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Optimal Channel Assignments for Lattices with Conditions at Distance Two *

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Abstract

The problem of radio channel assignments with multiple levels of interference can be modeled using graph theory. Given a graph G, possibly infinite, and real numbers $k_1, k_2, \ldots, k_p \geq 0$, a $L(k_1, k_2, \ldots, k_p)$-labeling of G assigns real numbers $f(x) \geq 0$ to the vertices x, such that the labels of vertices u and v differ by at least k_i if u and v are at distance i apart. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum span over such labelings f. We survey this new theory of real number labelings. When $p = 2$ it is enough to determine $\lambda(G; k, 1)$ for reals $k \geq 0$, which will be a piecewise linear function. We present the function for the square lattice (grid) and for the hexagonal lattice. For the triangular lattice, we have also solved it except for the range $1/2 \leq k \leq 4/5$.

1. Introduction

As wireless networks continue to grow rapidly and the radio frequency spectrum remains a scarce resource, efficient channel assignment algorithms are increasingly important.

The channel assignment problem is to assign channels to the transmitters in a network in a way which avoids interference and uses the spectrum as efficiently as possible. We consider the version suggested by Roberts (see [12]) in which the assignment must satisfy some separation constraints depending on the distance, and the goal is to make the assignment bandwidth as small as possible. The problem is modeled nicely with graph theory by letting each transmitter correspond to a vertex and representing by an edge each pair of nearby transmitters.

A $L(k_1, k_2, \cdots, k_p)$-labeling of a graph G is an assignment of nonnegative numbers to the vertices of G, with $x \in V(G)$ labeled by $f(x)$, such that $|f(u) - f(v)| \geq k_i$ if u and v are at distance i apart. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum span over such f, i.e., the $L(k_1, k_2, \cdots, k_p)$-labeling number of graph G, where the span is the difference between the supremum and the infimum of the labels $f(x)$.

Griggs and Yeh [12] (1992) introduced integer $L(k_1, k_2, \ldots, k_p)$-labelings of graphs, where all labels are integers, and they obtained many results for the particular case $L(2, 1)$.

Often the frequency channel separations k_i for nearby two transmitters are inversely proportional to the distance i between the two transmitters, so that we usually have $k_1 \geq k_2 \geq \ldots \geq k_p$. But this is not required in our theory.

The wireless networks include cellular mobile networks, wireless computer networks [3], wireless ATM networks [17], private mobile radio networks [23]. Bertossi, Maurizio and Bonuccelli [3](1995) introduced a kind of integer "control code" assignment in Packet Radio Networks to avoid hidden terminal interference, due to the stations (transmitters) which are outside the hearing range of each other and transmit to the same receiving stations (transmitters), i.e., graph $L(0, 1)$ labeling problem. Another engineering problem is to assign time slots without interference [2], which is modeled very well by graph labeling problems. Different channel assignment problems in the frequency, time and code domains (with a channel defined as a frequency, a time slot [2], or a control code [3], resp.) can be modeled by graph labeling problems. Ramanathan [20] mentions a unified framework of channel assignments by the similarity of the constraints for the channel assignment across these domains. Here we consider $L(k_1, k_2, \ldots, k_p)$ for all nonnegative real numbers $k_1, k_2, \ldots, k_p$.

Among all Δ-regular planar lattices (grids), we have $\Delta = 3, 4$ or 6. We determine the minimum spans $\lambda(G; k, 1)$ for the corresponding lattices.

2. Real Number Graph Labelings

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Since we can use any frequencies (channels) in the available continuous frequency spectrum instead of only discrete frequency channels, we extend the integer graph labeling problem to the real number graph labeling problem in which we allow the labels and constraints k_i to be nonnegative real numbers. Hence, the infimum span $\lambda(G; k_1, k_2, \ldots, k_p)$ may not be integer for some graphs G. For example, $\lambda(P_4; \sqrt{2}, 1) = \sqrt{2} + 1$ is not an integer for a path P_4 on four vertices.

