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Real number channel assignments
for lattices

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Real Number Channel Assignments for Lattices*

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Abstract

Numerical channels must be assigned to each transmitter in a large regular array such that multiple levels of interference, which depend on the distance between transmitters, are avoided by sufficiently separating the channels. The goal is to find assignments that minimize the span of the labels used. Our previous paper introduced a model for this problem using real number labelings of (possibly infinite) graphs G. Given reals $k_1, k_2, \ldots, k_p \geq 0$, we denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum of the spans of the labelings f of the vertices v of G, such that for any two vertices v and w, the difference in their labels is at least k_i, where i is the distance between v and w in G. When $p = 2$, it is enough to determine $\lambda(G; k, 1)$ for reals $k \geq 0$; For G of bounded maximum degree, this will be a continuous, piecewise linear function of k. Portions of it have been obtained by other researchers for infinite regular lattices that model large planar networks. Here we present the complete function $\lambda(G; k, 1)$, for G being the square lattice or the hexagonal lattice. For the triangular lattice, we have solved it, except for the range $1/3 \leq k \leq 4/5$.

1 Introduction.

Efficient channel assignment algorithms are increasingly important, as wireless networks rapidly proliferate while the radio frequency spectrum remains a scarce resource. It is often the situation that there is a large network of transmitters in the plane, and a numerical channel must be assigned to each transmitter, where channels for nearby vertices must be assigned so as to avoid interference. The goal is to minimize the portion of the frequency

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spectrum that must be allocated to the problem, so it is desired to minimize the span of a feasible labeling.

Hale [19] (1980) is credited with originally formulating such channel assignment problems in network engineering as graph labeling problems: Each transmitter is represented by a vertex, and any pair of vertices that may interfere is represented by an edge in the graph. All labels are integers. In his T-coloring model, there is a specified set T of integers of forbidden differences (that depends on the type of network): For any pair of adjacent vertices, the difference between their labels cannot belong to set T. This problem interested many graph theorists. See Roberts [27] for a survey of the research on the T-coloring problem.

In 1988 Lanfear proposed to Roberts [28] a new 2-level channel assignment problem of interest to NATO, in which integer labels are assigned to transmitters in the plane, with two levels of interference, depending on the distance between transmitters, say labels differ by at least two (respectively, one) when the transmitters are within some fixed distance A (resp., $2A$). In the language of communications engineering, what is happening is this: First, the *cochannel constraint* does not allow the same channel to be assigned to pairs of transmitters that are too close. Second, the *adjacent channel constraint* forbids the same or consecutive integer channels to be assigned to transmitters that are very close.

Griggs [18](1988) proposed studying the graph-theoretic analogue of the problem, which he extended in the natural way, by specifying separations $k_1, \ldots, k_p$ for vertices at distances $1, \ldots, p$: Specifically, we say a $L(k_1, k_2, \cdots, k_p)$**-labeling** of a graph G is an assignment of nonnegative numbers $f(v)$ to the vertices v of G, such that $|f(u) - f(v)| \geq k_i$ if u and v are at distance i in G. We say that labeling f belongs to the set $L(k_1, k_2, \cdots, k_p)(G)$. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the minimum span over such f, where the span is the difference between the largest and smallest labels $f(v)$. Griggs and Yeh [18] concentrated on the fundamental case of $L(2,1)$-labelings, and many authors have subsequently contributed to the literature on these labelings (see [14, 16, 21]). Increasing attention has been paid recently to more general $L(k_1, k_2, \cdots, k_p)$-labelings.

The frequency channel separations k_i for two transmitters are often inversely proportional to the distance i between them [4]. Most articles assume that the separations are nonincreasing, $k_1 \geq k_2 \geq \ldots \geq k_p$. But this is not required in our theory, and there are different settings in which these labelings are a good model, but without the added assumption on the separations k_i.

Wireless networks include cellular mobile networks, wireless computer networks [2], wireless ATM networks [23], and private mobile radio networks [30]. Bertossi and Bonuccelli [2] (1995) introduced an integer "control code" assignment in packet radio networks of computers to avoid hidden terminal interference. This occurs for stations (transmitters), which are outside the hearing range of each other, that transmit to the same receiving stations: It is the $L(0,1)$ graph-labeling problem.

Different channel assignment problems in the frequency, time and code domains (with a channel defined as a frequency, a time slot [1], or a control code [2]) can be modeled by graph labelings. Ramanathan [26] formulated a framework of channel assignments unified by the similarity of the constraints across these domains.

Since we can use any frequencies (channels) in the available continuous frequency spectrum, not only from a discrete set, Griggs [16] extended integer graph labelings to allow the labels and separations k_i to be nonnegative *real* numbers. We use the same notation as before, $L(k_1, \ldots, k_p)(G)$ and $\lambda(G; k_1, \ldots, k_p)$, but now the span of a real labeling is the difference between the supremum and the infimum of the labels used, and λ is the infimum of the spans of such labelings.

For graphs of bounded maximum degree, Griggs and Jin proved the existence of an optimal labeling of a nice form, in which all labels belong to the discrete set, denoted by $D(k_1, k_2, \ldots, k_p)$, of linear combinations $\sum_i a_i k_i$, with nonnegative integer coefficients a_i. We cannot ensure the existence of finite $\lambda(G; k_1, k_2, \ldots, k_p)$ for an infinite graph G without some restriction, such as on the degrees.

Theorem 1.1 (The D-Set Theorem [16]). *Let G be a graph, possibly infinite, with finite maximum degree. Let real numbers $k_i \geq 0$, $i = 1, 2, \ldots, p$. Then there exists a finite optimal $L(k_1, k_2, \ldots, k_p)$-labeling $f^* : V(G) \to [0, \infty)$ in which the smallest label is 0 and all labels belong to the set $D(k_1, k_2, \ldots, k_p)$. Hence, $\lambda(G; k_1, k_2, \ldots, k_p)$ belongs to $D(k_1, k_2, \ldots, k_p)$.*

Due to the D-set Theorem, previous optimal integer labeling results are compatible with our optimal real number labeling results. Some natural properties of distance-constrained labelings become more evident in the setting of real number labelings. In particular, we observe the following

Proposition 1.2 (Scaling Property). *For real numbers $d, k_i \geq 0$, $i = 1, 2, \ldots, p$,*

$$\lambda(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \lambda(G; k_1, k_2, \ldots, k_p).$$

In [16, 21] we proved $\lambda(G; k_1, k_2, \ldots, k_p)$ is a continuous function of the separations k_i for any graph G with finite maximum degree. Hence, results about the minimum spans $\lambda(G; k_1, k_2, \ldots, k_p)$ for k_i being rational numbers can often be extended into the results for k_i being real numbers. Indeed, by Scaling, it is usually enough to obtain results for integer k_i. But the analysis is more clear, and more results emerged, by considering real number labelings.

For any fixed p and any graph G with finite maximum degree, we conjectured [16] that $\lambda(G; k_1, k_2, \ldots, k_p)$ is a piecewise linear function of real numbers k_i, where the pieces have nonnegative integer coefficients and where there are only finitely many pieces. We proved this if G is finite or if $p = 2$.

By Scaling, we have that for $k_2 > 0$, $\lambda(G, k_1, k_2) = k_2 \lambda(G; k, 1)$, where $k = k_1/k_2$. This reduces the two-parameter function to a one parameter function, $\lambda(G; k, 1)$, $k \geq 0$. As just discussed, we can be sure it is a continuous, nondecreasing, piecewise linear function with finitely many pieces. Further, each piece has the form $ak + b$ for some nonnegative integers $a, b \geq 0$.

In this paper, we will discuss the function $\lambda(G; k, 1), k \geq 0$, for some infinite regular planar lattices. Despite their nice properties, we shall see that the functions are surprisingly complicated. So the proofs are necessarily complicated and require some care and detail.

The next section introduces some of the general methods used to obtain optimal lattice labelings. It also reviews some of the known results for labeling infinite trees with conditions at distance two, which are closely related to the lattice results.

The three following sections contain our results for the triangular, square, and hexagonal lattices. The detailed proofs for the three lattices, which make up most of the paper, are presented in the next three sections. Note that for the sake of brevity, we omit the details for cases which are very similar to ones already presented; the reader is referred to [21] for complete details in those cases.

The paper concludes with a brief section describing directions for future research. The problem of optimal labelings for lattices with conditions to distance three may be too broad to solve in general. We show how to extend the main known result, $\lambda(\Gamma_\Delta; 2, 1, 1) = 11$ due to Bertossi *et al.* [4] to give $\lambda(\Gamma_\Delta; k, 1, 1)$ for $1 \le k \le 2$.

2 Methods

The upper bounds are generally achieved by constructing an efficient labeling, sometimes discovered by computer search. We typically coordinatize the vertices of the lattice, give an explicit labeling for a small piece, and repeat the pattern, tiling the whole lattice with congruent pieces.

The lower bound proofs seem to be more difficult. There are crucial particular values of k where we need to prove a lower bound on $\lambda(G; k, 1)$. Such k are rational, say $k = a/b$ for some integers $a, b > 0$. By Scaling, it is equivalent to bound $\lambda(G; a, b)$ below, which has the advantage that we need only consider integer $L(a, b)$-labelings, which have integer spans. We then seek to prove an integer bound, say $\lambda(G; a, b) \ge c$, by contradiction: If it is not true, then $\lambda(G; a, b) \le c - 1$, and there must exist a labeling f of G using labels from the set $\{0, 1, \ldots, c - 1\}$. We restrict f to an appropriate finite induced subgraph of G, and argue that some label, call it L, must be avoided by f. We continue to eliminate possible labels, until there remains a set of labels for which it can be shown that in fact no feasible labeling exists. In some cases we had to write a computer program to check all possible labelings from a specified label set of a particular induced subgraph.

