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Triebel-Lizorkin and Besov spaces

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On the construction of frames on Triebel-Lizorkin and Besov spaces

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Abstract

We present a general method for construction of frames $\{\psi_I\}_{I \in \mathcal{D}}$ for Triebel-Lizorkin and Besov spaces, whose nature can be prescribed. In particular, our method allows for constructing frames consisting of rational functions or more general functions which are linear combinations of a fixed (small) number of shifts and dilates of a single smooth and rapidly decaying function θ such as the Gaussian $\theta(x) = \exp(-|x|^2)$. We also study the boundedness and invertibility of the frame operator $Sf = \sum_{I \in \mathcal{D}} \langle f, \psi_I \rangle \psi_I$ on Triebel-Lizorkin and Besov spaces and give necessary and sufficient conditions for the dual system $\{S^{-1}\psi\}_{I \in \mathcal{D}}$ to be a frame as well.

1 Introduction

Frames were introduced in the early 1950's by Duffin and Schaeffer [DS] as a means for studying the convergence and summability properties of non-harmonic Fourier series. Since then frames have attracted a lot of attention, which is partly due to their numerous applications in Signal analysis, Fourier analysis and Approximation theory. In particular, Gabor frames have been used for time-frequency analysis of functions (signals) (see [FG2]), while wavelet-type frames have been used in statistics, image processing, and nonlinear approximation (see e.g. [De, Do]).

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. A countable family of functions $\Psi \subset H$ is called a frame for H if there exist constants $A, B > 0$ such

that for all $f \in H$

$$A\|f\|_H^2 \leq \sum_{\psi \in \Psi} |\langle f, \psi \rangle|^2 \leq B\|f\|_H^2.$$

It is not hard to prove that the *frame operator* $S : H \rightarrow H$ defined by

$$Sf = \sum_{\psi \in \Psi} \langle f, \psi \rangle \psi$$

is a bounded linear operator and $AI \leq S \leq BI$; hence S is self-adjoint. Consequently, S is invertible and $B^{-1}I \leq S^{-1} \leq A^{-1}I$. Moreover, the family $S^{-1}\Psi := \{S^{-1}\psi\}_{\psi \in \Psi}$ is a frame for H as well.

Further, for every $f \in H$

$$f = SS^{-1}f = \sum_{\psi \in \Psi} \langle f, S^{-1}\psi \rangle \psi. \tag{1.1}$$

Thus Ψ provides a stable representation of all $f \in H$; it acts like a Riesz basis. However, unlike bases, in general, Ψ is redundant. Thus representation (1.1) is not necessarily unique. In many application this redundancy turns out to be of an advantage since it provides extra flexibility. We refer the reader to [HW] for a more detailed overview of the basic properties of frames for Hilbert spaces.

In the late 1980's it was recognized that certain elements of the abstract theory on Hilbert spaces could be extended to more general settings. Gröhenig [G], building upon the theory of the so called coorbit spaces and using group theoretic techniques, introduced frames for Banach spaces which included the modulation spaces and the Besov-Triebel-Lizorkin spaces (see also [FG1]).

Suppose that X is a quasi-Banach space and let Y be an associated sequence space. We say that a countable family of functions Ψ in the dual X^* of X is a frame for X if there exist constants $A, B > 0$ such that for all $f \in X$

$$A\|f\|_X \leq \|(\langle f, \psi \rangle)_{\psi \in \Psi}\|_Y \leq B\|f\|_X,$$

where $\langle f, \psi \rangle := \psi(f)$.

Our goal in this article is to develop a general method for construction of frames for homogeneous Triebel-Lizorkin and Besov spaces. An obvious modification of this method (which will not be considered here) produces

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frames for inhomogeneous Triebel-Lizorkin and Besov spaces as well. Our method utilizes a specific small perturbation principle which allows for a lot of flexibility. Thus the main feature of our method is that the nature of the frame elements can be prescribed. In particular, if θ is a sufficiently smooth and rapidly decaying function on $\mathbb{R}^d$, then our method allows for constructing a frame consisting of functions which are linear combinations of a fixed (small) number of shifts and dilates of θ. Typical examples of such functions are $\theta(x) = \exp(-|x|^2)$ and $\theta(x) = (1+|x|^2)^{-N}$. For construction of bases this method was first introduced in [Pet] and further generalized and refined in [KP].

Frames for Triebel-Lizorkin-Besov spaces can be valuable if the representation (1.1) holds; this is no longer a byproduct of the general theory. Moreover, it is important to know whether the membership of a distribution f to a certain Triebel-Lizorkin or Besov space can be characterized by the frame coefficients $\langle f, S^{-1}\psi \rangle, \psi \in \Psi$, i.e., if $S^{-1}\Psi$ is a frame itself. These are the primary objectives of our development of new frames in this article.

To better describe our results we begin with the introduction of some standard notation. As is customary we denote by $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^d$ and by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions. We also denote by $\mathcal{S}'/\mathcal{P}$ the space of equivalence classes of distributions in $\mathcal{S}'$ modulo polynomials, and by $\mathcal{S}_\infty$ its dual (for more details and references we refer the reader to [K]).

We let $\mathcal{D}$ denote the family of all dyadic cubes in $\mathbb{R}^d$ and let $\mathcal{D}_m$, $m \in \mathbb{Z}$, be the collection of all cubes $I \in \mathcal{D}$ of side-length $\ell(I) = 2^{-m}$. For any dyadic cube $I \in \mathcal{D}$, we use x_I to denote its lower-left corner and $|I|$ for its volume. For any $f \in \mathcal{S}'$, we define

$$f_I(\cdot) := |I|^{-1/2} f\left(\frac{\cdot - x_I}{\ell(I)}\right), \tag{1.2}$$

where the dilation and translation are considered in distributional sense.

