{\varepsilon_m}(f)\|_p = o(1).$$

Then

$$\|G_m(f) - f\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

We now present some results in the direction of weakening the assumption $\varepsilon_m = o(m^{1-p})$ in Corollary 3.1.

Theorem 3.2. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer, $\delta > 0$. Assume that $f \in L_p(\mathbb{T}^d)$ and there exists a sequence of positive integers $M(m) > m^{1+\delta}$ such that*

$$\|G_m(f) - G_{M(m)}(f)\|_p \to 0 \quad \text{as} \quad m \to \infty. \tag{3.6}$$

Then we have

$$\|G_m(f) - f\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

Proof. Let $m_0 := m$, $m_j := M(m_{j-1})$ for $j \in \mathbb{N}$. We have $m_j > m^{(1+\delta)^j}$. Fix $j_0 > \log(2p)/\log(1+\delta)$. Let $M_0(m) := m_{j_0}$. We have $M_0(m) > m^{2p}$. Also, by (3.6),

$$\|G_m(f) - G_{M_0(m)}(f)\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

Let Λ_m and Y_m be defined from $G_m(f) = S_{\Lambda_m}(f)$ and $G_{M_0(m)}(f) = S_{Y_m}(f)$. Using that $a_{M_0(m)}(f) = O(M_0(m)^{-1/2}) = O(m^{-p}) = o(m^{1-p})$, we complete the proof of Theorem 3.2 by Theorem 3.1.

Theorem 3.3. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer, $\delta > 0$. Assume that $f \in L_p(\mathbb{T}^d)$ and for any $\varepsilon > 0$ there is an $\eta(\varepsilon) < \varepsilon^{1+\delta}$ such that*

$$\|T_\varepsilon(f) - T_{\eta(\varepsilon)}(f)\|_p \to 0 \quad as \quad \varepsilon \to 0. \tag{3.7}$$

Then we have

$$\|T_\varepsilon(f) - f\|_p \to 0 \quad as \quad \varepsilon \to 0.$$

To prove this theorem we need the following simple lemma.

Lemma 3.2. *Let $p \ge 2$ and $\delta > 0$, For any $f \in L_p(\mathbb{T}^d)$ there is an $\varepsilon_{f,p} > 0$ with the following property. For any $\varepsilon \in (0, \varepsilon_{f,p})$ there exists an $m(\varepsilon)$ such that $\varepsilon^{-p/(p-1)+\delta} < m(\varepsilon) < \varepsilon^{-2}$ and*

$$\|G_{m(\varepsilon)}(f) - T_\varepsilon(f)\|_p \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. We have $G_{m_1(\varepsilon)}(f) = S_{\Lambda(\varepsilon)}(f)$ for $m_1(\varepsilon) = |\Lambda(\varepsilon)|$. Moreover, the condition $f \in L_2(\mathbb{T}^d)$ implies $m_1(\varepsilon) = o(\varepsilon^{-2})$. If $m_1(\varepsilon) > \varepsilon^{-p'+\delta}$, where $p' = p/(p-1)$, then we put $m(\varepsilon) = m_1(\varepsilon)$. Suppose that $m_1 \le \varepsilon^{-p'+\delta}$. Let $m_2(\varepsilon) = [\varepsilon^{-p'+\delta}]$, $m(\varepsilon) = m_1(\varepsilon) + m_2(\varepsilon)$. By the Hausdorff-Young theorem we have

$$\|G_{m(\varepsilon)}(f) - G_{m_1(\varepsilon)}(f)\|_p \le m_2(\varepsilon)^{1/p'}\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and, moreover, $\varepsilon^{-p/(p-1)+\delta} < m(\varepsilon) < \varepsilon^{-2}$ for small ε. This proves the lemma.

Proof of Theorem 3.3. By Lemma 3.2 we find $m(\varepsilon)$ satisfying $\varepsilon^{-p'+\delta} < m(\varepsilon) < \varepsilon^{-2}$ and

$$\|G_{m(\varepsilon)}(f) - T_\varepsilon(f)\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proceeding as in the proof of Theorem 3.2, for any $\varepsilon > 0$ we get the $\eta(\varepsilon) < \varepsilon^{2p} < m(\varepsilon)^{-p}$ such that

$$\|T_\varepsilon(f) - T_{\eta(\varepsilon)}(f)\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.8}$$

We now apply Theorem 3.1 with $\Lambda_{m(\varepsilon)}$ and $Y_{m(\varepsilon)}$ defined from

$$G_{m(\varepsilon)}(f) = S_{\Lambda_{m(\varepsilon)}}(f); \quad T_{\eta(\varepsilon)}(f) = S_{Y_{m(\varepsilon)}}(f).$$

The proof of Theorem 3.3 is complete.

Theorem 3.4. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer, $\delta > 0$. Assume that $f \in L_p(\mathbb{T}^d)$ and for any positive integer m there exists an $\varepsilon(m) < m^{1/p-1-\delta}$ such that*

$$\|G_m(f) - T_{\varepsilon(m)}(f)\|_p \to 0 \quad as \quad m \to \infty.$$

Then we have

$$\|G_m(f) - f\|_p \to 0 \quad as \quad m \to \infty.$$

Proof. It is clear that it suffices to prove the theorem for small δ. Let $0 < \delta < p' - 1/p'$. Applying Lemma 3.2 with $\varepsilon = \varepsilon(m)$ we get the existence of $M(m) > m^{1+\delta'}$ with some $\delta' > 0$ such that

$$\|G_{M(m)}(f) - G_m(f)\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

It remains to use Theorem 3.2.