To describe optimal real number labelings of graphs, we define the D-set for $k_1, k_2, \ldots, k_p$ to be the set of linear combinations $\sum_i a_i k_i$ of k_i with nonnegative integer coefficients a_i. We prove the existence of some optimal labeling $f \in L(k_1, k_2, \ldots, k_p)(G)$ with smallest label 0 and all labels in the D-set, and hence the span $\lambda(G; k_1, k_2, \ldots, k_p)$ exists in the D-set, with minimum instead of infimum, for G being any graph with finite maximum degree. We cannot ensure the existence of $\lambda(G; k_1, k_2, \ldots, k_p)$ for an infinite graph G with infinite maximum degree.

Theorem 1 (The D-set Theorem). Let G be a finite or infinite graph with maximum degree Δ. Let $k_i \in [0, \infty)$, $i = 1, 2, \ldots, p$. Then there exists a finite optimal $L(k_1, k_2, \ldots, k_p)$-labeling $f^* : V(G) \to [0, \infty)$ with the smallest label 0 and all labels in the D-set for $k_1, k_2, \ldots, k_p$. Hence $\lambda(G; k_1, k_2, \ldots, k_p) \in D_{k_1, k_2, \ldots, k_p}$.

For nonnegative integer separations k_i, $\lambda(G; k_1, k_2, \ldots, k_p)$ is a nonnegative integer which is attained by some optimal integer labeling. All previous optimal integer labeling results are compatible with our optimal real number labeling results.

In [16] we prove $\lambda(G; k_1, k_2, \ldots, k_p)$ is a nondecreasing, continuous function of real numbers k_i for a graph G with finite maximum degree. Hence the results $\lambda(G; k_1, k_2, \ldots, k_p)$ for k_i being integers or rational numbers can be pushed into the results for k_i being real numbers.

We also prove in [16] that $\lambda(G; k_1, k_2, \ldots, k_p)$ is a piecewise linear function of k_i with nonnegative integer coefficients and finitely many linear pieces for arbitrary p and any finite graph G, or for any infinite graph G with finite maximum degree when $p = 2$.

From our definition we have

The Scaling Property. We have $\lambda(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \lambda(G; k_1, k_2, \ldots, k_p)$ for real numbers $d, k_i \geq 0, i = 1, 2, \ldots, p$.

In particular, $\lambda(G; k_1, k_2) = k_2 \lambda(G; k, 1)$ for real numbers $k = k_1/k_2$ and $k_2 > 0$. We will give the minimum label span $\lambda(G; k, 1), k \geq 0$ for some infinite regular graphs, lattices.

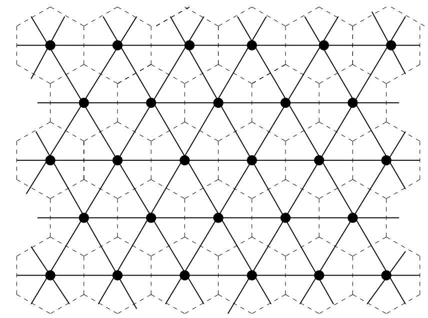

Figure 1: The Hexagonal Cell Covering

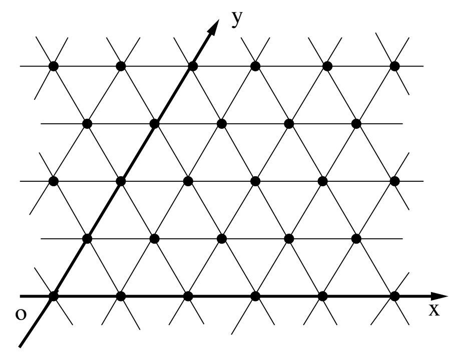

Figure 2: The Triangular Lattice $\Gamma_\triangle$

3. The Triangular Lattice

In a radio mobile network, large service areas are often covered by a network of almost congruent polygonal cells, with each transmitter at the center of a cell that it covers. A honeycomb of hexagonal cells provides the most economic covering of the whole plane [9] (i.e., cover the whole plane with smallest possible transmitters density), where the transmitters are placed in the triangular lattice $\Gamma_\triangle$ (see Figure 2). We fix a point to be the original point o and put a xoy coordinate system so that we can name each point by its xoy coordinate. We may place the transmitters in some subgraph of the triangular lattice for a big open area.