A nice way to expand the set of avoided labels by using symmetry was observed by one of the student teams we mention at the start of the next section, Broadhurst *et al.* [5]. A similar idea, though not formulated as explicitly, was used by another student team, Goodwin *et al.* [13]. Here we state the principle in our more general setting of general graphs and distance conditions:

Property 2.1 (The Symmetry Argument). *Let S, L, and $k_1, k_2, \ldots, k_p$ be nonnegative integers, and let G be a graph. If every $L(k_1, k_2, \ldots, k_p)(G)$-labeling f into $\{0, \ldots, S\}$ avoids (respectively, uses) label L, then every such labeling f avoids (respectively, uses) label $S - L$.*

We next describe a simple method for general graphs G that is surprisingly useful. It permits us to extend a bound at some particular value a of k to general values of k:

Lemma 2.2. *Let a, b be reals with $a > 0$.*
If $\lambda(G; a, 1) \le b$, then $\lambda(G; k, 1) \le \begin{cases} b & \text{if } 0 \le k \le a \\ \frac{b}{a}k & \text{if } k \ge a \end{cases}$.
If $\lambda(G; a, 1) \ge b$, then $\lambda(G; k, 1) \ge \begin{cases} \frac{b}{a}k & \text{if } 0 \le k \le a \\ b & \text{if } k \ge a \end{cases}$.
In particular, if $\lambda(G; a, 1) = b$, then
For $0 \le k \le a$, $\frac{b}{a}k \le \lambda(G; k, 1) \le b$;
For $k \ge a$, $b \le \lambda(G; k, 1) \le \frac{b}{a}k$.

Proof: If $\lambda(G; a, 1) \le b$, we have:

- For $0 \le k \le a$, the result follows from the fact $\lambda(G; k, 1)$ is nondecreasing.
- For $k \ge a$, we also use Scaling to obtain $\lambda(G; k, 1) \le \lambda(G; k, \frac{k}{a}) = \frac{k}{a}\lambda(G; a, 1) \le \frac{b}{a}k$.

The proof is similar, if $\lambda(G; a, 1) \ge b$. ■

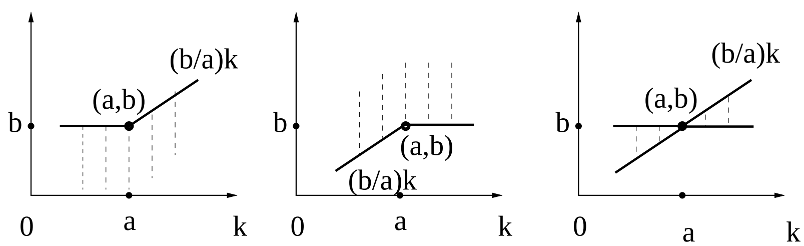


Figure 1: The bound on $\lambda(G; k, 1)$

It is interesting and productive to compare our lattice problems to those for infinite trees, so let us review results for trees. For integer $d > 0$, let T_d denote the tree that is regular of degree d. Note that T_d is infinite for $d \ge 2$ and T_2 is an infinite path. For the path P_n on n vertices, $n \ge 7$, we [21] have determined the minimum span $\lambda(P_n; k, 1), n \ge 7$ (see Figure 2).

Georges and Mauro [11] obtained the values of $\lambda(T_d; k_1, k_2)$ for integers $k_1 \ge k_2 \ge 0$. In a subsequent paper (with the same title!) Calamoneri, Pelc and Petreschi [8] gave the values for integers $0 \le k_1 \le k_2$. By continuity and scaling, these can be restated in terms of $\lambda(T_d; k, 1)$ for reals $k \ge 0$, which is neater, so we use this format here. As d grows, the functions get more and more complicated for $k \ge 1$, so we only state those for the values we require here, $d = 3, 4$:

Theorem 2.3 ([11]). *For real $k \ge 1$ we have*

$$\lambda(T_3; k, 1) = \begin{cases} 3k & \text{if } 1 \le k \le \frac{3}{2} \\ k+3 & \text{if } \frac{3}{2} < k \le 2 \\ 2k+1 & \text{if } 2 \le k \le 3 \\ k+4 & \text{if } k \ge 3 \end{cases}$$

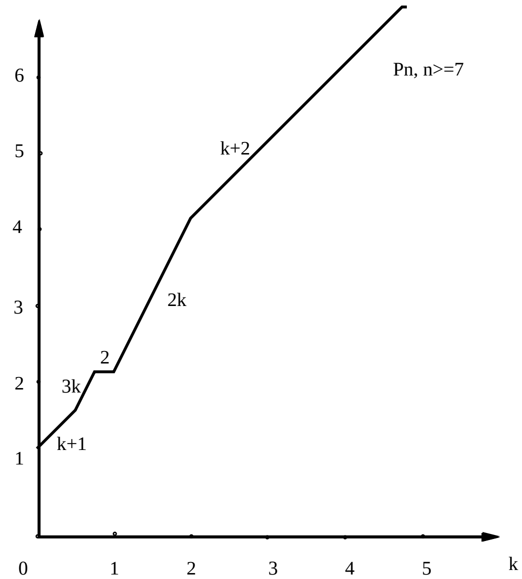


Figure 2: The minimum span $\lambda(P_n; k, 1)$ for path $P_n, n \geq 7$.

Theorem 2.4 ([11]). *For real $k \geq 1$, we have*

$$\lambda(T_4; k, 1) = \begin{cases} 4k & \text{if } 1 \leq k \leq \frac{4}{3} \\ k+4 & \text{if } \frac{4}{3} < k \leq \frac{3}{2} \\ 3k+1 & \text{if } \frac{3}{2} \leq k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 \leq k \leq \frac{5}{2} \\ k+5 & \text{if } \frac{5}{2} \leq k \leq 3 \\ 2k+2 & \text{if } 3 \leq k \leq 4 \\ k+6 & \text{if } k \geq 4 \end{cases}$$

Theorem 2.5 ([8]). *For real k, $0 \leq k \leq 1$, and integer $d \geq 2$, we have*

$$\lambda(T_d; k, 1) = \begin{cases} k + (d-1) & \text{if } 0 \leq k \leq \frac{1}{2} \\ (2d-1)k & \text{if } \frac{1}{2} < k \leq \frac{d}{2d-1} \\ d & \text{if } \frac{d}{2d-1} \leq k \leq 1 \end{cases}$$

Next we present results we need that relate the optimal spans of regular trees T_d to that of general d-regular graphs.

Theorem 2.6 ([12]). *Let G be a regular graph of degree $d \geq 2$. Then for all real $k \geq 1$, we have $\lambda(G; k, 1) \geq \lambda(T_d; k, 1)$.*

Proof: We define a graph homomorphism h from T_d to G. Begin with any arbitrary vertices $v \in V(T_d)$ and $v' \in V(G)$. Put $h(v) = v'$. Next, arbitrarily define h on the d neighbors w of v to range over the d neighbors w' of v' in G. Continue working through the vertices x of T_d in Breadth-First-Search order: Say we have $h(x) = x'$, which was defined when we considered the neighbors of some vertex y adjacent to x in T_d, with $h(y)$

denoted already by y'. Then define $h(z)$ for the other $d-1$ neighbors z of x other than y to range over the $d-1$ neighbors z' of x' in G other than y'. In particular, $h(z) \neq y'$. Continuing in this way we successively define h on all of T_d. We see that adjacent vertices of T_d are sent to adjacent vertices of G, *i.e.*, h is a homomorphism.

Suppose f' is an optimal $L(k,1)$-labeling of G. We obtain a labeling f of T_d by defining, for any vertex u of T_d, $f(u) = f'(h(u))$. If s and t are adjacent vertices in T_d, then $h(s)$ and $h(t)$ are adjacent in G, so that

$$|f(s) - f(t)| = |f'(h(s)) - f'(h(t))| \geq k.$$

On the other hand, if s and t are at distance two in T_d, we were careful to ensure that $h(s)$ and $h(t)$ would be at distance either one or two in G. Since $k \geq 1$, we find that

$$|f(s) - f(t)| = |f'(h(s)) - f'(h(t))| \geq 1.$$

Thus, f is a $L(k,1)$-labeling of T_d, so that

$$\lambda(T_d; k, 1) \leq \text{span}(f) \leq \text{span}(f') = \lambda(G; k, 1). \qquad \blacksquare$$

The condition $k \geq 1$ above is certainly necessary, since it could be for vertices s and t at distance two that $h(s)$ and $h(t)$ are adjacent, and we would only be certain that $|f(s) - f(t)| \geq k$, which is not strong enough, if $k < 1$. For instance, let $k < 1$. If $d = 2$, then T_d is an infinite path, and we may consider the 2-regular graph $G = C_3$. It is easily seen (by examining the two neighbors of a vertex with label 0) that $\lambda(T_2; k, 1) \geq 1 + k$, which exceeds $\lambda(C_3; k, 1) = 2k$.

However, if G is triangle-free, then it cannot be that $h(s)$ and $h(t)$ are adjacent in the problematic case above. We find that

Theorem 2.7. *Let G be a triangle-free regular graph of degree $d \geq 2$. Then for all real $k \geq 0$, we have $\lambda(G; k, 1) \geq \lambda(T_d; k, 1)$.* $\blacksquare$

3 The Triangular Lattice

In a radio mobile network, the large service areas are often covered by a network of nearly congruent polygonal cells, with each transmitter at the center of a cell that it covers. A honeycomb of hexagonal cells provides the most economic covering of the whole plane [10] (i.e., covers the plane with smallest possible transmitter density), where the transmitters are placed in the triangular lattice Γ_Δ (see Figure 3). We fix a point to be the original point o and impose an xoy coordinate system so that we can name each point by its xoy coordinate.