4. Necessary conditions in the case $p \in (2, \infty)$

Theorem 4.1. *For any $p > 2$ there exists a function $f \in L_p(\mathbb{T})$ such that*
1) if two sequences $\{\Lambda_j\}$ and $\{Y_j\}$ of sets of frequencies satisfy the conditions

$$\sup_{k \notin \Lambda_j} |\hat{f}(k)| \le \varepsilon_j := \inf_{k \in \Lambda_j} |\hat{f}(k)|,$$

$$\sup_{k \notin Y_j} |\hat{f}(k)| \le \delta_j := \inf_{k \in Y_j} |\hat{f}(k)|,$$

$$\Lambda_j \subset Y_j$$

and either

$$|Y_j| = |\Lambda_j|^{1+o(1)} \quad (j \to \infty)$$

or

$$\delta_j = \varepsilon_j^{1+o(1)} \quad (j \to \infty),$$

then

$$\|S_{\Lambda_j}(f) - S_{Y_j}(f)\|_p \to 0 \quad (j \to \infty);$$

2) $\liminf_{\varepsilon \to 0} \|f - \sum_{k:\, |\hat{f}(k)| \ge \varepsilon} \hat{f}(k) e^{ikx}\|_p > 0.$

Let M be a sufficiently large positive integer, η_k $(1 \le k \le M)$ be independent random variables such that each η_k takes value n, $1 \le n \le M$, with probability $1/M$. We will use the following probabilistic inequality.

Lemma 4.1. *There is a constant $C_1 = C_1(p)$ such that for any function $g : \{1, \dots, M\} \to \mathbb{R}$ with $\sum_{n=1}^{M} g(n) = 0$, independent random variables $\xi_k = g(\eta_k)$, and complex numbers $z_1, \dots, z_M$, with $|z_k| \le 1$, $(k = 1, \dots, M)$ we have*

$$\mathbf{E}\left(\left|\sum_{k=1}^{M} \xi_k z_k\right|^p\right) \le C_1 M^{p/2} \left(\mathbf{E}(\xi_1^2)\right)^{p/2}.$$

Proof. First assume that the numbers $z_1, \dots, z_M$ are real. We observe that $\mathbf{E}(\xi_k) = 0$ for $k = 1, \dots, M$. By Rosenthal's inequality, we have

$$\begin{aligned} \mathbf{E}\left(\left|\sum_{k=1}^{M} \xi_k z_k\right|^p\right) &\le C(p) \left(\sum_{k=1}^{M} |z_k|^p \mathbf{E}(|\xi_1|^p) + \left(\sum_{k=1}^{M} z_k^2 \mathbf{E}(\xi_1^2)\right)^{p/2}\right) \\ &\le C(p) \left(M \mathbf{E}(|\xi_1|^p) + M^{p/2} \left(\mathbf{E}(\xi_1^2)\right)^{p/2}\right). \end{aligned} \tag{4.1}$$

Further,

$$\mathbf{E}(|\xi_1|^p) = \frac{1}{M} \sum_{n=1}^{M} |g(n)|^p \le \frac{1}{M} \left(\sum_{n=1}^{M} g(n)^2\right)^{p/2} = M^{p/2-1} \left(\mathbf{E}(\xi_1^2)\right)^{p/2}.$$

After substitution of the last inequality into (4.1) we get

$$\mathbf{E}\left(\left|\sum_{k=1}^{M}\xi_k z_k\right|^p\right) \le 2C(p)M^{p/2}\left(\mathbf{E}(\xi_1^2)\right)^{p/2}.$$

Finally, if the numbers $z_1,\dots,z_M$ are complex then

$$\mathbf{E}\left(\left|\sum_{k=1}^{M}\xi_k z_k\right|^p\right) \le 2^p\mathbf{E}\left(\left|\sum_{k=1}^{M}\xi_k \Re z_k\right|^p\right) + 2^p\mathbf{E}\left(\left|\sum_{k=1}^{M}\xi_k \Im z_k\right|^p\right)$$
$$\le 2^{p+2}C(p)M^{p/2}\left(\mathbf{E}(\xi_1^2)\right)^{p/2},$$

and the lemma is proved.

We will need some properties of random trigonometric polynomials.

Lemma 4.2. *Let $b=(b_1,\dots,b_M)$ be real numbers such that $\sum_{k=1}^M b_k=0$. Then*

$$\mathbf{E}\|\sum_{k=1}^{M} b_{\eta_k}e^{ikx}\|_p^p \le C(p)\|b\|_{\ell_2}^p.$$

Proof. We use Lemma 4.1 with g: $g(n)=b_n$, $z_n=e^{inx}$, $n=1,\dots,M$. We get by Lemma 4.1 for each x

$$\mathbf{E}|\sum_{k=1}^{M} b_{\eta_k}e^{ikx}|^p \le C_1(p)M^{p/2}(\mathbf{E}(\xi_1^2))^{p/2}.$$

Therefore,

$$\mathbf{E}\|\sum_{k=1}^{M} b_{\eta_k}e^{ikx}\|_p^p = \|\mathbf{E}|\sum_{k=1}^{M} b_{\eta_k}e^{ikx}|^p\|_1 \le C_1(p)M^{p/2}(\mathbf{E}(\xi_1^2))^{p/2}.$$

We have

$$\mathbf{E}(\xi_1^2) = \frac{1}{M}\sum_{n=1}^{M} b_n^2 = \|b\|_{\ell_2}^2/M.$$

This completes the proof of Lemma 4.2.