Griggs [13] formulated an integer $L(k, 1)$-labeling problem on the triangular lattice $\Gamma_\triangle$for 2000 International Math Contest in Modeling. Among 243 teams which worked on this problem in four days, five teams [19, 5, 21, 8, 11](2002) won the contest and got their papers published. They gave results for $\lambda(\Gamma_\triangle; k, 1)$ for $k = 2, 3$, and most of them gave the upper bounds for $k \geq 4$ or $k = 1$. Af-

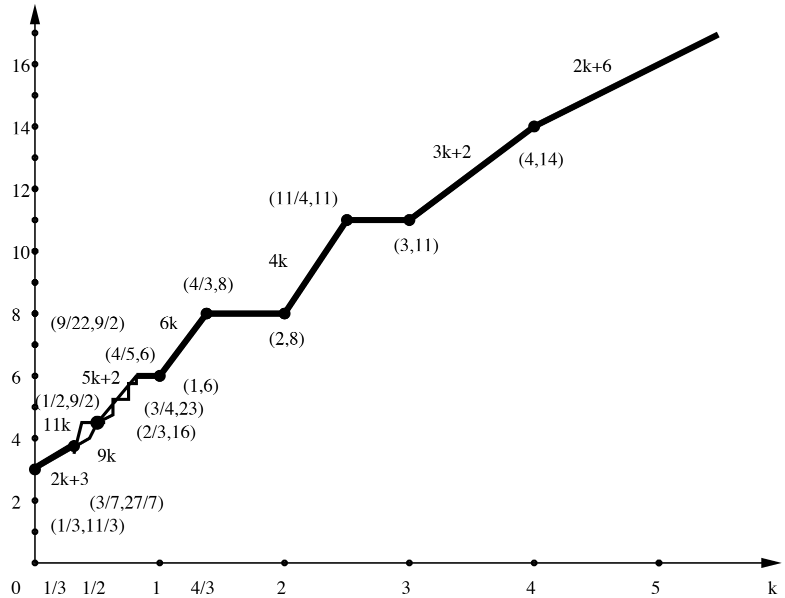

Figure 3: $\lambda(\Gamma_\triangle; k, 1)$.

ter that, Yeh [15] and Zhu and Shi [24] solved some special cases for the integer $L(k_1, k_2)$ labeling problems on the triangular lattice, for integers $k_1 \geq k_2$. Calamoneri [6] gives the minimum integer span for the triangular lattice for integers $k_1 \geq 3k_2$, and the bounds for $k_2 \leq k_1 \leq 3k_2$ independently of us.

Here we describe the full solution for $k \geq 1$ and the best current bounds we have for $k \leq 1$ (see Figure 3). In Section 6 we describe some of the proofs of this result.

Theorem 2. For real numbers $k \geq 0$ and the triangular lattice $\Gamma_\triangle$, we have the following minimum span:

$$\lambda(\Gamma_\triangle; k, 1) \begin{cases} = 3 & \text{if } k = 0 \\ = 2k+3 & \text{if } 0 \leq k \leq \frac{1}{3} \\ \in [2k+3, 11k] & \text{if } \frac{1}{3} \leq k \leq \frac{9}{22} \\ \in [9k, 11k] & \text{if } \frac{9}{22} \leq k \leq \frac{11}{27} \\ \in [9k, \frac{9}{2}] & \text{if } \frac{11}{27} \leq k \leq \frac{1}{2} \\ \in [\frac{9}{2}, 5k+2] & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ \in [\frac{16}{3}, 5k+2] & \text{if } \frac{2}{3} \leq k \leq \frac{3}{4} \\ \in [\frac{23}{4}, 5k+2] & \text{if } \frac{3}{4} \leq k \leq \frac{4}{5} \\ = 6 & \text{if } \frac{4}{5} \leq k \leq 1 \\ = 6k & \text{if } 1 \leq k \leq \frac{4}{3} \\ = 8 & \text{if } \frac{4}{3} \leq k \leq 2 \\ = 4k & \text{if } 2 \leq k \leq \frac{11}{4} \\ = 11 & \text{if } \frac{11}{4} \leq k \leq 3 \\ = 3k+2 & \text{if } 3 \leq k \leq 4 \\ = 2k+6 & \text{if } k \geq 4 \end{cases}$$

4. The Square Lattice

Inside cities, due to the high buildings which are obstacles in the signal path (as well as a limited range of a cell), a Manhattan fashion cellular system [4] can be modeled by the square lattice $\Gamma_\square$ (see Figure 4). Many graphs corresponding to cellular systems are the induced subgraphs of the square lattice or the triangular lattice. Calamoneri [6] gives the minimum integer span for the square lattice for