This problem has some history, owing to the fundamental nature of the triangular lattice for channel assignment problems. Griggs [14] formulated an integer $L(k,1)$-labeling problem on the triangular lattice Γ_Δ for the 2000 International Math Contest in Modeling (MCM). Among 271 teams which worked on this problem for four days and wrote papers,

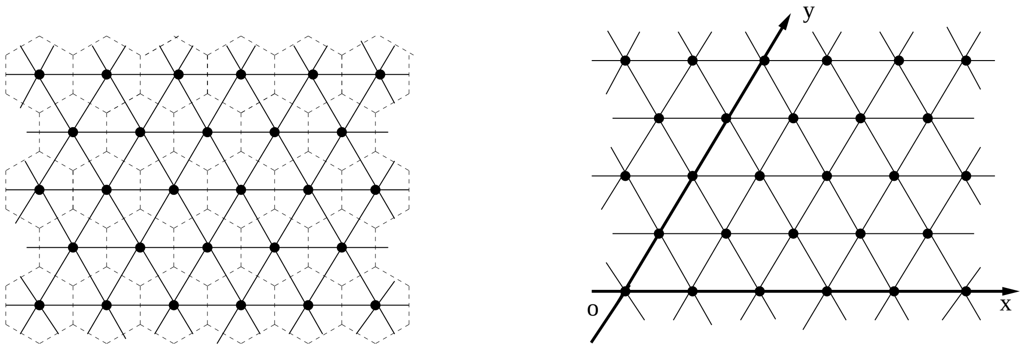


Figure 3: The Hexagonal Cell Covering and the Triangular Lattice Γ_Δ

five teams [5, 9, 13, 25, 29] won the contest and got their papers published. All winners found $\lambda(\Gamma_\Delta; k, 1)$ for $k = 2, 3$, and some gave labelings for $k = 1$ or for integers $k \geq 4$ that turn out to be optimal, but without proving the lower bound. Goodwin, Johnston and Marcus [13] proved the optimality for integers $k \geq 4$ (quite an achievement in such a short time) and considered the more general problem of $\lambda(\Gamma_\Delta; k_1, k_2)$ for integers k_1, k_2. Subsequently, Yeh [22] and Zhu and Shi [31] each solved some special cases for integers $k_1 \geq k_2$. Calamoneri [7] gave the minimum span for integers $k_1 \geq 3k_2$, and she gave bounds for $k_2 \leq k_1 \leq 3k_2$, independently of us.

Here we describe the solution of the $L(k, 1)$-labeling problem for the triangular lattice for real numbers $k \geq 1$, and we give bounds for $0 \leq k \leq 1$ (see Figure 4), where considerable effort has not yet led to a full solution. In Section 6 we describe the proof of this result.

Theorem 3.1. *For $k \geq 0$ the minimum span of any $L(k, 1)$-labeling of the triangular lattice is given by:*

$$\lambda(\Gamma_\Delta; k, 1) \begin{cases} = 2k+3 & \textit{if } 0 \leq k \leq \frac{1}{3} \\ \in [2k+3, 11k] & \textit{if } \frac{1}{3} \leq k \leq \frac{9}{22} \\ \in [2k+3, \frac{9}{2}] & \textit{if } \frac{9}{22} \leq k \leq \frac{3}{7} \\ \in [9k, \frac{9}{2}] & \textit{if } \frac{3}{7} \leq k \leq \frac{1}{2} \\ \in [\frac{9}{2}, \frac{16}{3}] & \textit{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ \in [\frac{16}{3}, \frac{23}{4}] & \textit{if } \frac{2}{3} \leq k \leq \frac{3}{4} \\ \in [\frac{23}{4}, 6] & \textit{if } \frac{3}{4} \leq k \leq \frac{4}{5} \\ = 6 & \textit{if } \frac{4}{5} \leq k \leq 1 \\ = 6k & \textit{if } 1 \leq k \leq \frac{4}{3} \\ = 8 & \textit{if } \frac{4}{3} \leq k \leq 2 \\ = 4k & \textit{if } 2 \leq k \leq \frac{11}{4} \\ = 11 & \textit{if } \frac{11}{4} \leq k \leq 3 \\ = 3k+2 & \textit{if } 3 \leq k \leq 4 \\ = 2k+6 & \textit{if } k \geq 4 \end{cases}.$$

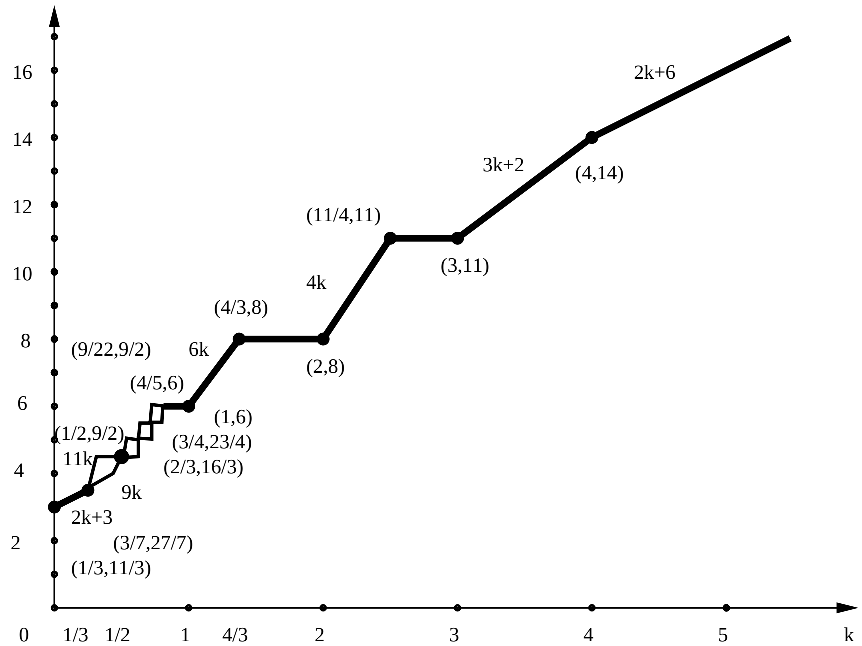


Figure 4: $\lambda(\Gamma_\Delta; k, 1)$ for $k \geq 0$.

We can use Lemma 2.2 to give a slight improvement to the stated bounds in the interval that is not yet resolved, $1/3 \leq k \leq 4/5$: Having the exact values of lambda at $k = 2/3,\ 3/4,\ 4/5$ means that there is a linear lower bound for k just below these values, of $8k$, if $k \in [\frac{9}{16}, \frac{2}{3}]$; of $\frac{23k}{3}$, if $k \in [\frac{16}{23}, \frac{3}{4}]$; and of $\frac{15k}{2}$, if $k \in [\frac{23}{30}, \frac{4}{5}]$. Similarly, there is a linear upper bound for k just above these values, of $9k$, if $k \in [\frac{1}{2}, \frac{16}{27}]$; of $8k$, if $k \in [\frac{2}{3}, \frac{23}{32}]$; and of $\frac{23k}{3}$, if $k \in [\frac{3}{4}, \frac{18}{23}]$.

We conjecture that the upper bound on $\lambda(\Gamma_\Delta; k, 1)$ is the actual value for $\frac{1}{3} \leq k \leq \frac{1}{2}$. For $\frac{1}{2} \leq k \leq \frac{4}{5}$, we conjecture that $\lambda(\Gamma_\Delta; k, 1) = 5k + 2$, a formula which works already in this interval at $k = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ and $\frac{4}{5}$.

Incidentally, we compared the formulas for the triangular lattice (which is 6-regular) to that of the regular infinite tree, T_6, and found they are quite different, not worth stating explicitly here.

4 The Square Lattice

Inside cities the high buildings can be obstacles in the signal path and limit the range of a cell. A Manhattan cellular system [4] can be used that is modeled by the square lattice $\Gamma_\Box$ (see Figure 4). Many graphs corresponding to cellular systems are the induced subgraphs of the square lattice and the triangular lattice.

Theorem 4.1 presents our full solution of the problem of determining $\lambda(\Gamma_\Box; k, 1)$ for real numbers $k \geq 0$ (see Figure 6). In Section 7 we describe the proof of this result. Previously, Calamoneri [7] independently gave the minimum (integer) span $\lambda(\Gamma_\Box; k_1, k_2)$

for integers $k_1 \geq 3k_2$, as well as bounds when $k_2 \leq k_1 \leq 3k_2$. (It should be noted that the stated bounds in the earlier extended abstract [6] are not entirely correct, such as the claim that $\lambda(\Gamma_\Box; 3, 2) = 12$, which is contradicted by the $L(3, 2)$-labeling of span only 11 from [20]. However, the bounds in the subsequent preprint [7] appear to be correct.)

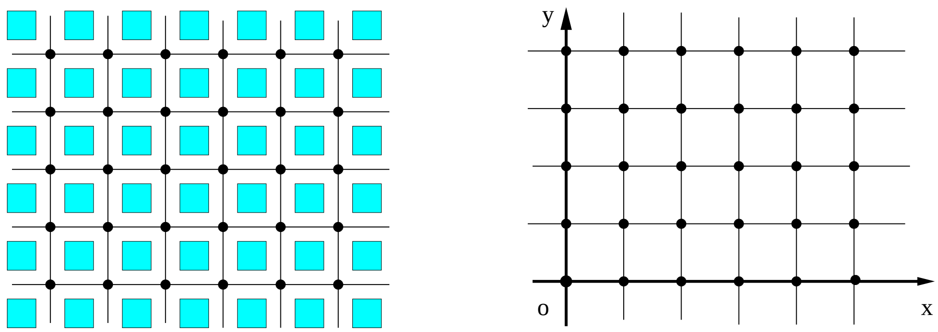


Figure 5: A Manhattan Fashion Network and the Square Lattice $\Gamma_\Box$

Theorem 4.1. *For $k \geq 0$ the minimum span of any $L(k, 1)$-labeling of the square lattice is given by:*

$$\lambda(\Gamma_\Box; k, 1) = \begin{cases} k+3 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 7k & \text{if } \frac{1}{2} < k \leq \frac{4}{7} \\ 4 & \text{if } \frac{4}{7} \leq k < 1 \\ 4k & \text{if } 1 \leq k \leq \frac{4}{3} \\ k+4 & \text{if } \frac{4}{3} < k \leq \frac{3}{2} \\ 3k+1 & \text{if } \frac{3}{2} < k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 < k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \\ 2k+2 & \text{if } 3 \leq k \leq 4 \\ k+6 & \text{if } k \geq 4 \end{cases}.$$

The full determination of $\lambda(\Gamma_\Box; k, 1)$ allows us now to answer a question posed by Georges and Mauro (private communication): Does $\lambda(\Gamma_\Box; k, 1)$ agree with $\lambda(T_4; k, 1)$ for all $k \geq 0$, which we stated in Theorems 2.4 and 2.5? Since $\Gamma_\Box$ is a triangle-free regular graph of degree 4, Theorem 2.7 is applicable, and tells us that $\lambda(\Gamma_\Box; k, 1) \geq \lambda(T_4; k, 1)$ for all $k \geq 0$. Indeed, they almost always agree.