For a given $a=(a_1,\dots,a_M)$ consider the following random polynomials

$$t_I^a(x) := \sum_{\eta_k\in I} a_{\eta_k}e^{ikx} - s_I D_M(x)/M$$

where $I\subseteq[1,M]$ is an interval and

$$s_I := \sum_{n\in I} a_n; \qquad D_M(x) := \sum_{k=1}^{M} e^{ikx}.$$

Below we use the notation log for logarithm with the base 2.

Lemma 4.3. *We have for any $A > 0$, $M \geq 8$,*

$$\mathbf{P}\{\max_{I\subseteq[1,M]} \|t_I^a\|_p \leq A^{1/p} 3\log M \|a\|_{\ell_2}\} \geq 1 - C_2(p)A^{-1}\log M.$$

Proof. First, by Lemma 4.2 with $b_n = a_n\chi_I(n) - s_I/M$, $n = 1, \dots, M$, we obtain

$$\mathbf{E}\|t_I^a\|_p^p \leq C(p)(\sum_{n=1}^{M} b_n^2)^{p/2}.$$

Next,

$$\sum_{n=1}^{M} b_n^2 \leq \sum_{n=1}^{M} 2((a_n\chi_I(n))^2 + (s_I/M)^2) = 2(\sum_{n\in I} a_n^2 + M(\sum_{n\in I} a_n)^2 M^{-2}) \leq 4\sum_{n\in I} a_n^2.$$

Hence,

$$\mathbf{E}\|t_I^a\|_p^p \leq 4C(p)(\sum_{n\in I} a_n^2)^{p/2}.$$

Denote $I(j,l) := (2^j l, 2^j(l+1)] \cap [1,M]$, $j = 0, \dots, J$, $l = 0, 1, \dots$ with $J := [\log M] + 1$. Then for any $j \in [0,J]$

$$\sum_{l=0}^{\infty} \mathbf{E}\|t_{I(j,l)}^a\|_p^p \leq 4C(p)\sum_{l=0}^{\infty}(\sum_{n\in I(j,l)} a_n^2)^{p/2} \leq 4C(p)\|a\|_{\ell_2}^p.$$

Using Markov's inequality: for any nonnegative random variable X, and $t > 0$

$$\mathbf{P}\{X \geq t\} \leq \mathbf{E}(X)/t$$

we get for each $j \in [0,J]$

$$\mathbf{P}\{\sum_{l=0}^{\infty} \|t_{I(j,l)}^a\|_p^p \geq A\|a\|_{\ell_2}^p\} \leq 4C(p)/A.$$

Since every interval $I \subseteq [1,M]$ with integer endpoints can be represented as a union of at most $2J+1$ disjoint dyadic intervals $I(j,l)$ we obtain

$$\mathbf{P}\{\max_{I\subseteq[1,M]} \|t_I^a\|_p \leq A^{1/p}(2\log M + 3)\|a\|_{\ell_2}\} \geq 1 - 4C(p)(\log M + 2)/A.$$

Lemma 4.3 is proved.

Lemma 4.4. *Let $a_1 > a_2 > \cdots > a_M \geq 0$. Then for each $n \in [1, M]$*

$$\mathbf{P}\{||\{k : a_{\eta_k} \geq a_n\}| - n| \geq M^{1/2} \log M\} \leq 2e^{-C(\log M)^2}.$$

Proof. We use the probabilistic Bernstein inequality. If ξ is a random variable (a real valued function on a probability space Z) then denote

$$\sigma^2(\xi) := \mathbf{E}(\xi - \mathbf{E}(\xi))^2.$$

The probabilistic Bernstein inequality states: if $|\xi - \mathbf{E}(\xi)| \leq B$ a.e. then for any $\varepsilon > 0$

$$\mathbf{P}_{z \in Z^m}\{|\frac{1}{m}\sum_{i=1}^{m} \xi(z_i) - \mathbf{E}(\xi)| \geq \varepsilon\} \leq 2\exp\left(-\frac{m\varepsilon^2}{2(\sigma^2(\xi) + B\varepsilon/3)}\right).$$

We define a random variable β as follows

$$\beta(k) = 1 \quad \text{if} \quad a_{\eta_k} \geq a_n; \quad \beta(k) = 0 \quad \text{otherwise.}$$

Then

$$\mathbf{P}\{\beta(k) = 1\} = \mathbf{P}\{\eta_k \in [1, n]\} = n/M.$$

Also

$$\mathbf{E}(\beta) = n/M; \quad \sigma^2(\beta) = (1 - n/M)n/M \leq 1/4,$$

and

$$|\{k : a_{\eta_k} \geq a_n\}| = \sum_{k=1}^{M} \beta(k).$$

Applying the Bernstein inequality for β with $m = M$ and $\varepsilon = M^{-1/2} \log M$ we obtain Lemma 4.4.

It will be convenient for us to use the following direct corollary of Lemma 4.4.