As usual the Fourier transform $\widehat{f}$ of an integrable function f is defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi} \, dx$ and can be extended by duality uniquely from $\mathcal{S}$ to $\mathcal{S}'$. Finally, we use $\langle f, \eta \rangle$ for the standard inner product $\int f\overline{\eta}$ of two functions (whenever this makes sense) and the same notation is employed for the action of a distribution $f \in \mathcal{S}'$ on $\overline{\eta} \in \mathcal{S}$.

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1.1 Frames for homogeneous Triebel Lizorkin and Besov spaces

Choose an arbitrary function $\varphi \in \mathcal{S}$ so that supp $\widehat{\varphi} \subset \{\xi : 2^{-1} \le |\xi| \le 2\}$ and $|\widehat{\varphi}(\xi)| \ge c > 0$ if $3/5 \le |\xi| \le 5/3$, and define $\varphi_\nu(\cdot) := 2^{d\nu} \varphi(2^\nu \cdot)$.

For $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, the homogeneous Triebel- Lizorkin space $\dot{F}^s_{pq}$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{F}^s_{pq}} := \left\| \left(\sum_{\nu \in \mathbb{Z}} [2^{\nu s} |\varphi_\nu * f|]^q \right)^{1/q} \right\|_{L_p} < \infty$$

with the usual modification when $q = \infty$. By interchanging the roles of integration and summation we get the homogeneous Besov spaces, that is, for $s \in \mathbb{R}$, $0 < p, q \le \infty$, $\dot{B}^s_{pq}$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}^s_{pq}} := \left(\sum_{\nu \in \mathbb{Z}} [2^{\nu s} \|\varphi_\nu * f\|_{L_p}]^q \right)^{1/q} < \infty.$$

It can be shown that the above definitions of Triebel-Lizorkin and Besov spaces are independent of the specific selection of φ (see e.g. [T]).

Note that the F- and B-spaces are quasi-Banach spaces (Banach spaces if $p, q \ge 1$) and by varying the indexes s, p, q one can recover from them most of the classical spaces (see e.g. [T]). In particular, $\dot{F}^0_{p2} \approx L_p$ if $1 < p < \infty$, and $\dot{F}^0_{p2} \approx H_p$ (the Hardy space) if $0 < p \le 1$. Also, for $s > 0$, $1 < p < \infty$, $\dot{F}^s_{p2}$ coincides with the potential space H^s_p and for integer values of s, $\dot{F}^s_{p2} \approx W^s_p$, the Sobolev space equipped with its seminorm. Here $\approx$ means that the spaces have equivalent (quasi-)norms.

Now, let $\varphi \in \mathcal{S}$ satisfy

$$\begin{aligned} &(i) \text{ supp } \widehat{\varphi} \subset \{\xi : 2^{-1} \le |\xi| \le 2\}, \\ &(ii) \ |\widehat{\varphi}(\xi)| \ge c > 0 \text{ if } 3/5 \le |\xi| \le 5/3, \\ &(iii) \ \sum_{\nu \in \mathbb{Z}} |\widehat{\varphi}(2^\nu \xi)|^2 = 1, \quad \xi \ne 0. \end{aligned} \qquad (1.3)$$

By (1.3(iii)) it readily follows that for $f \in \mathcal{S}'/\mathcal{P}$,

$$\sum_{\nu \in \mathbb{Z}} \widetilde{\varphi}_\nu * \varphi_\nu * f = f, \qquad \widetilde{\varphi}_\nu(x) := \overline{\varphi_\nu(-x)}, \qquad (1.4)$$

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in distributional sense. Identity (1.4) is the so called Calderón's reproducing formula. Using techniques reminiscent of the Shannon sampling theorem and expanding $\widetilde{\varphi}_\nu * \varphi_\nu * f$, one can further show [FJ2] that for all $f \in \mathcal{S}'/\mathcal{P}$,

$$f = \sum_{I \in \mathcal{D}} \langle f, \varphi_I \rangle \varphi_I, \tag{1.5}$$

where $\varphi_I(\cdot) := |I|^{-1/2}\varphi(\frac{\cdot - x_I}{\ell(I)})$ (see (1.2)).

The F- and B-spaces are naturally characterized by the sequence spaces $\dot{f}^s_{pq}$ and $\dot{b}^s_{pq}$ defined as follows: For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$,

$$\dot{f}^s_{pq} := \{h := (h_I)_{I \in \mathcal{D}} : \|h\|_{\dot{f}^s_{pq}} < \infty\},$$

where

$$\|h\|_{\dot{f}^s_{pq}} := \Big\|\Big(\sum_{I \in \mathcal{D}} (|I|^{-s/d-1/2}|h_I \mathbb{1}_I|)^q\Big)^{1/q}\Big\|_{L_p}.$$

Similarly, for $s \in \mathbb{R}$ and $0 < p, q \le \infty$,

$$\dot{b}^s_{pq} := \{h := (h_I)_{I \in \mathcal{D}} : \|h\|_{\dot{b}^s_{pq}} < \infty\},$$

where

$$\|h\|_{\dot{b}^s_{pq}} := \Big(\sum_{e \in E}\sum_{m \in \mathbb{Z}}\Big(\sum_{I \in \mathcal{D}_m}\big(|I|^{-s/d+1/p-1/2}|h_{Ie}|\big)^p\Big)^{q/p}\Big)^{1/q}.$$