However, there is one interval in which the inequality is strict: It is when $\frac{5}{2} < k < 3$. In this range, $\lambda(\Gamma_\Box; k, 1)$ is larger, and the answer to the question is negative.

5 The Hexagonal Lattice

Another interesting fundamental planar array is the hexagonal lattice Γ_H (see Figure 7), which is the dual of the triangular lattice. We are not aware of its being used in real life

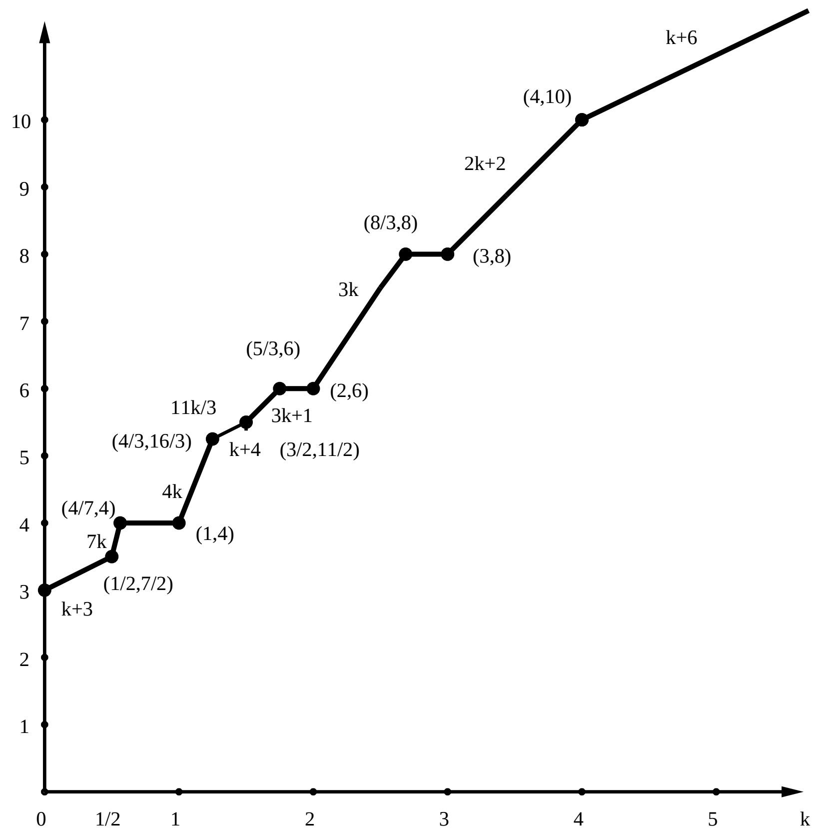


Figure 6: The Minimum Span $\lambda(\Gamma_{\Box}; k, 1)$

for wireless networks, but it is mentioned in the engineering literature.

We designate a point o to be the origin, and we impose a xoy coordinate system so that we can name each point by its xoy coordinate, where (i, j) are vertices (see Figure 7). The vertices (i, j) and $(i+1, j)$ are adjacent. The vertices (i, j) and $(i, j+1)$ are adjacent if and only if $i \equiv j(\mod 2)$. Calamoneri [7] gives the minimum span for the hexagonal lattice for integers $k_1 \geq 2k_2$ and bounds for $k_2 \leq k_1 \leq 2k_2$. We finish all the cases for real numbers $k \geq 0$(see Figure 5). In Section 8 we describe the proof of this result.

Theorem 5.1. *For $k \geq 0$ the minimum span of any $L(k, 1)$-labeling of the hexagonal lattice is given by:*

$$\lambda(\Gamma_H; k, 1) = \begin{cases} k+2 & \textit{if } 0 \leq k \leq \frac{1}{2} \\ 5k & \textit{if } \frac{1}{2} \leq k \leq \frac{3}{5} \\ 3 & \textit{if } \frac{3}{5} \leq k \leq 1 \\ 3k & \textit{if } 1 \leq k \leq \frac{5}{3} \\ 5 & \textit{if } \frac{5}{3} \leq k \leq 2 \\ 2k+1 & \textit{if } 2 \leq k \leq 3 \\ k+4 & \textit{if } k \geq 3 \end{cases}.$$

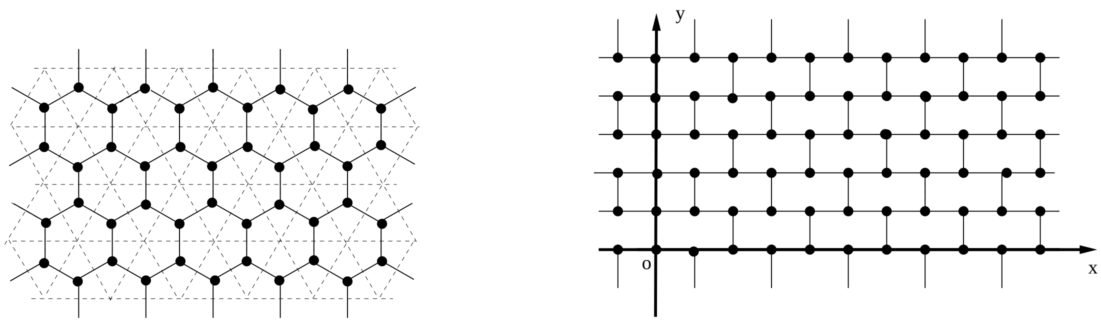


Figure 7: The Equilateral Triangle Cell Covering and the Hexagonal Lattice Γ_H

We may compare the spans of the hexagonal lattice and the regular tree of the same degree, T_3. As before, the fact that Γ_H is triangle-free allows us to apply Theorem 2.7 to see that $\lambda(\Gamma_H; k, 1) \geq \lambda(T_3; k, 1)$ for all $k \geq 0$. Comparing the formula above for Γ_H to those from Theorems 2.3 and 2.5, we see that Γ_H agrees with T_3 except in the range $\frac{3}{2} < k < 2$, where the inequality is strict, and the hexagonal lattice has larger span.

6 The Proof for the Triangular Lattice

Generally, we get upper bounds by constructing feasible labelings and lower bounds by deriving contradictions on induced subgraphs for labelings of smaller span. Lemma 2.2 is useful in obtaining bounds. Here we present proofs of bounds in Theorem 3.1 for various cases.

We need some notation. Given a vertex v, let B_7, (resp., B_{17}, B_{37}) be the induced subgraphs of Γ_Δ on all vertices which are at distance at most one (resp., two, three) from the vertex v.

To find an upper bound on $\lambda(\Gamma_\Delta; k, 1)$, one construction method is to tile the whole lattice by a labeled parallelogram described by a matrix of labels. We define a doubly periodic labeling of the triangular lattice by an $m \times n$ labeling matrix $A := [a_{i,j}]$, such that we label point (i, j) by $a_{m-(j \bmod m),\ (i \bmod n)+1}$, where i, j are integers.

For example, the following labeling (see Figure 9) is defined by the labeling matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then Figure 9 shows how the labels are assigned, where $a_{3,1}$ is at the vertex with coordinates $(0, 0)$ in the triangular lattice. The whole lattice is tiled with copies of the 3×3 tile as shown.

A special case of matrix labeling is defined simply by "arithmetic progressions": For positive integers k_1, k_2, we construct a labeling $f \in L(k_1, k_2)$ by taking $f(i, j) = (ai + bj) \bmod l$, for positive integers a, b, l, where " mod l" is taken to be the element in the

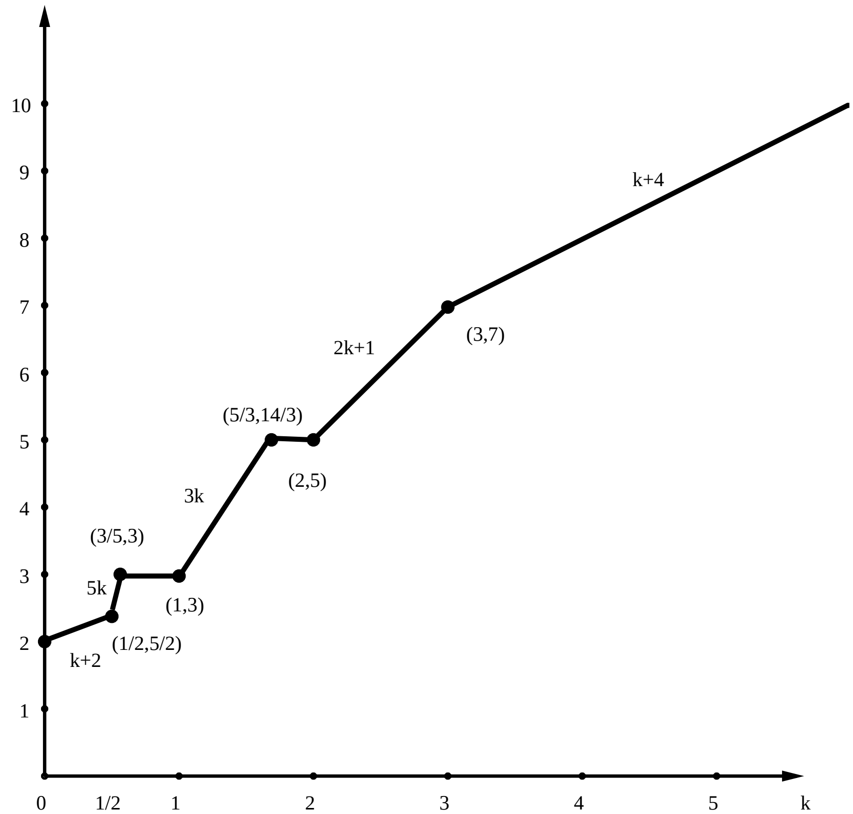


Figure 8: The Minimum Span $\lambda(\Gamma_H; k, 1)$ for $k \geq 0$.

congruence class that is in $\{0, \dots, l-1\}$. When such f is feasible, we obtain $\lambda(\Gamma_\Delta; k_1, k_2) \leq l-1$. Some labelings of this kind were given for the triangular and square lattices in [20]. We found some new arithmetic progression labelings by computer search. We begin our constructions at $k = 0$:

Proposition 6.1. *For* $0 \leq k \leq \frac{1}{3}$*, we have* $\lambda(\Gamma_\Delta; k, 1) \leq 2k+3$*. For* $\frac{1}{3} \leq k \leq \frac{9}{22}$*, we have* $\lambda(\Gamma_\Delta; k, 1) \leq 11k$*.*

Proof: From an optimal $L(0,1)$-labeling, we shift up some labels by $k, 2k$ or $3k$ to satisfy the $L(k,1)$ conditions. We get the upper bound $\lambda(\Gamma_\Delta; k, 1) \leq 2k+3$ for $0 \leq k \leq \frac{1}{3}$ by defining the labeling matrix

$$A = \begin{bmatrix} k+1 & 2k & 0 & k+3 & 2k+2 & 2 \\ 2k+1 & 1 & k & 2k+3 & 3 & k+2 \\ 0 & k+3 & 2k+2 & 2 & k+1 & 2k \\ k & 2k+3 & 3 & k+2 & 2k+1 & 1 \\ 2k+2 & 2 & k+1 & 2k & 0 & k+3 \\ 3 & k+2 & 2k+1 & 1 & k & 2k+3 \end{bmatrix}.$$

In particular, $\lambda(\Gamma_\Delta; \frac{1}{3}, 1) \leq \frac{11}{3}$, and Lemma 2.2 implies that $\lambda(\Gamma_\Delta; k, 1) \leq 11k$ for $k \geq \frac{1}{3}$. ■

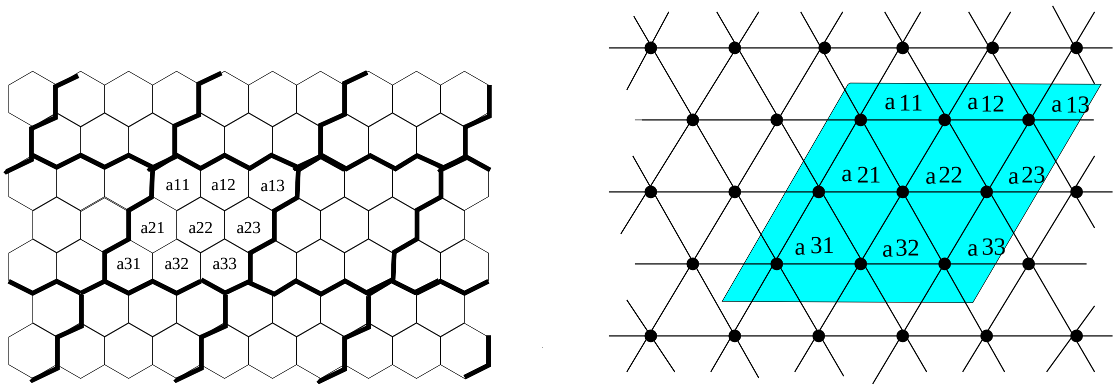


Figure 9: The Matrix Labeling

Next, we can improve upon the $11k$ upper bound for k between 9/22 and 1/2:

Proposition 6.2. *For* $\frac{9}{22} \leq k \leq \frac{1}{2}$ *we have* $\lambda(\Gamma_\Delta; k, 1) \leq \frac{9}{2}$.

Proof: The upper bound $\lambda(1,2) \leq 9$ is given in [22] by an arithmetic progression labeling in $L(1,2)$: Label point (i,j) by $(i+4j) \mod 10$. By scaling, this gives the bound on $\lambda(\Gamma_\Delta; \frac{1}{2}, 1)$, which then extends to $k \leq \frac{1}{2}$ by Lemma 2.2. ■

The upper bound for k between $\frac{1}{2}$ and $\frac{3}{4}$ follows from the bounds at $k = \frac{2}{3}$ and $\frac{3}{4}$ by the fact that $\lambda(\Gamma_\Delta; k, 1)$ is nondecreasing (Lemma 2.2):

Proposition 6.3. *1. We have* $\lambda(\Gamma_\Delta; 2, 3) \leq 16$. *Hence,* $\lambda(\Gamma_\Delta; \frac{2}{3}, 1) \leq \frac{16}{3}$.

2. We have $\lambda(\Gamma_\Delta; 3, 4) \leq 23$. *Hence,* $\lambda(\Gamma_\Delta; \frac{3}{4}, 1) \leq \frac{23}{4}$.

Proof: By computer search of arithmetic progression labelings, we discovered $f_1 \in L(2,3)(\Gamma_\Delta)$ given by $f_1(i,j) = (2i+7j) \mod 17$ and $f_2 \in L(3,4)(\Gamma_\Delta)$ given by $f_2(i,j) = (3i+10j) \mod 24$. Hence, $\lambda(\Gamma_\Delta; 2, 3) \leq 16$ and $\lambda(\Gamma_\Delta; 3, 4) \leq 23$. ■

Next we obtain the upper bound out to $k = \frac{4}{3}$ by applying Lemma 2.2 with the upper bound on $\lambda(\Gamma_\Delta; 1, 1)$. Note that the upper bounds we are giving here for $k = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1 are matched by the lower bounds, so give the correct values of $\lambda(\Gamma_\Delta; k, 1)$ for these k.

Proposition 6.4 ([5, 13]). *We have* $\lambda(\Gamma_\Delta; 1, 1) = 6$.

Hence, $\lambda(G; k, 1) \leq \begin{cases} 6 & \text{if } \frac{3}{4} \leq k \leq 1 \\ 6k & \text{if } 1 \leq k \leq \frac{4}{3} \end{cases}$.

Proof: We get the upper bound, $\lambda(B_7; 1, 1) \leq 6$, from the arithmetic progression labeling $f(i,j) = (i+3j) \mod 7$. The rest follows from Lemma 2.2. ■

We now use the construction of numerous MCM teams at $k = 2$ to extend our upper bound out to $k = \frac{11}{4}$:

Proposition 6.5 ([5, 9, 13, 25, 29]). *We have* $\lambda(\Gamma_\Delta; 2, 1) \leq 8$.

Hence, $\lambda(G; k, 1) \leq \begin{cases} 8 & \text{if } \frac{4}{3} \leq k \leq 2 \\ 4k & \text{if } 2 \leq k \leq \frac{11}{4} \end{cases}$.

Proof: Label point (i, j) by $(2i + 5j) \bmod 9$. ■

Next we continue out to $k = 4$:

Proposition 6.6. *1. For $3 \le k \le 4$, we have $\lambda(\Gamma_\Delta; k, 1) \le 3k + 2$.*

2. For $\frac{11}{4} \le k \le 3$, we have $\lambda(G; k, 1) \le 11$.

Proof: 1. We rewrite the proof of [13, 25]. We get the bound by defining the labeling matrix

$$A = \begin{bmatrix} 3k & 0 & k & 2k \\ 1 & k+1 & 2k+1 & 3k+1 \\ k+2 & 2k+2 & 3k+2 & 2 \end{bmatrix}.$$

2. Using $\lambda(\Gamma_\Delta; 3, 1) \le 11$, this extends to smaller k by Lemma 2.2. ■

A construction from the winning MCM papers takes care of all large k:

Proposition 6.7. *For $k \ge 4$, we have $\lambda(\Gamma_\Delta; k, 1) \le 2k + 6$.*

Proof [5, 9, 13, 25, 29]: We get the labeling from the matrix

$$A = \begin{bmatrix} 2k+5 & 0 & k+4 \\ 1 & k+2 & 2k+6 \\ k+3 & 2k+4 & 2 \end{bmatrix}. \quad ■$$

We verify the lower bounds using proofs by contradiction (which can be rather complicated) and Lemma 2.2. We shall postpone the small values, $k \le \frac{3}{4}$. We demonstrate two main methods of proof. The first method, for integers k_1, k_2, involves the successive elimination of possible labels, until a contradiction is reached. This method was used in the contest paper of Goodwin *et al.* to handle the case of integer $k \ge 4$ (see our comments before Proposition 6.11). We also drew ideas from [31] for the proof of the following important case.

Proposition 6.8. *We have $\lambda(\Gamma_\Delta; 4, 3) \ge 24$.*

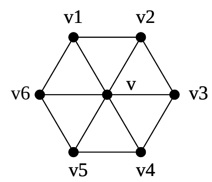
Hence, $\lambda(\Gamma_\Delta; k, 1) = \begin{cases} 6k & \text{if } \frac{3}{4} \le k \le \frac{4}{3} \\ 8 & \text{if } \frac{4}{3} \le k \le 2 \end{cases}$

Proof: The first statement implies the second by Lemma 2.2. It suffices to prove that $\lambda(\Gamma_\Delta; 4, 3) \ge 24$. Assume to the contrary that there exists a labeling $f \in L(4, 3)(\Gamma_\Delta)$ with its labels in $\{0, 1, \ldots, 23\}$. The series of claims that follows restricts the labels f one can use until we find that no such f can exist at all, proving the proposition.

Claim 1. *The labeling f cannot use label* 3 *or* 20.

Proof: Assume f uses label 3 at v. By the separation conditions, the six labels around v belong to $\{7, 8, \ldots, 23\}$, and the difference between any pair of them is at least 3.

Among all 49 possible labelings of B_7 with central label 0 by symmetry, we found by computer that there are just five feasible labelings of subgraph B_{19} that use 3 at the center (B_7, B_{19} are shown in Figure 10), and none of these can be extended to B_{37}. Full details are in [21].



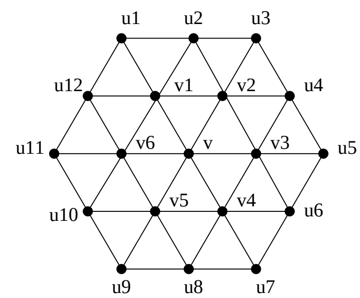


Figure 10: The Subgraphs B_7 and B_{19} of the Triangular Lattice.