Lemma 4.5. *Let $a_1 > a_2 > \cdots > a_M \geq 0$. Then*

$$\mathbf{P}\{\max_{1 \leq n \leq M} ||\{k : a_{\eta_k} \geq a_n\}| - n| \geq M^{1/2} \log M\} \leq 2Me^{-C(\log M)^2}.$$

We will now consider some specific polynomials that will be used as building blocks of a counterexample. For a given $p \in (2, \infty)$ we take $\gamma \in (\max(3/4, 2/p), 1)$. For $M \in \mathbb{N}$ we denote $m_1 := m_1(M) := [M^\gamma] + 1$. Let $m_2 := m_2(M)$ be such that

$$\sum_{n=1}^{m_2-1} (n + m_1)^{-1} < \frac{1}{2}\sum_{n=1}^{M}(n + m_1)^{-1} \leq \sum_{n=1}^{m_2}(n + m_1)^{-1}. \tag{4.2}$$

We define $a_n := a_n(M) := (n+m_1)^{-1}$ for $1 \le n \le m_2$, and $a_n := a_n(M) := -(n+m_1)^{-1}$ for $m_2 < n \le M$. We consider the following random trigonometric polynomials

$$P_M(x) := \sum_{k=1}^{M} a_{\eta_k} e^{ikx}.$$

We also need some polynomials associated with P_M. For arbitrary integers n_1 and n_2, $0 \le n_1 < n_2 \le M$, we define $I := (n_1, n_2]$,

$$S_I := S_{n_1,n_2} := \sum_{n=n_1+1}^{n_2} a_n.$$

We consider the following function $g : \{1, \dots, M\} \to \mathbb{R}$:

$$g(n) = \begin{cases} a_n - S_I/M, & n \in I; \\ -S_I/M, & \text{otherwise}, \end{cases}$$

the following random variable $\xi_k = g(\eta_k)$, $(1 \le k \le M)$, and the random trigonometric polynomial

$$t_I^a(x) = \sum_{k=1}^{M} \xi_k e^{ikx}.$$

It is easy to see that

$$P_I(x) := \sum_{\eta_k \in I} a_{\eta_k} e^{ikx} = t_I^a(x) + S_I D_M(x)/M. \tag{4.3}$$

We need the following well-known lemma.

Lemma 4.6. *Let*

$$D_M(x) = \sum_{k=1}^{M} e^{ikx}$$

Then

$$C_2 M^{1-1/p} \le \|D\|_p \le C_3 M^{1-1/p}$$

for some positive $C_2 = C_2(p)$ and $C_3 = C_3(p)$.

Applying Lemma 4.3 with $A = (\log M)^2$ we obtain

$$\mathbf{P}\{\max_{I \subseteq [1,M]} \|t_I^a\|_p \le 3(\log M)^2 m_1^{-1/2}\} \ge 1 - C_2(p)/\log M. \tag{4.4}$$

By Lemma 4.5

$$\mathbf{P}\{\max_{1 \le n \le M} ||\{k : |\hat{P}_M(k)| \ge (m_1+n)^{-1}\}| - n| \ge M^{1/2}\log M\} \le 2Me^{-C(\log M)^2}. \tag{4.5}$$

Therefore, for $M \geq M_0(p)$ there exists a realization $a_{\eta_1}, \dots, a_{\eta_M}$ such that for the polynomial P_M we have: for any $I \subseteq [1, M]$

$$\|t_I^a\|_p \leq 3(\log M)^2 M^{-\gamma/2} \tag{4.6}$$

and for any $n \in [1, M]$

$$||\{k : |\hat{P}_M(k)| \geq (m_1 + n)^{-1}\}| - n| \leq M^{1/2} \log M. \tag{4.7}$$

We will use polynomials satisfying (4.6), (4.7). We also need some other properties of these polynomials. We begin with two simple properties:

$$\|P_M\|_p \leq 3(\log M)^2 M^{-\gamma/2} + C(p) M^{-1/p-\gamma} \tag{4.8}$$

and for $I = (n_1, n_2]$

$$\|P_I\|_p \leq 3(\log M)^2 M^{-\gamma/2} + CM^{-1/p}(\ln(m_1 + n_2) - \ln(m_1 + n_1)). \tag{4.9}$$

The estimate (4.8) follows from (4.3) with $I = [1, M]$, (4.6), Lemma 4.6, and (4.2). The estimate (4.9) follows from (4.3), (4.6), Lemma 4.6, and the inequality

$$|S_I| \leq \sum_{n \in I} (n + m_1)^{-1} \leq C(\ln(m_1 + n_2) - \ln(m_1 + n_1)).$$

Let $\varepsilon_0 := (m_1 + m_2)^{-1}$. Then

$$T_{\varepsilon_0}(P_M) = \sum_{\eta_k \in [1, m_2]} a_{\eta_k} e^{ikx} = P_{[1, m_2]}.$$

Using (4.3), Lemma 4.6, and (4.6) we obtain

$$\|T_{\varepsilon_0}(P_M)\|_p \geq C_1 S_{[1,m_2]} M^{-1/p} - 3(\log M)^2 M^{-\gamma/2} \geq C_2 M^{-1/p} \ln M \tag{4.10}$$

provided $M \geq M_1(p, \gamma)$.