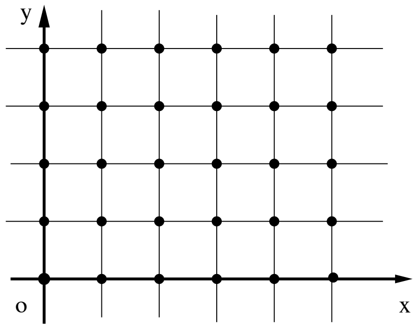

Figure 4: The Square Lattice $\Gamma_\Box$

integers $k_1 \geq 3k_2$, and the bounds for $k_2 \leq k_1 \leq 3k_2$ independently of us.

Theorem 3. For real numbers $k \geq 0$ and the square lattice $\Gamma_\Box$, we have the following minimum span.

$$\lambda(\Gamma_\Box; k, 1) = \begin{cases} k+3 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 7k & \text{if } \frac{1}{2} < k \leq \frac{4}{7} \\ 4 & \text{if } \frac{4}{7} \leq k < 1 \\ 4k & \text{if } 1 \leq k \leq \frac{4}{3} \\ k+4 & \text{if } \frac{4}{3} < k \leq \frac{3}{2} \\ 3k+1 & \text{if } \frac{3}{2} < k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 < k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \\ 2k+2 & \text{if } 3 \leq k \leq 4 \\ k+6 & \text{if } k \geq 4 \end{cases}$$

5. The Hexagonal Lattice

One may place the transmitters at nodes in the hexagonal lattice Γ_H (see Figure 6), which is the dual of the triangular lattice. Calamoneri [7] gives the minimum span for the hexagonal lattice for $k \geq 2$ and the bounds for $1 \leq k \leq 2$. We finish all the cases for $k \geq 0$ in [16].

Theorem 4. For real numbers $k \geq 0$ and the hexagonal lattice Γ_H we have the following minimum span.

$$\lambda(\Gamma_H; k, 1) \begin{cases} = k+2 & \text{if } 0 \leq k \leq \frac{1}{2} \\ = 5k & \text{if } \frac{1}{2} \leq k \leq \frac{3}{5} \\ = 3 & \text{if } \frac{3}{5} \leq k \leq 1 \\ = 3k & \text{if } 1 \leq k \leq \frac{5}{3} \\ = 5 & \text{if } \frac{5}{3} \leq k \leq 2 \\ = 2k+1 & \text{if } 2 \leq k \leq 3 \\ = k+4 & \text{if } k \geq 3 \end{cases}$$