A fundamental result due to Frazier and Jawerth [FJ1, FJ2] says that the family $\Phi := \{\varphi_I\}_{I \in \mathcal{D}}$ is a frame for both the Triebel-Lizorkin and Besov spaces: If $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$, there exist constants $c_1, c_2 > 0$ such that for $f \in \dot{F}^s_{pq}$,

$$c_1\|f\|_{\dot{F}^s_{pq}} \le \|(\langle f, \varphi_I \rangle)_I\|_{\dot{f}^s_{pq}} \le c_2\|f\|_{\dot{F}^s_{pq}}. \tag{1.6}$$

Similarly, if $s \in \mathbb{R}$, and $0 < p, q \le \infty$, then for $f \in \dot{B}^s_{pq}$,

$$c_1\|f\|_{\dot{B}^s_{pq}} \le \|(\langle f, \varphi_I \rangle)_I\|_{\dot{b}^s_{pq}} \le c_2\|f\|_{\dot{B}^s_{pq}}. \tag{1.7}$$

1.2 The construction of new frames

The construction of affine frames in the literature follows in the footsteps of the main scheme for constructing wavelet bases, namely, by making use of

the notion of a *Multiresolution analysis* (see e.g. [BL]). As a consequence, most of the constructed frames are of a very particular nature.

Here instead we propose a method for constructing frames based on the idea of the small perturbation principle. The starting point of our method is the selection of a frame $\Phi := \{\varphi_I\}_{I\in\mathcal{D}}$ such as the Frazier-Jawerth frame described above, consisting of sufficiently smooth and rapidly decaying functions. Then the main step of our scheme is to approximate every frame element φ_I of Φ by a function ψ_I of the *new system* $\Psi := \{\psi_I\}_{I\in\mathcal{D}}$.

The grand question is: *What kind of approximation should be used in this construction in order that the new system Ψ be a frame for the spaces of interest?*

Following the idea introduced in [Pet] and further developed in [KP], we propose the following approximation scheme: For suitable values of the parameters $\varepsilon, r, k, M > 0$ we require that all ψ_I in Ψ satisfy the conditions:

A1.
$$|\varphi_I^{(\alpha)}(x) - \psi_I^{(\alpha)}(x)| \le \varepsilon |I|^{-1/2 - |\alpha|/d}\big(1 + \ell(I)^{-1}|x - x_I|\big)^{-M}, \quad |\alpha| \le r.$$

A2.
$$\int_{\mathbb{R}^d} x^\alpha \psi_I(x)\, dx = 0, \quad |\alpha| \le k.$$

For simplicity we shall assume in the following that Φ is the Frazier-Jawerth frame described above. However, any other frame consisting of sufficiently smooth and rapidly decaying functions with enough vanishing moments can do the job.

We shall prove in Theorem 3.1 that if $\varepsilon > 0$ is sufficiently small and r, k, M are sufficiently large (depending on s, p, q), then the family Ψ is a frame for the Besov-Triebel-Lizorkin spaces. The questions that naturally arise here are, whether $S^{-1}\Psi$ is a frame itself and if the representation

$$f = \sum_{I\in\mathcal{D}} \langle f, S^{-1}\psi_I\rangle \psi_I \tag{1.8}$$

holds? It turns out that the key to both questions is the boundedness of the operator with matrix

$$\mathbf{B} = (b_{IJ})_{I,J\in\mathcal{D}}, \quad \text{where} \quad b_{IJ} := \langle \varphi_J, S^{-1}\psi_I\rangle,$$

on $\dot{f}_{pq}^s$ (or $\dot{b}_{pq}^s$). As we show in Theorem 3.3 the boundedness of $\mathbf{B}$ is a necessary and sufficient condition for $S^{-1}\Psi$ to be a frame for the Triebel-Lizorkin and Besov spaces. Moreover, the same condition guarantees that (1.8) holds in distributional sense.

Since the family Φ is generated by dilations and shifts of a single function φ, it suffices to approximate φ by a function ψ satisfying the conditions:

B1.
$$|\varphi^{(\alpha)}(x) - \psi^{(\alpha)}(x)| \leq \varepsilon\big(1 + |x|\big)^{-M}, \quad |\alpha| \leq r.$$

B2.
$$\int_{\mathbb{R}^d} x^\alpha \psi(x)\, dx = 0, \quad |\alpha| \leq k.$$

After that we define the new system Ψ by $\Psi := \{\psi_I\}_{I \in \mathcal{D}}$ (see (1.2)). Then by dilation and translation it follows that Ψ satisfies **A1-2**.

The construction of functions ψ satisfying **B1-2** has been explored in [KP]. In particular, it is shown that if θ is an arbitrary sufficiently smooth and rapidly decaying function on $\mathbb{R}^d$, then one can devise ψ satisfying **B1-2** using a finite linear combination of shifts and dilates of θ. Typical examples include the rational function $\theta(\cdot) := (1 + |\cdot|^2)^{-N}$ and the Gaussian $\theta(\cdot) := e^{-|\cdot|^2}$.

2 Auxiliary results

In this section we present some basic results concerning the algebra of almost diagonal operators on $\dot{f}^s_{pq}$ and $\dot{b}^s_{pq}$ and their interplay with the study of the boundedness of linear operators on Triebel-Lizorkin and Besov spaces. These results have been established by Frazier and Jawerth within the framework of the so-called ϕ-transform (see [FJ1, FJ2, FJW]) and will be needed for our developments in the next section.