We now estimate from above the $\|T_\delta(P_M) - T_\varepsilon(P_M)\|_p$ for arbitrary $\varepsilon > \delta > 0$. It is clear that it is sufficient to consider the case $a_1 \geq \varepsilon > \delta \geq |a_M|$. We define the numbers $1 \leq n_1 \leq n_2 \leq M$ as follows

$$|a_{n_1}| \geq \varepsilon > |a_{n_1+1}|, \quad |a_{n_2}| \geq \delta > |a_{n_2+1}|$$

(we set $a_{M+1} := 0$). Let $I = (n_1, n_2]$. Then

$$T_\delta(P_M) - T_\varepsilon(P_M) = P_I.$$

By (4.9) we get

$$(4.11) \qquad \|T_\delta(P_M) - T_\varepsilon(P_M)\|_p \le 3(\log M)^2 M^{-\gamma/2} + CM^{-1/p}(\ln\varepsilon - \ln\delta).$$

We note that the condition $\delta \ge \varepsilon^{1+\alpha}$ implies

$$(4.12) \qquad \|T_\delta(P_M) - T_\varepsilon(P_M)\|_p \le 3(\log M)^2 M^{-\gamma/2} + C\alpha M^{-1/p}\log M.$$

We now set $\varepsilon_n := |a_n|$ and estimate $\|G_n(P_M) - T_{\varepsilon_n}(P_M)\|_p$. We have

$$T_{\varepsilon_n}(P_M) = P_{[1,n]}.$$

Let

$$G_n(P_M) = \sum_{k\in\Lambda_n} \hat{P}_M(k)e^{ikx}, \quad |\Lambda_n| = n,$$

and let I_n be such that

$$T_{\varepsilon_n}(P_M) = \sum_{k\in I_n} \hat{P}_M(k)e^{ikx}.$$

It is clear that we have either $\Lambda_n \subseteq I_n$ or $I_n \subseteq \Lambda_n$. Hence, for

$$Z_n := (\Lambda_n \setminus I_n) \cup (I_n \setminus \Lambda_n)$$

we get

$$|Z_n| \le ||\Lambda_n| - |I_n||.$$

By property (4.7) we obtain

$$|Z_n| \le M^{1/2}\log M,$$

and

$$(4.13) \qquad \|G_n(P_M) - T_{\varepsilon_n}(P_M)\|_p \le C(M^{1/2}\log M)^{1-1/p}M^{-\gamma}.$$

We now take two numbers $1 \le n < m \le M$ and estimate $\|G_m(P_M) - G_n(P_M)\|_p$. By (4.13) we have

$$(4.14) \quad \|G_m(P_M) - G_n(P_M)\|_p \le 2C(M^{1/2}\log M)^{1-1/p}M^{-\gamma} + \|T_{\varepsilon_m}(P_M) - T_{\varepsilon_n}(P_M)\|_p.$$

Using (4.11) we continue

$$(4.15) \qquad \le 2C(M^{1/2}\log M)^{1-1/p}M^{-\gamma} + 3(\log M)^2 M^{-\gamma/2}$$

$$+C_1 M^{-1/p}(\ln(m+m_1) - \ln(n+m_1)).$$

Proof of Theorem 4.1. We define two sequences of natural numbers. Let M_1 be a big enough number to guarantee that there are polynomials P_M, $M \geq M_1$, satisfying (4.6)–(4.15). For $\nu \geq 1$ we define

$$M_{\nu+1} = 4M_\nu^2.$$

We define $N_1 = 0$ and for $\nu \geq 1$ we set

$$N_{\nu+1} = N_\nu + M_\nu.$$

Let

$$f(x) := \sum_{\mu=1}^{\infty} M_\nu^{1/p}(\log M_\nu)^{-1} e^{iN_\nu x} P_{M_\nu}(x). \tag{4.16}$$

It follows from (4.8) and the inequality $\gamma > 2/p$ that the series (4.16) converges in the L_p norm. It follows from (4.10) that the statement 2) from Theorem 4.1 is satisfied. We now proceed to the proof of part 1) of Theorem 4.1. Let $\Lambda := \Lambda_j$, $Y := Y_j$, $\varepsilon := \varepsilon_j$, $\delta := \delta_j$ be from Theorem 4.1. We assume that j is big enough to guarantee that $|Y| \leq |\Lambda|^2$ and $\delta \geq \varepsilon^2$. Denote

$$U_\nu := \cup_{\mu=1}^{\nu}(N_\mu, N_\mu + M_\mu].$$

We note that

$$\min_{k \in (N_\nu, N_\nu + M_\nu]} |\hat{f}(k)| > \max_{k \in (N_{\nu+1}, N_{\nu+1} + M_{\nu+1}]} |\hat{f}(k)|.$$

Let ν be such that

$$U_{\nu-1} \subset \Lambda \subseteq U_\nu.$$

We will prove that $Y \subseteq U_{\nu+1}$. Indeed, if to the contrary $U_{\nu+1} \subset Y$ then

$$|Y| \geq M_{\nu+1} \geq 4M_\nu^2; \quad |\Lambda| \leq \sum_{\mu=1}^{\nu} M_\mu < 2M_\nu$$

which contradicts to $|Y| \leq |\Lambda|^2$. Also, $U_{\nu+1} \subset Y$ implies

$$\delta \leq M_{\nu+2}^{-\gamma+1/p}(\log M_{\nu+2})^{-1} \tag{4.17}$$

and $\Lambda \subseteq U_\nu$ implies that

$$\varepsilon \geq M_\nu^{1/p}(\log M_\nu)^{-1}(2M_\nu)^{-1}. \tag{4.18}$$

The relations (4.17) and (4.18) for big ν contradict to our assumption that $\delta \geq \varepsilon^2$. Thus we have $Y \subseteq U_{\nu+1}$. There are two cases: $Y \subseteq U_\nu$ or $U_\nu \subset Y$. In both cases the proof is similar. Let us begin with the first one: $Y \subseteq U_\nu$. In this case