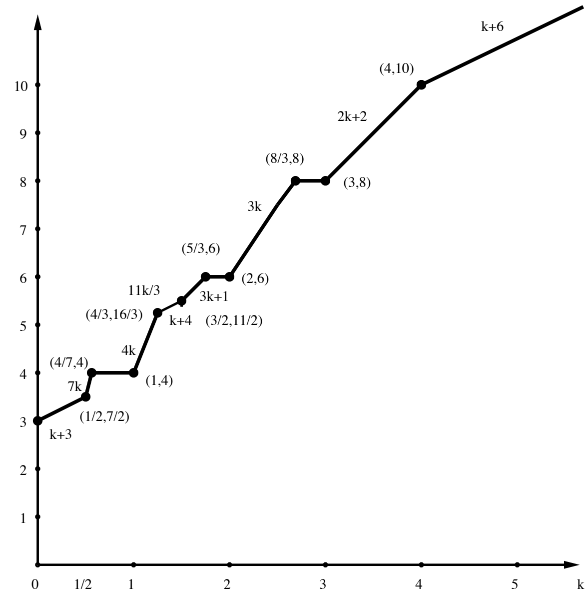

Figure 5: $\lambda(\Gamma_\Box; k, 1)$

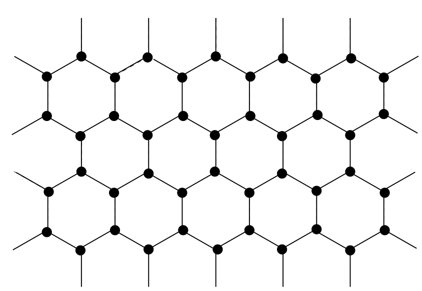

Figure 6: The Hexagonal Lattice Γ_H

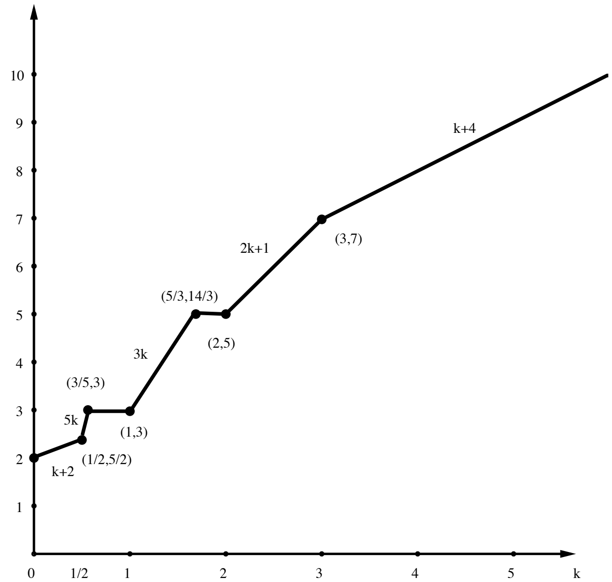

Figure 7: $\lambda(\Gamma_H; k, 1)$

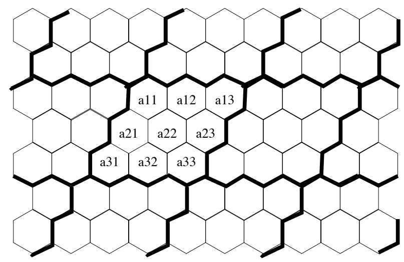

Figure 8: The Matrix Labeling

6. Proofs for the Triangular Lattice

Generally, we get upper bounds by the constructions of the labelings, and lower bounds by contradictions on induced subgraphs.

We will find the upper bound on $\lambda(\Gamma_\triangle; k, 1)$, $k \geq 1$, by constructions and Lemma 7 below. One method is to tile the whole lattice by a labeled parallelogram described by a matrix of labels.

Definition. Given an $m \times n$ labeling matrix $A := [a_{i,j}]$, we label point (i, j) by $a_{n-j (\text{mod } n), i+1 (\text{mod } m)}$, for $i, j \in \mathbb{Z}$.

For example, if we have labeling matrix $A := [a_{i,j}]_{3\times 3}$, then Figure 8 shows how the labels are assigned, where $a_{3,1}$ is at the vertex with coordinates $(0, 0)$ in the triangular lattice. The whole lattice is tiled with copies of the 3×3 tile shown.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Proposition 5 [19, 11]. For $3 \leq k \leq 4$, $k \in \mathbb{R}$, we have $\lambda(\Gamma_\triangle; k, 1) \leq 3k + 2$.

Proof [19, 11]: For $3 \leq k \leq 4$, we get the upper bound by defining labeling matrix $A := [a_{i,j}]_{3\times 4}$, where

$$A = \begin{bmatrix} 3k & 0 & k & 2k \\ 1 & k+1 & 2k+1 & 3k+1 \\ k+2 & 2k+2 & 3k+2 & 2 \end{bmatrix}$$

$\square$

Proposition 6 [19, 5, 21, 8, 11]. For $k \geq 4$, $k \in \mathbb{R}$, we have $\lambda(\Gamma_\triangle; k, 1) \leq 2k + 6$.

Proof [19, 5, 21, 8, 11]: We get the upper bound by defining labeling matrix $A := [a_{i,j}]_{3\times 3}$ where

$$A = \begin{bmatrix} 2k+5 & 0 & k+4 \\ 1 & k+2 & 2k+6 \\ k+3 & 2k+4 & 2 \end{bmatrix}$$

$\square$

Note if we find some upper (or lower) bound on some $k_1 = a$, then we can get some upper (or lower) bounds for $k_1 \leq a$ or $k_1 \geq a$. For the case $p = 2$, we have the following results (see Figure 9), by which we prove the bounds later .