Let $\mathbf{A}$ be a linear operator acting on $\dot{f}^s_{pq}$ or $\dot{b}^s_{pq}$ with associated matrix $(a_{IJ})_{I,J \in \mathcal{D}}$. We say that $\mathbf{A}$ is *almost diagonal* on $\dot{f}^s_{pq}$ or $\dot{b}^s_{pq}$ if there exist $\delta > 0$ and $c > 0$ such that

$$|a_{IJ}| \leq c\omega_\delta(I, J), \quad I, J \in \mathcal{D},$$

with

$$
\begin{aligned}
\omega_\delta(I, J) \ &:= \ \left(\frac{\ell(I)}{\ell(J)}\right)^s \left(1 + \frac{|x_I - x_J|}{\max\{\ell(I), \ell(J)\}}\right)^{-\mathcal{J} - \delta} \\
&\quad \times \ \min\left\{\left(\frac{\ell(I)}{\ell(J)}\right)^{(d+\delta)/2}, \left(\frac{\ell(J)}{\ell(I)}\right)^{(\delta-d)/2 + \mathcal{J}}\right\},
\end{aligned}
$$

where $\mathcal{J} := d/\min\{1, p, q\}$ for $\dot{f}^s_{pq}$ and $\mathcal{J} := d/\min\{1, p\}$ for $\dot{b}^s_{pq}$.

We denote by $\mathbf{ad}^s_{pq}$ the collection of all almost diagonal operators on $\dot{f}^s_{pq}$ or $\dot{b}^s_{pq}$. It turns out that $\mathbf{ad}^s_{pq}$ is closed under composition and therefore it is an algebra. Moreover, almost diagonal operators are bounded on $\dot{f}^s_{pq}$ and $\dot{b}^s_{pq}$.

Proposition 2.1. *Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and $\mathbf{A} \in \mathbf{ad}^s_{pq}$. Then $\mathbf{A}$ is bounded on $\dot{b}^s_{pq}$ and, if $p < \infty$, on $\dot{f}^s_{pq}$.*

It is not hard to construct almost diagonal operators for the f- or b-spaces. Consider for instance the following construction: Let $\mathcal{J}$ be as above and $\mathcal{N} := \max\{[\mathcal{J} - d - s], -1\}$. Suppose that for some $M > \max\{\mathcal{J}, d + s\}$, $k \geq [s]$, and $r > \mathcal{N}$, the functions $\{b_I\}_{I \in \mathcal{D}}$ satisfy the conditions

$$|b_I^{(\alpha)}(x)| \leq c|I|^{-1/2 - |\alpha|/d}\left(1 + \ell(I)^{-1}|x - x_I|\right)^{-M}, \quad |\alpha| \leq r, \quad (2.1)$$

$$\int_{\mathbb{R}^d} x^\alpha b_I(x)\, dx = 0, \quad |\alpha| \leq k, \quad (2.2)$$

where (2.2) is void if $s < 0$, then the operator $\mathbf{A}$ with matrix

$$\mathbf{A} = (a_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad a_{IJ} := \langle \varphi_J, b_I \rangle, \quad (2.3)$$

is almost diagonal on $\dot{f}^s_{pq}$ or $\dot{b}^s_{pq}$.

An important consequence of the boundedness of $\mathbf{A}$ is that the family $\{b_I\}_{I \in \mathcal{D}}$ is a *norming family* for the Triebel-Lizorkin and Besov spaces, that is, for all $f \in \dot{F}^s_{pq}$

$$\|(\langle f, b_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq c\|f\|_{\dot{F}^s_{pq}}, \quad (2.4)$$

and similarly for $f \in \dot{B}^s_{pq}$

$$\|(\langle f, b_I \rangle)_I\|_{\dot{b}^s_{pq}} \leq c\|f\|_{\dot{B}^s_{pq}}. \quad (2.5)$$

We say that $\{m_I\}_{I \in \mathcal{D}}$ is a family of *smooth molecules* for $\dot{F}^s_{pq}$ or $\dot{B}^s_{pq}$ if for some $r > [s]$, $k \geq \mathcal{N}$, and $M > \mathcal{J}$,

$$|m_I(x)| \leq c|I|^{-1/2}\left(1 + \ell(I)^{-1}|x - x_I|\right)^{-\max(M, M - s)}, \quad (2.6)$$

$$|m_I^{(\alpha)}(x)| \leq c|I|^{-1/2 - |\alpha|/d}\left(1 + \ell(I)^{-1}|x - x_I|\right)^{-M}, \quad |\alpha| \leq r, \quad (2.7)$$

$$\int_{\mathbb{R}^d} x^\alpha m_I(x)\, dx = 0, \quad |\alpha| \leq k, \quad (2.8)$$

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where (2.7) is void if $s < 0$.