$$S_Y(f) - S_\Lambda(f) = M_\nu^{1/p}(\log M_\nu)^{-1} e^{iN_\nu x}(S_{Y'}(P_{M_\nu}) - S_{\Lambda'}(P_{M_\nu}))$$

where $\Lambda' := \{k - N_\nu,\ k \in \Lambda\}$, $Y' := \{k - N_\nu,\ k \in Y\}$. By (4.12) we get

$$\|S_Y(f) - S_\Lambda(f)\|_p = o(1) \tag{4.19}$$

if $\delta = \varepsilon^{1+o(1)}$. By (4.14)–(4.15) we also obtain (4.19) if $|Y| = |\Lambda|^{1+o(1)}$. This completes the proof of 1) from Theorem 4.1 in the first case.

We now proceed to the second case: $U_\nu \subset Y \subseteq U_{\nu+1}$. This case reduces to the first one by rewriting

$$S_Y(f) - S_\Lambda(f) = S_Y(f) - S_{U_\nu}(f) + S_{U_\nu}(f) - S_\Lambda(f).$$

The proof of Theorem 4.1 is complete.

5. Necessary and sufficient conditions in the case $p = \infty$

If W is any set and $f : W \to W$ is any operator then by f_k ($k \in \mathbb{N}$) we denote the k-fold iteration of f.

Theorem 5.1. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then the following conditions are equivalent:*
a) for some $k \in \mathbb{N}$ and for any sufficiently large $m \in \mathbb{N}$ we have $\alpha_k(m) > e^m$;
b) if $f \in C(\mathbb{T})$ and

$$\left\|G_{\alpha(m)}(f) - G_m(f)\right\|_\infty \to 0 \quad (m \to \infty) \tag{5.1}$$

then

$$\|f - G_m(f)\|_\infty \to 0 \quad (m \to \infty). \tag{5.2}$$

Proof. 1) a) implies b). Denote $\gamma = \alpha_{2k}$. Then

$$\gamma(m) > e^{e^m} \quad (m \geq m_0). \tag{5.3}$$

Let $f \in C(\mathbb{T})$ and let (5.1) hold. Then

$$\left\|G_{\gamma(m)}(f) - G_m(f)\right\|_\infty \to 0 \quad (m \to \infty). \tag{5.4}$$

Let us estimate $\|V_m(f) - G_m(f)\|_\infty$, where $V_m(f)$ is the de la Vallée Poussin sum

$$V_m(f) = \sum_{|k| \leq 2m} \min\left(1, \frac{2m - |k|}{m}\right) \hat{f}(k) e^{ikx}.$$

For $m \geq m_0$ we denote

$$h_1 := G_m(f) - V_m(f), \quad h_2 := G_{\gamma(m)}(f) - G_m(f), \quad h_3 := G_{\gamma(m)}(f), \quad h_4 := f - G_{\gamma(m)}(f).$$

It will be convenient for us to use the following notation

$$\|f\|_{\hat{\ell}_\infty} := \|\{\hat{f}(k)\}\|_{\ell_\infty} := \sup_k |\hat{f}(k)|.$$

We have

$$\inf_{\hat{h}_3(k)\neq 0} |\hat{h}_3(k)| \le \|h_3\|_2(\gamma(m))^{-1/2} \le \|f\|_2 e^{-e^m/2}, \tag{5.5}$$

and, hence,

$$\|h_4\|_{\hat{\ell}_\infty} \le \|f\|_2 e^{-e^m/2}. \tag{5.6}$$

By Theorem 2.1 with $K = 2$, we get

$$\|h_1 + h_4\|_\infty \ge \|h_1\|_\infty/4 - e^{Cm}\|h_4\|_{\hat{\ell}_\infty}.$$

By (5.6), we obtain

$$\|h_1 + h_4\|_\infty \ge \|h_1\|_\infty/4 - o(1) \quad (m \to \infty).$$

Therefore, using (5.4), we have for $m \to \infty$

$$\|h_1\|_\infty \le 4\|h_1 + h_4\|_\infty + o(1) = 4\|f - V_m(f) - h_2\|_\infty + o(1) = o(1).$$

We have used above the well known fact that $\|f - V_m(f)\|_\infty \to 0$ with $m \to 0$ (see [14,Chap.3,S.13]). Using it again we complete the proof of the first implication: a) implies b).

2) b) implies a). We assume that a function α does not satisfy a), and we shall show that b) does not hold. If α is identical on $\mathbb{N}$, then the statement trivially follows from existence of a continuous function with divergent greedy approximations. Otherwise there is $m_0 \in \mathbb{N}$ such that $\alpha(m_0) \neq m_0$. Since α is strictly increasing, we have $\alpha(m_0) > m_0$ and, moreover, $\alpha(m) > m$ for $m \ge m_0$. Let $m_j = \alpha_j(m_0) = \alpha(m_{j-1})$ for $j \in \mathbb{N}$. Then the sequence $\{m_j\}$ is strictly increasing. Moreover, the sequence $\{m_{j+1} - m_j\}$ is nondecreasing. By our supposition, for any $k \in \mathbb{N}$ there is $m > m_0$ such that $\alpha_{k+1}(m) < e^m$. Let $m_{j-1} < m \le m_j$. Then $\alpha_{k+1}(m) > m_{j+k}$ and thus, $m_{j+k} < e^{m_j}$. Therefore, there is an unbounded nondecreasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that for infinitely many $j \in \mathbb{N}$ we have