Lemma 7. Let $a > 0$.
If $\lambda(G; a, 1) \leq b$, then
$$\lambda(G; k, 1) \leq \begin{cases} b & \text{if } 0 \leq k \leq a \\ \frac{b}{a}k & \text{if } k \geq a \end{cases}$$
If $\lambda(G; a, 1) \geq b$, then
$$\lambda(G; k, 1) \geq \begin{cases} \frac{b}{a}k & \text{if } 0 \leq k \leq a \\ b & \text{if } k \geq a \end{cases}$$
Hence, if $\lambda(G; a, 1) = b$, then $\frac{b}{a}k \leq \lambda(G; k, 1) \leq b$, while if $0 \leq k \leq a$ and $b \leq \lambda(G; k, 1) \leq \frac{b}{a}k$ if $k \geq a$.

Proof: If $\lambda(G; a, 1) \leq b$, we have:

For $0 \leq k \leq a$, the result follows from the fact $\lambda(G; k, 1)$ is nondecreasing.

For $k \geq a$, $1 \leq \frac{k}{a}$, we have $\lambda(G; k, 1) \leq \lambda(G; k, \frac{k}{a}) = \frac{k}{a}\lambda(G; a, 1) \leq \frac{b}{a}k$.

The proof is similar, if $\lambda(G; a, 1) \geq b$. $\square$

For positive integers k_1, k_2, if we have a feasible labeling $f \in L(k_1, k_2)$ with arithmetic progression $f(i, j) \equiv ai + bj \pmod{l}$ for some positive integers a, b, l, then $\lambda(\Gamma_\triangle; k_1, k_2) \leq l - 1$. Hence we get $\lambda(\Gamma_\triangle; k, 1) \leq (l-1)/k_2$, where $k = k_1/k_2$. We wrote a computer program in C language to find feasible $L(k_1, k_2)$-labelings by arithmetic progression, which

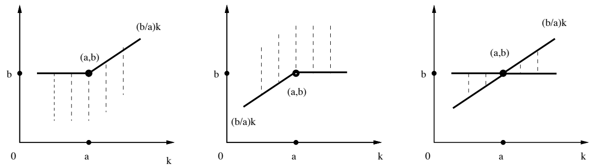

Figure 9: $\lambda(G;k,1)$

are better to describe as follows. Then by Lemma 7, we have the results.

Proposition 8 [15]. We have $\lambda(\Gamma_\triangle;\frac{1}{2},1) \leq \frac{9}{2}$. Hence, $\lambda(G;k,1) \leq \frac{9}{2}$ if $0 \leq k \leq 1/2$.

Proof : We get the upper bound $\lambda(\Gamma_\triangle;1,2) \leq 9$ by labeling point (i,j) by $i+4j(\text{mod } 10)$. □

Proposition 9. We have $\lambda(\Gamma_\triangle;1,1) = 6$. Hence, $\lambda(G;k,1) \leq \begin{cases} 6 & \text{if } \frac{4}{5} \leq k \leq 1 \\ 6k & \text{if } 1 \leq k \leq \frac{4}{3} \end{cases}$

Proof: The lower bound comes from $\lambda(B_7;1,1) = 6$.

We get the upper bound by defining integer labeling $f(i,j) \equiv i+3j \pmod 7$, which is unique by the symmetry of the Triangular Lattice. □

Proposition 10. We have $\lambda(\Gamma_\triangle;2,1) \leq 8$. Hence, $\lambda(G;k,1) \leq \begin{cases} 8 & \text{if } \frac{4}{3} \leq k \leq 2 \\ 4k & \text{if } 2 \leq k \leq \frac{11}{4} \end{cases}$

Proof: We get the upper bound by defining the integer labeling $f(i,j) \equiv 2i+5j \pmod 9$, which is unique by the symmetry of the Triangular Lattice. □

We have $\lambda(\Gamma_\triangle;3,1) \leq 11$ by Proposition 5. Hence, the following proposition follows by Lemma 7

Proposition 11. We have $\lambda(\Gamma_\triangle;k,1) \leq 11$, if $\frac{11}{4} \leq k \leq 3$.

We verify the lower bounds by contradiction and Lemma 7. Sometimes we wrote C programs to verify all possible cases. We give two kinds of proofs for lower bounds.

The Symmetry Argument [5]. Let G be a graph and $k_1,k_2,\ldots,k_p \in [0,\infty)$. If there exists a $L(k_1,k_2,\ldots,k_p)(G)$ labeling f using label x, then there exists a $L(k_1,k_2,\ldots,k_p)(G)$ labeling using label $\text{span}(f)-x$ with the same span, where $\text{span}(f) = \max\{f(v)\} - \min\{f(v)\}$, for all vertices $v \in V(G)$.