Similarly as above, molecules give rise to almost diagonal operators. For instance, the operator $\mathbf{C}$ with matrix

$$\mathbf{C} = (c_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad c_{IJ} := \langle m_J, \varphi_I \rangle, \tag{2.9}$$

is almost diagonal on $\dot{f}^s_{pq}$ and $\dot{b}^s_{pq}$. Consequently, for every sequence $(s_I)_I \in \dot{f}^s_{pq}$,

$$\Big\| \sum_I s_I m_I \Big\|_{\dot{F}^s_{pq}} \leq c \|(s_I)_I\|_{\dot{f}^s_{pq}}, \tag{2.10}$$

and similarly if $(s_I)_I \in \dot{b}^s_{pq}$, then

$$\Big\| \sum_I s_I m_I \Big\|_{\dot{B}^s_{pq}} \leq c \|(s_I)_I\|_{\dot{b}^s_{pq}}. \tag{2.11}$$

Let us now assume that T is a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}'$. If the operator with matrix

$$\mathbf{D} = (d_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d_{IJ} := \langle T\varphi_J, \varphi_I \rangle, \tag{2.12}$$

is almost diagonal on $\dot{f}^s_{pq}$ or $\dot{b}^s_{pq}$ then T extends to a bounded operator on $\dot{F}^s_{pq}$ or $\dot{B}^s_{pq}$. Indeed, employing (1.5)-(1.6), we have for $f \in \dot{F}^s_{pq}$,

$$\|Tf\|_{\dot{F}^s_{pq}} \leq c \|(\langle Tf, \varphi_I \rangle)_I\|_{\dot{f}^s_{pq}} = c \Big\| \Big(\sum_J d_{IJ} \langle f, \varphi_J \rangle \Big)_I \Big\|_{\dot{f}^s_{pq}}$$

$$\leq c \|(\langle f, \varphi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq c \|f\|_{\dot{F}^s_{pq}}.$$

A similar result holds for Besov spaces as well.

3 Main results

We recall some of our notation from the previous section: $\mathcal{J} := d/\min\{1, p, q\}$ in the case of Triebel-Lizorkin spaces, while $\mathcal{J} := d/\min\{1, p\}$ for Besov spaces. Also $\mathcal{N} := \max\{[\mathcal{J} - d - s], -1\}$.

Theorem 3.1. *Suppose that $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and let $\Psi := \{\psi_I\}_{I \in \mathcal{D}}$ satisfy* **A1-2** *for some $r > \mathcal{N}$, $k \geq [s]$, and $M > max\{\mathcal{J}, d + s\}$. If $\varepsilon > 0$ is sufficiently small, then there exist constants $A, B > 0$ such that for $f \in \dot{B}^s_{pq}$,*

$$A\|f\|_{\dot{B}^s_{pq}} \leq \|(\langle f, \psi_I \rangle)_I\|_{\dot{b}^s_{pq}} \leq B\|f\|_{\dot{B}^s_{pq}} \tag{3.1}$$

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and, if $p < \infty$, for $f \in \dot{F}^s_{pq}$,

$$A\|f\|_{\dot{F}^s_{pq}} \leq \|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq B\|f\|_{\dot{F}^s_{pq}}. \qquad (3.2)$$

Proof. We shall prove only (3.2) since (3.1) follows similarly. From **A1-2** it follows that Ψ is a norming family for $\dot{F}^s_{pq}$ (see (2.4)), i.e. there exists a constant B such that for $f \in \dot{F}^s_{pq}$,

$$\|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq B\|f\|_{\dot{F}^s_{pq}}.$$

Thus we only have to establish the left-hand-side estimate in (3.2). For this we note that since (φ_I) satisfies (2.1)-(2.2) for all $r > 0$ the family $((\varphi_I - \psi_I)\varepsilon^{-1})$ is a norming family as well and, therefore, for some $c > 0$

$$\|(\langle f, \varphi_I - \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq c\varepsilon\|f\|_{\dot{F}^s_{pq}}.$$

Therefore,

$$\|(\langle f, \varphi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq c(\|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} + \|(\langle f, \varphi_I - \psi_I \rangle)_I\|_{\dot{f}^s_{pq}})$$
$$\leq c(\|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} + \varepsilon\|f\|_{\dot{F}^s_{pq}}).$$

Taking now into account that the family Φ is a frame, we obtain using (1.6)

$$c_1\|f\|_{\dot{F}^s_{pq}} \leq c\|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} + c\varepsilon\|f\|_{\dot{F}^s_{pq}}.$$

Choosing $\varepsilon > 0$ sufficiently small, we have $A := (c_1 - c\varepsilon)/c > 0$ and hence

$$A\|f\|_{\dot{F}^s_{pq}} \leq \|(\langle f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}},$$

which completes the proof of the theorem. $\qquad \square$

Assume now that $\Psi := \{\psi_I\}_{I \in \mathcal{D}}$ is a frame for $\dot{F}^s_{pq}$ (or $\dot{B}^s_{pq}$) satisfying (2.1)-(2.2) (with $\{b_I\}$ replaced by $\{\psi_I\}$) and let k, M, r be sufficiently large. If $k \geq 0$, $M > d$, and $r > 0$, Ψ constitutes a frame for $F^0_{22} = B^0_{22} = L_2(\mathbb{R}^d)$ and from the Hilbert space theory we know that the frame operator $S : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ defined by

$$Sf := \sum_{I \in \mathcal{D}} \langle f, \psi_I \rangle \psi_I,$$

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is self-adjoint and invertible. Moreover, the family $S^{-1}\Psi := \{S^{-1}\psi_I\}_{I\in\mathcal{D}}$ is also a frame for $L_2(\mathbb{R}^d)$ and for $f \in L_2(\mathbb{R}^d)$,

$$f = \sum_{I\in\mathcal{D}} \langle f, S^{-1}\psi_I\rangle\psi_I.$$

We would like to devise conditions on Ψ which would guarantee that $S^{-1}\Psi$ is a frame for the Triebel-Lizorkin and Besov spaces as well. As already mentioned in the introduction, in essence one has to study the boundedness of the operator $\mathbf{B}$ with matrix

$$\mathbf{B} := (b_{IJ})_{I,J\in\mathcal{D}}, \quad \text{where} \quad b_{IJ} := \langle \varphi_J, S^{-1}\psi_I\rangle, \tag{3.3}$$

on the corresponding sequence spaces.