By the Symmetry Argument 2.1, f is also excluded from using the complementary label $23 - 3 = 20$. $\square$

Claim 2. *The labeling f cannot use label 7 or 16.*

Proof: Assume f uses label 7 at $v \in V(\Gamma_\Delta)$. Denote the six labels around v by $x_1 < x_2 \cdots < x_6$. By the separation conditions, $x_{i+1} \geq x_i + 3$ for $i = 1, 2, \ldots, 5$, and each $x_i \in \{0, 1, 2, 11, 12, \ldots, 19, 21, 22, 23\}$ (recall we cannot use 3 or 20). Then, even if $x_1 \leq 2$, we must have $x_2 \geq 11$, $x_3 \geq 14$, $x_4 \geq 17$, $x_5 \geq 21$, $x_6 \geq 24$, a contradiction. $\square$

Now f has no label $3, 7, 16, 20$.

Claim 3. *The labeling f cannot use label 6 or 17.*

Proof: Assume $f(v) = 6$ for some $v \in V(\Gamma_\Delta)$. Let the six labels around v be $x_1 < x_2 \cdots < x_6$. By the condition, $x_{i+1} \geq x_i + 3$ for $i = 1, 2, \ldots, 5$, and each $x_i \in \{0, 1, 2, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23\}$ for $i = 1, 2, \ldots, 6$. Then $x_2 \geq 10$, $x_3 \geq 13$, $x_4 \geq 17$, $x_5 \geq 21$, $x_6 \geq 24$, which is too large. Neither can f use 17, by the Symmetry Argument. $\square$

Now f has no labels $3, 6, 7, 16, 17, 20$.

Claim 4. *The labeling f cannot use label 10 or 13.*

Proof: Assume $f(v) = 10$ for some $v \in V(\Gamma_\Delta)$. Let the six labels around v be $x_1 < x_2 \cdots < x_6$. Then $x_{i+1} \geq x_i + 3$ for $i = 1, 2, \ldots, 5$, and each $x_i \in \{0, 1, 2, 4, 5, 14, 15, 18, 19, 21, 22, 23\}$. Now $x_3 \geq 14$, $x_4 \geq 18$, $x_5 \geq 21$, $x_6 \geq 24$, which is again too large. By the Symmetry Argument, we cannot use 13 either. $\square$

Now f has no label $3, 6, 7, 10, 13, 16, 17, 20$.

Claim 5. *The labeling f cannot use label 11 or 12.*

Proof: Assume $f(v) = 11$ for some $v \in V(\Gamma_\Delta)$. Let the labels around v be $x_1 < x_2 \cdots < x_6$, so $x_{i+1} \geq x_i + 3$, and each $x_i \in \{0, 1, 2, 4, 5, 15, 18, 19, 21, 22, 23\}$. Then $x_3 \geq 15$, $x_4 \geq 18$, $x_5 \geq 21$, $x_6 \geq 24$, once again a contradiction. We eliminate 12 by the Symmetry Argument. $\square$

Now the set of all possible labels is $\{0, 1, 2, 4, 5, 8, 9, 14, 15, 18, 19, 21, 22, 23\}$. We cannot find seven distinct labels, such that the difference between any two of them is at least 3. So we cannot label B_7, which is a contradiction. Thus, $\lambda(\Gamma_\Delta; 4, 3) \geq 24$. $\blacksquare$

By similar proofs, we have the following bounds. See [21] for full details.

Proposition 6.9. *1. We have* $\lambda(\Gamma_\Delta; 11, 4) \geq 44$.

Hence, $\lambda(\Gamma_\Delta; k, 1) \geq \begin{cases} 4k & \text{if } 2 \leq k \leq \frac{11}{4} \\ 11 & \text{if } \frac{11}{4} \leq k \leq 3 \end{cases}$.

2. We have $\lambda(\Gamma_\Delta; 1, 2) \geq 9$.

Hence, $\lambda(\Gamma_\Delta; k, 1) \geq \begin{cases} 9k & \text{if } \frac{3}{7} \leq k \leq \frac{1}{2} \\ \frac{9}{2} & \text{if } k \geq \frac{1}{2} \end{cases}$

3. We have $\lambda(\Gamma_\Delta; 2, 3) \geq 16$*. Hence,* $\lambda(\Gamma_\Delta; k, 1) \geq \frac{16}{3}$ *for* $k \geq \frac{2}{3}$.

4. We have $\lambda(\Gamma_\Delta; 3, 4) \geq 23$*. Hence,* $\lambda(\Gamma_\Delta; x, 1) \geq \frac{21}{4}$ *for* $k \geq \frac{3}{4}$.

5. We have $\lambda(\Gamma_\Delta; 4, 5) \geq 30$*. Hence,* $\lambda(\Gamma_\Delta; x, 1) \geq 6$ *for* $k \geq \frac{4}{5}$.

The next result, which takes care of all k in the interval $(3, 4)$, can be derived by continuity and scaling from the corresponding result by Calamoneri [7] for integer labelings that give $\lambda(\Gamma_\Delta; k_1, k_2)$ for integers k_1, k_2 with $3k_2 \leq k_1 \leq 4k_2$. Her lower bound method involves looking at a small induced subgraph of the lattice and checking cases according to the numerical order of the labels. This is similar to the method devised independently by Georges and Mauro for labeling trees [11]. We discovered the result independently (but waited on the rest of this project before writing it up here). Because our proof illustrates a different method with some potential for future value, we include it here. It involves the successive removal of intervals of possible labels until there is a contradiction.

Proposition 6.10. *For* $3 < k < 4$*, we have* $\lambda(\Gamma_\Delta; k, 1) = 3k + 2$.

Proof: The upper bound comes from Proposition 6.6. We prove the lower bound by contradiction: Assume $\lambda(\Gamma_\Delta; k, 1) = l < 3k + 2$. By the D-Set Theorem, there is an optimal labeling $f \in L(k, 1)(\Gamma_\Delta)$ with $\text{span}(f) = l < 3k + 2$ and $f(u) = 0$ for some $u \in V(\Gamma_\Delta)$.

Claim 1. *The labeling* f *cannot use labels in* $[k - 1, k) \cup (l - k, l - k + 1]$.

Proof: Assume $f(v) \in [k - 1, k)$ for some $v \in V(\Gamma_\Delta)$. The neighbors of v induce a C_6 subgraph, and their labels are all $\geq f(v)+k$. Hence, $\lambda(\Gamma_\Delta; k, 1) \geq f(v)+k+\lambda(C_6; k, 1) \geq (k - 1) + k + (k + 3) = 3k + 2$ (because $\lambda(C_6; k, 1) = k + 3$ for $k \geq 3$, see [21]). It gives a contradiction. Thus $f(v) \notin [k - 1, k)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l - k, l - k + 1]$ for all $v \in V(\Gamma_\Delta)$.□

Now, $f(v) \in I_1 \cup I_2 \cup I_3$ for all $v \in V(\Gamma_\Delta)$, where $I_1 = [0, k - 1), I_2 = [k, l - k], I_3 = (l - k + 1, l]$. Then $|I_1| = k - 1 < k, |I_2| = l - 2k < k + 2, |I_3| = k - 1 < k$.

Claim 2. *The labeling* f *cannot use labels in* $[k, k + 1) \cup (l - k - 1, l - k]$.

Proof: Assume $f(v) \in [k, k + 1)$ for some $v \in V(\Gamma_\Delta)$. Among the six distinct labels around v, at most one label is in $I_1 = [0, k - 1)$ (because this label is $\leq f(v) - k < 1$), at most two labels are in $I_2 = [k, l - k]$ (because these two labels are $\geq f(v) + k \geq 2k$ and $|[2k, l - k]| = l - 3k < 2$), and at most three labels are in $I_3 = (l - k + 1, l]$ (because $|I_3| < k$, these labels cannot be adjacent). Thus, one label is in I_1, two labels are in I_2, and three labels are in I_3. The three labels in I_3 are for vertices that aren't adjacent. The smallest of the three labels then must be next to at least one of the labels in I_2. This smallest label in I_3 is $\leq l - 2 < 3k$. But the two labels in I_2 are $\geq f(v) + k \geq 2k$, and the smallest label in I_3, being next to one of these, must then be at least $3k$, a contradiction.

Thus, $f(v) \notin [k, k+1)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l-k-1, l-k]$.□

Now, $f(v) \in I_1 \cup I_2' \cup I_3 = [0, k-1) \cup [k+1, l-k-1] \cup (l-k+1, l]$, where $I_2' = [k+1, l-k-1]$. Then $|I_2'| \le l - 2k - 2 < k$.

Claim 3. *The labeling f cannot use labels in $[k+1, k+2) \cup (l-k-2, l-k-1]$.*

Proof: Assume $f(v) \in [k+1, k+2)$ for some $v \in V(\Gamma_\Delta)$. Among the six labels around v, at most two labels are in $I_1 = [0, k-1)$ (because these two labels are $\le f(v) - k < 2$), no label is in $I_2' = [k+1, l-k-1]$ (because if it exists, it would be $\ge f(v)+k \ge 2k+1 > l-k-1$, a contradiction), and at most three labels are in $I_3 = (l-2, l]$. We cannot label all six vertices.

Thus $f(v) \notin [k+1, k+2)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l-k-2, l-k-1]$. □

Now, $f(v) \in I_1 \cup I_2'' \cup I_3 = [0, k-1) \cup [k+2, l-k-2] \cup (l-k+1, l]$ for all $v \in V(\Gamma_\Delta)$, where $I_2'' = [k+2, l-k-2]$. Then $|I_2''| = l - 2k - 4 < k - 2 < 2$ for $k < 4$.

Since $f(u) = 0$, among the six distinct labels around u (the difference between any pair of them is at least 1), no label is in I_1, at most two labels are in I_2'' (because $|I_2''| < 2$), and at most three labels are in $I_3 = (l-2, l]$ (because $|I_3| < k$ means no two of its labels are for adjacent vertices). So we cannot label all six neighbors of u, a contradiction. ■

We next address $k \ge 4$. One of the winning teams in the modeling contest, Goodwin, Johnston and Marcus (2000) [13], obtained the correct values for the integer cases, that is, for integer $k \ge 4$. It is a pity that, due to space limitations, the elegant proof in their contest paper was omitted from the published version! It is the same method we used to prove Proposition 6.8 above.