Proposition 12 [11]. For integers $k \geq 6$, we have $\lambda(\Gamma_\triangle;k,1) \geq 2k+6$. $\lambda(\Gamma_\triangle;k,1) = 2k+6$.

We drew some ideas from [24] for the proof of the following Proposition. We used a program to finish the proof of the propositions (to find contradictions for all possible cases if the span is less than the lambda number).

Proposition 13 [16]. We have $\lambda(\Gamma_\triangle;4,3) \geq 24$, i.e., $\lambda(\Gamma_\triangle;\frac{4}{3},1) = 8$.
Hence, $\lambda(\Gamma_\triangle;k,1) = \begin{cases} 6k & \text{if } \frac{3}{4} \leq k \leq \frac{4}{3} \\ 8 & \text{if } \frac{4}{3} \leq k \leq 2 \end{cases}$

Proof: It suffices to prove $\lambda(\Gamma_\triangle;4,3) \geq 24$. Assume $\lambda(\Gamma_\triangle;4,3) < 24$. By the D-set Theorem, there exists optimal labeling $f \in L(4,3)(\Gamma_\triangle)$ with the smallest label 0 and all integer labels between 0 and 23 in the D-set. $\text{span}(f) = 23$.

Claim 1. Labeling f cannot use label 3.

Proof of Claim 1: Assume f uses label 3 at some vertex v. By the conditions, the six distinct labels around v belong to $\{7,8,\ldots,23\}$, the difference between any pair of them is at least 3.

Let $B_7 (B_{17}, B_{37}$ resp.) be the induced subgraph of $\Gamma_\triangle$ by all vertices which are at distance at most one (two, three resp.) from the vertex v.

Consider all possible labelings on B_7 with center label 3, by using a computer program we wrote, we cannot find feasible labelings on B_{17} on most of the cases, except five cases [16]. But for the five cases with feasible labelings on B_{17}, we cannot label B_{37} without contradictions. Thus integer labeling f cannot use label 3. □

By the Symmetry Argument, labeling f cannot use label $20 = 23-3$, which is the span of f less 3. Now f has no label $3, 20$.

Claim 2. Labeling f cannot use label 7, and hence no label $16 = 23-7 = \text{span}(f)-7$.

Proof of Claim 2: Assume f uses label 7 at some vertex v. Let the six distinct labels around v be $x_1 < x_2 \cdots < x_6$. By the condition, $x_{i+1} \geq x_i+3$ for $i = 1,2,\ldots,5$, and $x_i \leq 2 < 3 = 7-4$ (because no label 3), $x_i \geq 11 = 7+4$. Now $x_i \in \{0,1,2,11,12,\ldots,19,21,22,23\}$ for $i = 1,2,\ldots,6$. Then $x_1 = 0, 1$, or 2, $x_2 \geq 11$, $x_3 \geq 14$, $x_4 \geq 17$, $x_5 \geq 21$ (because no label 20), $x_6 \geq 24$. Contradiction. □

Now f has no label $3, 7, 16, 20$.

Claim 3. Labeling f cannot use label 6, and hence no label $17 = 23-6 = \text{span}(f)-6$.

Claim 4. Labeling f cannot use label 10, and hence no label $13 = 23-10 = \text{span}(f)-10$.

Claim 5. Labeling f cannot use label 11, and hence no label $12 = 23-11 = \text{span}(f)-11$.

Proof of Claim 3, 4, 5: Similar to the proof of Claim 2. □

Now the set of all possible labels is:
$\{0, 1, 2, 4, 5, 8, 9, 14, 15, 18, 19, 21, 22, 23\}$.

We cannot find seven distinct labels for subgraph B_7, such that the difference between any two of them is at least 3. It gives a contradiction. Thus $\lambda(\Gamma_\triangle; 4, 3) \geq 24$. $\square$

By the similar argument, we have the following propositions.