We first have to give meaning to the coefficients $\langle f, S^{-1}\psi_I\rangle$. Since $S^{-1}\Psi$ is a frame for $L_2(\mathbb{R}^d)$, if $f \in L_2(\mathbb{R}^d)$, then $f = \sum_J \langle f, \varphi_J\rangle\varphi_J$ in L_2, and hence

$$\langle f, S^{-1}\psi_I\rangle = \sum_{J\in\mathcal{D}} \langle f, \varphi_J\rangle\langle \varphi_J, S^{-1}\psi_I\rangle, \quad I \in \mathcal{D}. \tag{3.4}$$

Assuming that $\mathbf{B}$ is bounded on $\dot{f}^s_{pq}$, (3.4) yields that for $f \in L_2(\mathbb{R}^d) \cap \dot{F}^s_{pq}$ and $I \in \mathcal{D}$,

$$|\langle f, S^{-1}\psi_I\rangle| \le \|(\langle f, S^{-1}\psi_I\rangle)_I\|_{\dot{f}^s_{pq}} \le c\|(\langle f, \varphi_I\rangle)_I\|_{\dot{f}^s_{pq}} \le c\|f\|_{\dot{F}^s_{pq}}. \tag{3.5}$$

A similar argument applies to the $\dot{b}^s_{pq}$ and $\dot{B}^s_{pq}$ spaces as well. Using the density of $L_2(\mathbb{R}^d)$ in $\dot{F}^s_{pq}$ (or $\dot{B}^s_{pq}$) and a standard argument one now easily extends the linear functional $S^{-1}\psi_I : f \to \langle f, S^{-1}\psi_I\rangle$ to $\dot{F}^s_{pq}$ (or $\dot{B}^s_{pq}$).

Lemma 3.2. *Suppose that $s \in \mathbb{R}$, $0 < p,q \le \infty$ and let $\Psi := \{\psi_I\}_{I\in\mathcal{D}}$ be a frame for $L_2(\mathbb{R}^d)$ satisfying $(2.6) - (2.8)$ for some $r > [s]$, $k \ge \mathcal{N}$, and $M > \mathcal{J}$. If $\mathbf{B}$ is bounded on $\dot{b}^s_{pq}$, then for $f \in \dot{B}^s_{pq}$,*

$$f = \sum_{I\in\mathcal{D}} \langle f, S^{-1}\psi_I\rangle\psi_I, \tag{3.6}$$

in the sense of $\mathcal{S}'/\mathcal{P}$. Similarly, if $p < \infty$ and $\mathbf{B}$ is bounded on $\dot{f}^s_{pq}$, then (3.6) holds for $f \in \dot{F}^s_{pq}$.

Proof. We shall again treat only the case of Triebel-Lizorkin spaces. Since Ψ is a frame for $L_2(\mathbb{R}^d)$, we know that for every $I \in \mathcal{D}$,

$$\varphi_I = \sum_{J \in \mathcal{D}} \langle \varphi_I, S^{-1} \psi_J \rangle \psi_J$$

in L_2 (and consequently in distributional sense).

For $f \in \dot{F}_{pq}^s$, we have

$$\begin{aligned} f &= \sum_{I \in \mathcal{D}} \langle f, \varphi_I \rangle \varphi_I = \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} \langle f, \varphi_I \rangle \langle \varphi_I, S^{-1} \psi_J \rangle \psi_J \\ &= \sum_{J \in \mathcal{D}} \sum_{I \in \mathcal{D}} \langle f, \varphi_I \rangle \langle \varphi_I, S^{-1} \psi_J \rangle \psi_J = \sum_{J \in \mathcal{D}} \langle f, S^{-1} \psi_J \rangle \psi_J. \end{aligned}$$

To justify the shift of the order of the summation we note that since $(\langle f, \varphi_I \rangle)_{I \in \mathcal{D}} \in \dot{f}_{pq}^s$ and $\mathbf{B}$ is bounded on $\dot{f}_{pq}^s$, the sequence

$$d := (d_J)_{J \in \mathcal{D}} \quad \text{with } d_J := \sum_{I \in \mathcal{D}} |\langle f, \varphi_I \rangle| |\langle \varphi_I, S^{-1} \psi_J \rangle|$$

belongs to $\dot{f}_{pq}^s$. Using this fact one easily obtains (see [K], Lemma 4.1) that for $\eta \in \mathcal{S}_\infty$,

$$\sum_{J \in \mathcal{D}} |d_J| |\langle \psi_J, \eta \rangle| < \infty.$$

$\square$

Theorem 3.3. *Suppose that $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and let $\Psi := \{\psi_I\}_{I \in \mathcal{D}}$ be a frame for $L_2(\mathbb{R}^d)$ satisfying $(2.6) - (2.8)$ for some $r > [s]$, $k \geq \mathcal{N}$, and $M > \mathcal{J}$. Then $S^{-1}\Psi$ is a frame for $\dot{B}_{pq}^s$ or $\dot{F}_{pq}^s$, when $p < \infty$, if and only if the operator $\mathbf{B}$ defined by (3.3) is bounded on $\dot{b}_{pq}^s$ or $\dot{f}_{pq}^s$, respectively.*

Proof. We again treat only the case of Triebel-Lizorkin spaces. Let us first assume that $\mathbf{B}$ is a bounded operator on $\dot{F}_{pq}^s$. We shall prove that there exist constants $A, B > 0$ such that for $f \in \dot{F}_{pq}^s$,

$$A\|f\|_{\dot{F}_{pq}^s} \leq \|(\langle f, S^{-1}\psi_I \rangle)_I\|_{\dot{f}_{pq}^s} \leq B\|f\|_{\dot{F}_{pq}^s}. \tag{3.7}$$

The right-hand-side estimate in (3.7) has already been established in (3.5).