$$m_j < e^{m_{j-\tau(j)}}, \quad \tau(j) < j. \tag{5.10}$$

Define a sequence $\{A_n\}$. Let $A_n = 1$ for $n \le m_1$ and $A_n = (\tau(j))^{-1}(m_{j+1} - m_j)^{-1}$ for $m_j < n \le m_{j+1}$. Clearly $\{A_n\}$ is nonincreasing. Then we have

$$\sum_{n=m_{j-\tau(j)}+1}^{m_j} A_n = \sum_{i=j-\tau(j)}^{j-1} \sum_{n=m_i+1}^{m_{i+1}} A_n = \sum_{i=j-\tau(j)}^{j-1} \tau(i)^{-1} \ge \sum_{i=j-\tau(j)}^{j-1} \tau(j)^{-1} = 1.$$

If, moreover, j satisfies (5.10), then for $M = m_{j-\tau(j)}$ we get

$$\sum_{M<n\le e^M} A_n \ge 1.$$

We now use Theorem 4 from [10] (see Theorem 3 from Introduction): there is a function $f \in C(\mathbb{T})$ such that $a_n(f) \le A_n$ and (5.2) fails. We take $m > m_1$ and let $m_j < m \le m_{j+1}$. We have

$$\|G_{\alpha(m)}(f) - G_m(f)\| \le \sum_{n=m+1}^{\alpha(m)} a_n(f) \le \sum_{n=m_j+1}^{m_{j+2}} A_n$$
$$= \tau(j)^{-1} + \tau(j+1)^{-1} = o(1) \quad (m \to \infty).$$

This completes the proof of the theorem.

Theorem 5.2. *Let $\beta : (0, +\infty) \to$ be a nondecreasing function such that*

$$\limsup_{\varepsilon\to 0+} \beta(\varepsilon)/\varepsilon < 1. \tag{5.11}$$

Then the following conditions are equivalent:
a) for some $k \in \mathbb{N}$ and for any sufficiently large $u > 0$ we have $\beta_k(1/u) < e^{-u}$;
b) if $f \in C(\mathbb{T})$, and

$$\left\|T_{\beta(\varepsilon)}(f) - T_\varepsilon(f)\right\|_\infty \to 0 \quad (\varepsilon \to 0) \tag{5.12}$$

then

$$\|f - T_\varepsilon(f)\|_\infty \to 0 \quad (\varepsilon \to 0). \tag{5.13}$$

Proof. 1) a) implies b). Denote $\gamma = \beta_{2k}$. Then

$$\gamma(1/u) < e^{-e^u} \quad (u \ge u_0). \tag{5.14}$$

Let $f \in C(\mathbb{T})$ satisfy (5.12). Then

$$\left\|T_{\gamma(\varepsilon)}(f) - T_\varepsilon(f)\right\|_\infty \to 0 \quad (\varepsilon \to 0). \tag{5.15}$$

For $\varepsilon \ge \varepsilon_0$ we denote $m(\varepsilon) := [1/\varepsilon]$ and

$$h_1 := T_\varepsilon(f) - V_{m(\varepsilon)}, \quad h_2 := T_{\gamma(\varepsilon)}(f) - T_\varepsilon(f), \quad h_3 := T_{\gamma(\varepsilon)}(f), \quad h_4 := f - T_{\gamma(\varepsilon)}(f).$$

We have

$$|\{k : \hat{h}_1(k) \ne 0\}| \le |\{k : \hat{T}_\varepsilon(f)(k) \ne 0\}| + 4m(\varepsilon) \le \|f\|_2^2/\varepsilon^2 + 4m(\varepsilon).$$

The rest of the proof for the implication a)→ b) repeats the proof for the same implication in Theorem 5.1.

2) b) implies a). We assume that a function β does not satisfy a), and we shall show that b) does not hold. By supposition (5.11), there are numbers $\theta < 1$ and $\varepsilon_0 > 0$ such that

$$\beta(\varepsilon) \le \theta\varepsilon \quad (0 < \varepsilon \le \varepsilon_0).$$

For $j \in \mathbb{N}$ denote $\varepsilon_j = \beta_j(\varepsilon_0) = \beta(\varepsilon_{j-1})$. We have

$$\varepsilon_j \le \theta\varepsilon_{j-1}. \tag{5.16}$$

By our assumption, for any $k \in \mathbb{N}$ there is $\varepsilon < \varepsilon_0$ such that $\beta_{k+1}(\varepsilon) \ge e^{-1/\varepsilon}$. Let $\varepsilon_{j-1} \ge \varepsilon > \varepsilon_j$. Then $\beta_{k+1}(\varepsilon) \le \varepsilon_{j+k}$ and thus, $\varepsilon_{j+k} > e^{-1/\varepsilon_j}$. Therefore, there is an unbounded nondecreasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that for infinitely many $j \in \mathbb{N}$ we have

$$\varepsilon_j > e^{-1/\varepsilon_{j-\tau(j)}}. \tag{5.17}$$

Also, we can assume that the inequality

$$\tau(j) \le j \tag{5.18}$$

holds for all j. Let

$$m_j := \left[\frac{1}{\varepsilon_j \tau(j)}\right], \quad M_j := \sum_{i=1}^{j} m_i.$$