Proposition 14 [15]. We have $\lambda(\Gamma_\triangle; 1, 2) \geq 9$, i.e., $\lambda(\Gamma_\triangle; \frac{1}{2}, 1) \geq \frac{9}{2}$. Hence, by Lemma 7,

$$\lambda(\Gamma_\triangle; k, 1) = \begin{cases} 9k & \text{if } \frac{3}{7} \leq k \leq \frac{1}{2} \\ \frac{9}{2} & \text{if } \frac{1}{2} \leq k \leq \frac{3}{4} \end{cases}$$

For the remaining cases, we eliminate the intervals to reach the contradiction.

Proposition 15. For $3 < k < 4$, we have $\lambda(\Gamma_\triangle; k, 1) \geq 3k + 2$. Hence $\lambda(\Gamma_\triangle; k, 1) = 3k + 2$.

Proof: Assume $\lambda(\Gamma_\triangle; k, 1) = l < 3k+2$, By the D-set Theorem, there is an optimal labeling $f \in L(k, 1)(\Gamma_\triangle)$ with $\text{span}(f) = l < 3k + 2$ and the smallest label zero.

Claim 1. Labeling f can not use label in the interval $[k - 1, k)$.

Proof of Claim 1: Assume $f(v) \in [k - 1, k)$ for some $v \in V(\Gamma_\triangle)$. The six distinct labels around v are $\geq f(v) + k$. Since the induced subgraph by $N(v)$ is cycle C_6 on six vertices, $\lambda(\Gamma_\triangle; k, 1) \geq f(v) + k + \lambda(C_6; k, 1) \geq (k - 1) + k + (k + 3) = 3k + 2$ (because $\lambda(C_6; k, 1) = k + 3$ for $k \geq 3$). It gives a contradiction.$\square$

Thus, $f(v) \notin [k - 1, k)$ for all $v \in V(\Gamma_\triangle)$. By the symmetry argument, $f(v) \notin (l - k, l - k + 1]$ for all $v \in V(\Gamma_\triangle)$.

Now, $f(v) \in I_1 \cup I_2 \cup I_3 = [0, k - 1) \cup [k, l - k] \cup (l - k + 1, l]$ for all $v \in V(\Gamma_\triangle)$, where $I_1 = [0, k - 1), I_2 = [k, l - k], I_3 = (l - k + 1, l]$. Then $|I_1| = k - 1 < k, |I_3| = k - 1 < k$.

Claim 2. Labeling f cannot use labels in the interval $[k, k + 1)$.

Claim 3. Labeling f cannot use labels in the interval $[k + 1, k + 2)$.

Proof of Claims 2, 3: Similar to the proof of Claim 1. $\square$

Thus, $f(v) \notin [k-1, k+2)$ for all $v \in V(\Gamma_\triangle)$. By the symmetry argument, $f(v) \notin (l-k-2, l-k+1]$ for all $v \in V(\Gamma_\triangle)$.

Now, $f(v) \in I_1 \cup I_2'' \cup I_3 = [0, k-1) \cup [k+2, l - k - 2] \cup (l - k + 1, l]$ for all vertices v, where $I_2'' = [k+2, l-k-2]$. Then $|I_2''| \leq l-2k-4 < k-2 < 2$ for $k < 4$.

Let u be a vertex with label zero. Among the six distinct labels around u (the difference between any pair of them is at least 1 by the constraints), no label is in $I_1 = [0, 2)$ (because $|I_1| < k$), at most two labels are in $I_2'' = [k+2, l-k-2]$ (because $|I_2''| < 2$), and at most three labels are in $I_3 = (l - k + 1, l]$ (because $|I_3| < k$,these labels can not be adjacent each other). It gives a contradiction. $\square$

7. Further Research

After finding the minimum label span, we want to find all optimal labelings to see what they look like. We find all integer optimal $L(0, 1), L(1, 1), L(2, 1)$ labelings for the triangular lattice. We want to find some properties of optimal labelings of the triangular lattice (or any other infinite regular lattice), which help us to find optimal labelings in straightforward ways. We will discuss these properties and the relations among them in a later paper.

Bertossi, Pinotti, Tan [4] give the result $\lambda(\Gamma_\triangle; 2, 1, 1) = 11$. We may seek real number values $\lambda(G; k_1, k_2, k_3)$ for the triangular lattice or the square lattice in the future.

Since the graph labeling problem comes from the radio channel assignment problem in wireless communications, we wish to promote its applications by communication with the field engineers. So we offer our results here.

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