We next prove the left-hand-side estimate in (3.7). Consider the operator $\mathbf{C}$ with matrix

$$\mathbf{C} := (c_{IJ})_{I, J \in \mathcal{D}}, \quad \text{where} \quad c_{IJ} = \langle \psi_J, \varphi_I \rangle.$$

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As in (2.9) we know that $\mathbf{C}$ is bounded on $\dot{f}_{pq}^s$. Since $\langle f, \varphi_I \rangle = \sum_{J \in \mathcal{D}} \langle f, S^{-1}\psi_J \rangle \langle \psi_J, \varphi_I \rangle$ for $I \in \mathcal{D}$, putting $c_f := (\langle f, \varphi_I \rangle)_{I \in \mathcal{D}}$ and $d := (\langle f, S^{-1}\psi_I \rangle)_{I \in \mathcal{D}}$, we can write

$$\mathbf{C}d = c_f.$$

Then

$$\|f\|_{\dot{F}_{pq}^s} \le c\|(\langle f, \varphi_I \rangle)_I\|_{\dot{f}_{pq}^s} = c\|\mathbf{C}d\|_{\dot{f}_{pq}^s} \le c\|d\|_{\dot{f}_{pq}^s} = c\|(\langle f, S^{-1}\psi_I \rangle)_I\|_{\dot{f}_{pq}^s}.$$

Assume now that $S^{-1}\Psi$ is a frame for $\dot{F}_{pq}^s$. We claim that the operator $\mathbf{B}$ (see (3.3)) is necessarily bounded on $\dot{f}_{pq}^s$. To see this let $(d_I)_{I \in \mathcal{D}} \in \dot{f}_{pq}^s$. From (2.10) we know that the distribution $f = \sum_{I \in \mathcal{D}} d_I \varphi_I$ belongs to $\dot{F}_{pq}^s$ and

$$\|\sum_{I \in \mathcal{D}} d_I \varphi_I\|_{\dot{F}_{pq}^s} \le c\|(d_I)_I\|_{\dot{f}_{pq}^s}. \tag{3.8}$$

Since $S^{-1}\Psi$ is a frame for $\dot{F}_{pq}^s$, we also have

$$\|\langle f, S^{-1}\psi_I \rangle\|_{\dot{f}_{pq}^s} \le B\|f\|_{\dot{F}_{pq}^s}. \tag{3.9}$$

Finally, we use that $\langle f, S^{-1}\psi_I \rangle = \sum_{J \in \mathcal{D}} d_J \langle \varphi_J, S^{-1}\psi_I \rangle = \sum_{J \in \mathcal{D}} d_J b_{IJ} = (\mathbf{B}d)_I$ and (3.8)-(3.9) to obtain

$$\|\mathbf{B}d\|_{\dot{f}_{pq}^s} = \|(\langle f, S^{-1}\psi_I \rangle)_I\|_{\dot{f}_{pq}^s} \le c\|f\|_{\dot{F}_{pq}^s} \le c\|d\|_{\dot{f}_{pq}^s}.$$

$\square$

We next prove that if Ψ satisfies **A1-2** for sufficiently small $\varepsilon > 0$, then $\mathbf{B}$ is indeed a bounded operator on $\dot{b}_{pq}^s$ and $\dot{f}_{pq}^s$.

Theorem 3.4. *Suppose $s \in \mathbb{R}$, $0 < p, q \le \infty$, and let $\Psi := \{\psi_I\}_{I \in \mathcal{D}}$ be a family of functions satisfying* **A1-2** *for some $r, k > \max\{[s], \mathcal{N}\}$ and $M > \max\{\mathcal{J}, d + s\}$. If $\varepsilon > 0$ is sufficiently small, then $S^{-1}\Psi$ is a frame for $\dot{B}_{pq}^s$ and $\dot{F}_{pq}^s$, if $p < \infty$.*

Proof. According to Theorem 3.3 we need to establish that for sufficiently small $\varepsilon > 0$ the operator $\mathbf{B}$ defined in (3.3) is bounded on $\dot{f}_{pq}^s$ or $\dot{b}_{pq}^s$, respectively.

We begin by showing that the operator S^{-1} is bounded on $\dot{F}^s_{pq}$. To this end it suffices to show that

$$\|I - S\|_{\dot{F}^s_{pq} \to \dot{F}^s_{pq}} < 1.$$

In particular, we shall show that there exists a constant $c > 0$ independent of ε such that for $f \in \dot{F}^s_{pq}$,

$$\|(I - S)f\|_{\dot{F}^s_{pq}} \le c\varepsilon \|f\|_{\dot{F}^s_{pq}}. \tag{3.10}$$

According to (2.12) it suffices to prove that the operator $\mathbf{D}$ with matrix

$$\mathbf{D} := (d_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d_{IJ} := \langle (I - S)\varphi_J, \varphi_I \rangle,$$

is bounded on $\dot{f}^s_{pq}$ and

$$\|\mathbf{D}d\|_{\dot{f}^s_{pq}} \le c\varepsilon \|d\|_{\dot{f}^s_{pq}} \quad \text{for } d \in \dot{f}^s_{pq}. \tag{3.11}$$