We set $M_0 := 0$. Let us estimate M_j from above and from below. We have

$$M_j \le \sum_{i=1}^{j} \frac{1}{\varepsilon_j},$$

and, by (5.16),

$$M_j \le \frac{1}{(1-\theta)\varepsilon_j}. \tag{5.19}$$

Also, (5.16) and divergence $\tau(j)$ to ∞ as $j \to \infty$ imply

$$M_j = o\left(\varepsilon_j^{-1}\right) \quad (j \to \infty). \tag{5.20}$$

By (5.16), for sufficiently large j we have $\varepsilon_j < j^{-2}/4$, and, taking into account (5.18) we get

$$m_j \ge \frac{1}{2\varepsilon_j \tau(j)} \tag{5.21}$$

and also

$$M_j \geq m_j \geq (\varepsilon_j)^{-1/2}. \tag{5.22}$$

Now define a sequence $\{A_n\}$ as $A_n = \varepsilon_j$ for $M_{j-1} < n \leq M_j$. If $j - \tau(j)$ is large enough (observe that this is true if j is large itself and (5.17) holds), then, by (5.21), we have

$$\begin{aligned} \sum_{n=M_{j-\tau(j)}+1}^{M_j} A_n &= \sum_{i=j-\tau(j)}^{j-1} \sum_{n=M_i+1}^{M_{i+1}} A_n = \sum_{i=j-\tau(j)}^{j-1} m_i \varepsilon_i \\ &\geq \sum_{i=j-\tau(j)}^{j-1} (2\tau(i))^{-1} \geq \sum_{i=j-\tau(j)}^{j-1} (2\tau(j))^{-1} = \frac{1}{2}. \end{aligned} \tag{5.23}$$

We now assume that (5.17) holds and denote $\varepsilon := \varepsilon_{j-\tau(j)}$. Using (5.17), (5.19), and (5.22), we have

$$M_j < \frac{e^{1/\varepsilon}}{1-\theta}, \quad M_{j-\tau(j)} \geq \varepsilon^{-1/2}.$$

Therefore, if j is large enough (and, thus, ε is small), we have

$$M_j < \exp\left(\left[\exp(M_{j-\tau(j)})\right]\right).$$

We now take M equal to one of the numbers

$$M_{j-\tau(j)}, \quad \left[\exp(M_{j-\tau(j)})\right].$$

Then by (5.23) we get the inequality

$$\sum_{M<n\leq e^M} A_n \geq 1/4.$$

Similarly to the proof of Theorem 5.1 we now use Theorem 3: there is a function $f \in C(\mathbb{T})$ such that $a_n(f) \leq A_n$ and (5.2) fails. We shall take sufficiently small ε and estimate $\|T_{\beta(\varepsilon)}(f) - T_\varepsilon(f)\|_\infty$. Let $\varepsilon_{j-1} > \varepsilon \geq \varepsilon_j$. We have

$$\begin{aligned} \|T_{\beta(\varepsilon)}(f) - T_\varepsilon(f)\|_\infty &\leq \sum_{\beta(\varepsilon)\leq|\hat{f}(k)|<\varepsilon} |\hat{f}(k)| \leq \sum_{\varepsilon_{j+1}\leq|\hat{f}(k)|<\varepsilon_{j-1}} |\hat{f}(k)| \\ &\leq \Sigma_1 + \Sigma_2, \end{aligned} \tag{5.24}$$

where

$$\Sigma_1 = \sum_{\substack{n>M_{j-1}, \\ \varepsilon_{j+1}\leq a_n(f)<\varepsilon_{j-1}}} a_n(f),$$

$$\Sigma_2 = \sum_{\substack{n \le M_{j-1}, \\ \varepsilon_{j+1} \le a_n(f) < \varepsilon_{j-1}}} a_n(f).$$

We observe that in the case $n > M_{j+1}$

$$a_n(f) \le A_n < \varepsilon_{j+1}.$$

Hence,

$$\Sigma_1 = \sum_{\substack{M_{j-1} < n \le M_{j+1}, \\ \varepsilon_{j+1} \le a_n(f) < \varepsilon_{j-1}}} a_n(f) \le \sum_{M_{j-1} < n \le M_{j+1}} a_n(f) \tag{5.25}$$

$$\le \sum_{M_{j-1} < n \le M_{j+1}} A_n = m_j \varepsilon_j + m_{j+1} \varepsilon_{j+1} \le \tau(j)^{-1} + \tau(j+1)^{-1} \to 0 \quad (j \to \infty).$$

Further, by (5.20),

$$\Sigma_2 < \sum_{n \le M_{j-1}} \varepsilon_{j-1} \le M_{j-1} \varepsilon_{j-1} \to 0 \quad (j \to \infty). \tag{5.26}$$

Thus, by (5.24)–(5.26),

$$\lim_{\varepsilon \to 0} \| T_{\beta(\varepsilon)}(f) - T_\varepsilon(f) \|_\infty = 0, \tag{5.27}$$

and (5.12) holds. Moreover, (5.27) clearly implies that

$$\lim_{\delta \to 0} \sum_{|\hat{f}(k)| = \delta} |\hat{f}(k)| = 0,$$

and thus for f convergence of greedy and thresholding approximations are equivalent. But we know that (5.2) fails. Therefore, (5.13) does not hold either. Theorem 5.2 is proved.

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