Moreover, Goodwin *et al.* gave what is equivalent to the correct formula, $\lambda(\Gamma_\Delta; k_1, k_2) = 2k_1 + 6k_2$, for arbitrary integers k_1, k_2 with $k_1 > 6k_2 + 1$. By scaling and continuity, this implies the correct formula, $\lambda(\Gamma_\Delta; k, 1) = 2k + 6$, for all real $k \ge 6$. There appear to be some technical errors in their lower bound proof (quite understandable, since they had just four days to produce their entire paper from scratch!). However, we discovered that if one uses the D-Set Theorem, some small changes will fix their proof. We present below our own verification of the lower bound, which we need more generally for all real $k \ge 4$. We follow this with the much shorter proof, which is based on the method of Goodwin *et al.*, that only works for $k \ge 6$: It will be apparent that the method does not depend on the structure of the triangular lattice, so that it can be used on other graphs, for sufficiently large real k, provided that there is a linear bound for all large integers k.

Proposition 6.11. *For $k \ge 4$ we have $\lambda(\Gamma_\Delta; k, 1) \ge 2k + 6$.*

Proof: Assume for contradiction that $\lambda(\Gamma_\Delta; k, 1) = l < 2k + 6$ for some $k \ge 4$. By the D-Set Theorem, there is an optimal labeling $f \in L(k, 1)(\Gamma_\Delta)$ with span and largest label l and smallest label 0.

Claim 1. *The labeling f cannot use labels in $[3, k)$.*

Proof: If some $f(v) \in [3, k)$, then the labels on the vertices of the C_6 neighboring v are all at least $f(v) + k$. The largest of these labels is then at least $f(v) + k + \lambda(C_6; , k, 1) \ge 3 + k + (k+3) = 2k + 6 > l$, a contradiction (where $\lambda(C_6; , k, 1)$ is given in [21]).□

By symmetry, none of the labels in f belongs to $(l-k, l-3]$. So all labels belong to the union $I_1 \cup I_2 \cup I_3$, where $I_1 = [0, 3)$, $I_2 = [k, l-k]$, and $I_3 = (l-3, l]$.

Claim 2. *The labeling f cannot use labels in $[k, k+1)$.*

Proof: Assume some label $f(v) \in [k, k+1)$. At most one of the six vertices next to v has a label in I_1 because any such label is $\leq f(v) - k < 1$. At most three of the six vertices have labels in I_3 as any two must be at least one apart.

First suppose three of these labels are in I_3. They cannot be at adjacent vertices, so suppose they are at vertices v_1, v_3, and v_5, with reference to the graph B_7 in Figure 10. Two of the other labels next to v must belong to $[f(v)+k, l-k]$, so the larger of the two, say it is at v_2, must be at least $f(v) + k + 1 \geq 2k + 1$. Then both $f(v_1)$ and $f(v_3)$ are at least $f(v_2) + k$, and the larger of the two is at least $f(v_2) + k + 1 \geq 3k + 2 \geq 2k + 6 > l$, a contradiction.

Next suppose just two of these labels next to v lie in I_3. The two vertices are not adjacent. There must be at least three labels next to v in $[f(v) + k, l - k]$, and, because this interval has length $< k$, no two of the three are adjacent–say they are at v_1, v_3, v_5. The largest of the three labels is at least $f(v) + k + 2$, and its neighbor with label in I_3 has label at least $f(v) + k + 2 + k \geq 3k + 2$, which is again a contradiction.

Finally, suppose at most one label next to v lies in I_3. Then at least four labels next to v are in $[f(v) + k, l - k]$, so some two are adjacent–but this is impossible since they must differ by at least k (as $(l - k) - (f(v) + k) \leq l - 3k < 2 < k$). □

Hence, f has no labels in $[k, k+1)$ nor, by symmetry, in $(l-k-1, l-k]$. So all of its labels belong to $I_1 \cup I_2' \cup I_3$, where here $I_2' = [k+1, l-k-1]$.

Claim 3. *The labeling f cannot use labels in $[k+1, k+2)$.*

Proof: Suppose some $f(v) \in [k+1, k+2)$. Then labels used next to v in I_1 are at most $f(v) - k < 2$, so there can be at most two such labels. On the other hand, at most three labels next to v can come from I_3. Then some label used next to v lies in I_2'. But such a label must be at most $l - k - 1$ and at least $f(v) + k \geq 2k + 1 \geq k + 5 > l - k - 1$, a contradiction. □

By symmetry, no label of f belongs to $(l - k - 2, l - k - 1]$. Then all of its labels belong to $I_1 \cup I_2'' \cup I_3$, where $I_2'' = [k + 2, l - k - 2]$. Let u be a vertex with $f(u) = 0$. Then its six neighbors all have labels in $I_2'' \cup I_3$. But I_3 can contain at most three of the labels, as they must be at least one apart from each other. So some three of the labels are in I_2''. However, $(l - k - 2) - (k + 2) = l - 2k - 4 < 2$, so I_2'' can contain at most two of the labels, a contradiction, and no such f exists. ■

Here is the shorter proof of the restriction of the Proposition above to $k \geq 6$.

Proposition 6.12. *For $k \geq 6$ we have $\lambda(\Gamma_\Delta; k, 1) \geq 2k + 6$.*

Proof: Let us assume the result of Goodwin *et al.* that $\lambda(\Gamma_\Delta; k, 1) \geq 2k + 6$ for integers $k \geq 4$. Now consider any non-integer $k > 6$. Let $m = \lceil k \rceil - k$, so that $m \in (0, 1)$ and $k + m = \lceil k \rceil \geq 7$. Hence, $\lambda(\Gamma_\Delta; k + m, 1) \geq 2k + 2m + 6$.

Assume for contradiction that $\lambda(\Gamma_\Delta; k, 1) < 2k + 6$. Let f be an optimal labeling in $L(k, 1)(\Gamma_\Delta)$ as in the D-Set Theorem, with minimum value 0 at some vertex u and maximum value span(f) at some vertex w. Define a labeling f_1 by $f_1(v) = f(v) +$

$m\lfloor f(v)/k \rfloor$. We can check that $f_1 \in L(k+m,1)(\Gamma_\Delta)$. Further, the minimum value of f_1 is 0, which occurs at u, and its maximum occurs at v, which thus has value $f_1(v) = \mathrm{span}(f_1) < (2k+6) + m\lfloor (2k+6)/k \rfloor = 2k+6+2m$ (since $k > 6$ by assumption). This contradicts the lower bound in the previous paragraph. ■

We cannot see how to extend the argument in the last proof to work for k between 4 and 6.

It remains to do the lower bound for small k. Similar to the proof of Proposition 6.10, we can show (see [21]):

Proposition 6.13. *For $0 < k \le \frac{1}{2}$, we have $\lambda(\Gamma_\Delta; k, 1) \ge 2k+3$. Hence $\lambda(\Gamma_\Delta; k, 1) = 2k+3$ for $0 \le k \le \frac{3}{7}$.* ■

This completes the proof of Theorem 3.1.

7 The Proof for the Square Lattice.

We begin by establishing the claimed upper bounds on $\lambda(\Gamma_\Box; k, 1)$ for reals $k \ge 0$.

In many cases, we provide an explicit construction based on a modular construction, in which a particular matrix of labels is used for a rectangle of lattice points and then repeated over and over. This is described most conveniently by thinking of a $m \times n$ matrix A as having entries $a_{x,y}$, and the lattice point with coordinates (i,j) receives label $a_{j \bmod n+1,\, i \bmod m+1}$.

Proposition 7.1. *For $0 \le k \le \frac{1}{2}$, we have $\lambda(\Gamma_\Box; k, 1) \le k+3$.*

Proof: Starting from an optimal $L(0,1)$-labeling and shifting up some labels by k, in order to satisfy the $L(k,1)$ conditions, we came up with the following labeling matrix that attains the upper bound:

$$A = \begin{bmatrix} 0 & k & 1 & k+1 \\ k+3 & 2 & k+2 & 3 \\ 1 & k+1 & 0 & k \\ k+2 & 3 & k+3 & 2 \end{bmatrix}. \quad ■$$

Next, applying the result above at $k = 1/2$, Lemma 2.2 yields this upper bound for larger k:

Proposition 7.2. *For $\frac{1}{2} \le k \le \frac{4}{7}$, we have $\lambda(\Gamma_\Box; k, 1) \le 7k$.* ■

We can use arithmetic progression labelings, analogous to those for the triangular lattice in the previous section. Van den Heuvel, Leese and Shepherd [20] give a circular integer labeling result which is helpful for our real number labelings, as it suggests some arithmetic progression labelings that turn out to be optimal for our problem:

Proposition 7.3. *We have*
$\lambda(\Gamma_\Box;1,1)=4$,
$\lambda(\Gamma_\Box;2,1)\le 6$,
$\lambda(\Gamma_\Box;3,1)\le 8$, *and*
$\lambda(\Gamma_\Box;3,2)\le 11$.

Proof: From [20], we have these labelings:
$\lambda(\Gamma_\Box;1,1)\le 4$ by labeling f with $f(i,j)=(i+2j) \bmod 5$
$\lambda(\Gamma_\Box;2,1)\le 6$ by labeling f with $f(i,j)=(2i+3j) \bmod 7$
$\lambda(\Gamma_\Box;3,1)\le 8$ by labeling f with $f(i,j)=(3i+4j) \bmod 9$
$\lambda(\Gamma_\Box;3,2)\le 11$ by labeling f with $f(i,j)=(3i+5j) \bmod 12$.
It is easy to show $\lambda(\Gamma_\Box;1,1)\ge 4$. ■

Applying the preceding two propositions and Lemma 2.2, we have the following upper bounds.

Proposition 7.4. *We have* $\lambda(\Gamma_\Box;k,1)\le \begin{cases} 4 & \text{if } \frac{4}{7}\le k\le 1 \\ 4k & \text{if } 1\le k\le \frac{5}{3} \\ 6 & \text{if } \frac{5}{3}\le k\le 2 \\ 3k & \text{if } 2\le k\le \frac{8}{3} \\ 8 & \text{if } \frac{8}{3}\le k\le 3 \end{cases}$. ■

The upper bounds in the proposition above are weaker than what we want for $\frac{4}{3}<k<\frac{5}{3}$. Let us consider one value in this gap, $k=11/8$.