Using that $\varphi_J = \sum_Q \langle \varphi_J, \varphi_Q \rangle \varphi_Q$ and $S\varphi_J = \sum_Q \langle \varphi_J, \psi_Q \rangle \psi_Q$, we have

$$
\begin{aligned}
d_{IJ} &= \langle (I - S)\varphi_J, \varphi_I \rangle = \sum_Q \langle \varphi_J, \varphi_Q \rangle \langle \varphi_Q, \varphi_I \rangle - \sum_Q \langle \varphi_J, \psi_Q \rangle \langle \psi_Q, \varphi_I \rangle \\
&= \sum_Q \langle \varphi_J, \varphi_Q \rangle \langle \varphi_Q - \psi_Q, \varphi_I \rangle + \sum_Q \langle \varphi_J, \varphi_Q - \psi_Q \rangle \langle \psi_Q, \varphi_I \rangle.
\end{aligned}
$$

Setting now

$$
\begin{aligned}
\mathbf{D}_1 &:= (d^1_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d^1_{IJ} := \langle \varphi_J - \psi_J, \varphi_I \rangle, \\
\mathbf{D}_2 &:= (d^2_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d^2_{IJ} := \langle \varphi_J, \varphi_I \rangle, \\
\mathbf{D}_3 &:= (d^3_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d^3_{IJ} := \langle \psi_J, \varphi_I \rangle, \\
\mathbf{D}_4 &:= (d^2_{IJ})_{I,J \in \mathcal{D}}, \quad \text{where} \quad d^4_{IJ} := \langle \varphi_J, \varphi_I - \psi_I \rangle,
\end{aligned}
$$

we obtain

$$\mathbf{D} = \mathbf{D}_1 \mathbf{D}_2 + \mathbf{D}_3 \mathbf{D}_4.$$

The operators associated with all four matrices are bounded on $\dot{f}^s_{pq}$. Indeed, note that $\mathbf{D}_1, \mathbf{D}_2$, and $\mathbf{D}_3$ give rise to bounded operators as for the operator $\mathbf{C}$ in (2.9) because the families Φ, Ψ, and $\{\varepsilon^{-1}(\varphi_I - \psi_I)\}_{I \in \mathcal{D}}$ consist of smooth molecules. In particular, the last family gives us that for $d \in \dot{f}^s_{pq}$,

$$\|\mathbf{D}_1 d\|_{\dot{f}^s_{pq}} \le c\varepsilon \|d\|_{\dot{f}^s_{pq}}. \tag{3.12}$$

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As far as $\mathbf{D}_4$ is concerned, since $\{\varepsilon^{-1}(\varphi_I - \psi_I)\}_{I \in \mathcal{D}}$ is a norming family, then as for the operator $\mathbf{A}$ defined in (2.3) we obtain that for $d \in \dot{f}^s_{pq}$,

$$\|\mathbf{D}_4 d\|_{\dot{f}^s_{pq}} \leq c\varepsilon \|d\|_{\dot{f}^s_{pq}}. \tag{3.13}$$

Taking into account (3.12)-(3.13) it follows that for $d \in \dot{f}^s_{pq}$,

$$\|\mathbf{D}d\|_{\dot{f}^s_{pq}} \leq c(\|\mathbf{D}_1\|\|\mathbf{D}_2\| + \|\mathbf{D}_3\|\|\mathbf{D}_3\|)\|d\|_{\dot{f}^s_{pq}} \leq c\varepsilon\|d\|_{\dot{f}^s_{pq}},$$

which establishes (3.11). Consequently, (3.10) holds and hence for sufficiently small $\varepsilon > 0$ the operator S^{-1} is bounded on $\dot{F}^s_{pq}$, i.e. for $f \in \dot{F}^s_{pq}$,

$$\|S^{-1}f\|_{\dot{F}^s_{pq}} \leq c\|f\|_{\dot{F}^s_{pq}}. \tag{3.14}$$

Finally, in order to show that the operator $\mathbf{B}$ is bounded on $\dot{f}^s_{pq}$ we recall from (2.10) that if $d := (d_I)_{I \in \mathcal{D}} \in \dot{f}^s_{pq}$ and $f = \sum_I d_I \varphi_I$, then

$$\|f\|_{\dot{F}^s_{pq}} \leq c\|d\|_{\dot{f}^s_{pq}}. \tag{3.15}$$

Clearly, for $I \in \mathcal{D}$,

$$\langle S^{-1}f, \psi_I \rangle = \sum_{J \in \mathcal{D}} d_J \langle S^{-1}\varphi_J, \psi_I \rangle = \sum_{J \in \mathcal{D}} d_J \langle \varphi_J, S^{-1}\psi_I \rangle = (\mathbf{B}d)_I.$$

Therefore,

$$\|\mathbf{B}d\|_{\dot{f}^s_{pq}} = \|(\langle S^{-1}f, \psi_I \rangle)_I\|_{\dot{f}^s_{pq}} \leq c\|S^{-1}f\|_{\dot{F}^s_{pq}} \leq c\|f\|_{\dot{F}^s_{pq}} \leq c\|d\|_{\dot{f}^s_{pq}}.$$

Here for the first inequality we used that Ψ is a frame for $\dot{F}^s_{pq}$ if $\varepsilon > 0$ is sufficiently small, which follows by Theorem 3.1; we also used (3.14)-(3.15). $\qquad\square$

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