By Proposition 7.4, we get the upper bound $\lambda(\Gamma_\Box;\frac{11}{8},1)\le\frac{11}{2}$, so by scaling, $\lambda(\Gamma_\Box;11,8)\le 44$. To determine whether this is best-possible, we searched for a better labeling: We managed to construct a $L(11,8)$-labeling based on a matrix A in which the entries are elements of the D-set in $[0,43]$. Since 43 can be expressed in terms of 11 and 8 in just one way, $43=11+4\times 8$, we easily saw how to extend this matrix labeling to cases in the range $\frac{4}{3}\le k\le\frac{3}{2}$, as given in the following proposition. We next took the resulting labeling at $k=\frac{3}{2}$, and found a way to extend it to the range $\frac{3}{2}\le k\le\frac{5}{3}$ in a way that maintains the order of the labels, while expanding their pairwise differences, to maintain feasibility as k grows. This gives Proposition 7.6. Notice that these formulas for $\lambda(\Gamma_\Box;k,1)$ around $k=11/8$ are not of the simple form ck for some c, so we could not simply apply Lemma 2.2.

Proposition 7.5. *For* $\frac{4}{3}\le k\le\frac{3}{2}$, *we have* $\lambda(\Gamma_\Box;k,1)\le k+4$.

Proof: The upper bound is attained by the following labeling matrix:

$$\begin{bmatrix} 5 & k & 4 & 0 & 3 & k{+}4 & 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 \\ 0 & 3 & k{+}4 & 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 & 5 & k & 4 \\ 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 & 5 & k & 4 & 0 & 3 & k{+}4 \\ k{+}2 & 0 & k{+}1 & 5 & k & 4 & 0 & 3 & k{+}4 & 2 & k{+}3 & 1 \end{bmatrix}.\ \blacksquare$$

Proposition 7.6. *For* $\frac{3}{2}\le k\le\frac{5}{3}$, *we have* $\lambda(\Gamma_\Box;k,1)\le 3k+1$.

Proof: The upper bound is attained by the following labeling matrix:

$$\begin{bmatrix} 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 & 2 & 3k & 1 & k+2 & 0 & k+1 \\ 0 & 2k & 3k+1 & 2 & 3k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 \\ 2 & 3k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 \\ k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 & 2 & 3k & 1 \end{bmatrix}. \quad \blacksquare$$

For larger k, we first adapt the construction given by Calamoneri for integers k_1, k_2 with $3k_2 \leq k_1 \leq 4k_2$. We then present a simple matrix $L(k,1)$-labeling that turns out to be optimal for all $k \geq 4$.

Proposition 7.7. *For $3 \leq k \leq 4$ we have $\lambda(\Gamma_\square; k, 1) \leq 2k+2$.*

Proof: Adapting the construction in [7], the upper bound is attained by a $L(k,1)$-labeling matrix:

$$A = \begin{bmatrix} 2k+2 & k & 2k+1 & 2 & 2k & 1 & k+2 & 0 & k+1 \\ k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 2 & 2k & 1 \\ 2 & 2k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 \end{bmatrix}. \quad \blacksquare$$

Proposition 7.8. *For $k \geq 0$ we have $\lambda(\Gamma_\square; k, 1) \leq k+6$.*

Proof: The upper bound is attained by the following labeling matrix:

$$A = \begin{bmatrix} 0 & k+3 & 1 & k+4 \\ k+6 & 2 & k+5 & 3 \\ 1 & k+4 & 0 & k+3 \\ k+5 & 3 & k+6 & 2 \end{bmatrix}. \quad \blacksquare$$

We now work on the lower bounds to complete the proof of the formulas. It is helpful to compare our graph to the regular infinite tree T_4 of degree 4, discussed in the section on methods. By Theorem 2.7 we get that for all $k \geq 0$, $\lambda(\Gamma_\square; k, 1) \geq \lambda(T_4; k, 1)$. From the values of $\lambda(T_4; k, 1)$ presented in Theorems 2.4 and 2.5, we obtain the claimed values of $\lambda(\Gamma_\square; k, 1)$ for all k outside the interval $[\frac{5}{2}, 3]$. In this remaining interval, we must improve the lower bound on $\lambda(\Gamma_\square; k, 1)$. In view of Lemma 2.2, all that remains to prove the theorem is to establish the lower bound at $k = 8/3$:

Proposition 7.9. *We have $\lambda(\Gamma_\square; 8, 3) \geq 24$. Consequently, for $2 \leq k \leq 3$, we have* $\lambda(\Gamma_\square; k, 1) \geq \begin{cases} 3k & \text{if } 2 \leq k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \end{cases}$.

Proof: The second statement follows from the first by Lemma 2.2.

Assume for contradiction that the first statement fails. Then there exists a labeling $f \in L(8,3)(\Gamma_\square)$ with all labels in $\{0, \ldots, 23\}$. The series of claims that follows restricts the labels f one can use until we find that no such f can exist at all, proving the proposition.

Let $v_0 = (i_0, j_0) \in V(\Gamma_\square)$. Let B_{12} be the induced subgraph as in Figure 11.

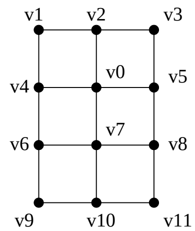


Figure 11: The Subgraph B_{12} of the Square Lattice

Claim 1. *The labeling f cannot use label 7 or 16.*
Proof: Assume $f(v_0) = 16$. Since no label can exceed 23, the four distinct labels around v_0 are each $\leq f(v_0) - 8 = 8$, which is impossible since any two must be at least 3 apart.

By the Symmetry Argument 2.1, labeling f is also excluded from using the complementary label $23 - 16 = 7$. $\square$
Claim 2. *The labeling f cannot use label 8 or 15.*
Proof: Assume some $f(v_0) = 8$. The four labels around v_0 are each $\geq f(v_0) + 8 = 16$ or $\leq f(v_0) - 8 = 0$, hence are 0 or ≥ 17 (because by Claim 1, f cannot use 16). Suppose they are labels $x < y < z < w$. Since the difference between any pair of the four labels is ≥ 3, it must be that $x = 0, y = 17, z = 20, w = 23$. Suppose without loss of generality that $f(v_7) = y = 17$. Since $f(v_7) + 8 = 25$ is too large, it must be that the neighboring labels $f(v_6), f(v_8), f(v_{10})$ are all $\leq f(v_7) - 8 = 9$, and hence, all $\leq f(v_0) - 3 = 8 - 3 = 5$. But this is impossible since the difference between any pair of the three labels must be at least 3. By symmetry, we must also exclude 15. $\square$
Claim 3. *The labeling f cannot use label 9 or 14.*
Proof: Assume some $f(v_0) = 9$. The four labels around v_0 are $\geq f(v_0) + 8 = 17$ or $\leq f(v_0) - 8 = 1$. Suppose they are labels $x < y < z < w$. Since the difference between any pair of the four labels is ≥ 3 (because they are at distance two each other), $x \in \{0, 1\}, y = 17, z = 20, w = 23$.

We may suppose in B_{12} that $f(v_2) = x \in \{0, 1\}$. Then $f(v_1), f(v_3)$ are $\geq f(v_2)+8 \geq 8$, hence are $\geq f(v_0)+3 = 12$. Since $17 \leq f(v_4), f(v_5) \leq 23$, it forces $f(v_1), f(v_3) \leq 23-8 = 15$, so ≤ 14, as 15 is excluded by Claim 2. But now we have$|f(v_1) - f(v_3)| \leq 2$, which is not allowed. We can exclude 14 by symmetry.$\square$
Claim 4. *The labeling f cannot use label 11 or 12.*
Proof: Assume some $f(v_0) = 11$. The four distinct labels around v_0 are each $\geq f(v_0)+8 = 19$ or $\leq f(v_0) - 8 = 3$. Suppose they are labels $x < y < z < w$. Since the difference between any pair is ≥ 3, we must have $x = 0, y = 3, z \in \{19, 20\}, w \in \{22, 23\}$.

Suppose $f(v_2) = y = 3$. Then $f(v_1), f(v_3)$ are $\geq f(v_2)+8 = 11$, so are $\geq f(v_0)+3 = 14$. But the previous claims exclude 14, 15, and 16, so they are ≥ 17. At least one of v_4, v_5 must have label z or w, so is between 19 and 23, but this is too close to the labels at the vertices v_1, v_3, one of which is adjacent. We have a contradiction, and exclude label 12 by symmetry. $\square$

By the D-Set Theorem, there exists optimal labeling $f^* \in L(8,3)(\Gamma_\square)$ with smallest label 0 and all labels in $D_{8,3} \cap [0,23] = \{0,3,6,8,9,11,12,14,15,16,17,18,19,20,21,22,23\}$. Applying the Claims above to f^*, we find that $f^*(v) \in \{0,3,6,17,18,19,20,21,22,23\}$ for all $v \in V(\Gamma_\square)$.

Let $f(v_0) = 0$. The four labels around v_0 are each $\geq f(v_0) + 8 = 8$. Their labels belong to $\{17,18,19,20,21,22,23\}$, a contradiction since the difference between any pair of them is ≥ 3. Thus, it must be that $\lambda(\Gamma_\square; 8,3) \geq 24$. ■

We have now completed the proof of the formulas for the square lattice, Theorem 4.1.

8 The Proof for the Hexagonal Lattice

We will find the upper bound on $\lambda(\Gamma_H; k, 1)$, $k \geq 0$, by constructions and Lemma 2.2. One construction method is to tile the whole lattice by a labeled parallelogram described by a matrix of labels. We define a doubly periodic labeling of the Hexagonal Lattice by an $m \times n$ labeling matrix $A := [a_{i,j}]$, for m, n even, such that we label point (i,j) by $a_{(n-j) \bmod n\ +1,\ i \bmod m\ +1}$, where i, j are even.

For example, the following labeling (see Figure 12) is defined by the labeling matrix A, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix}$$

Then Figure 12 shows how the labels are assigned, where $a_{4,1}$ is at the vertex with coordinates $(0,0)$ in the hexagonal lattice. The whole lattice is tiled with copies of the 4×6 tile as shown:

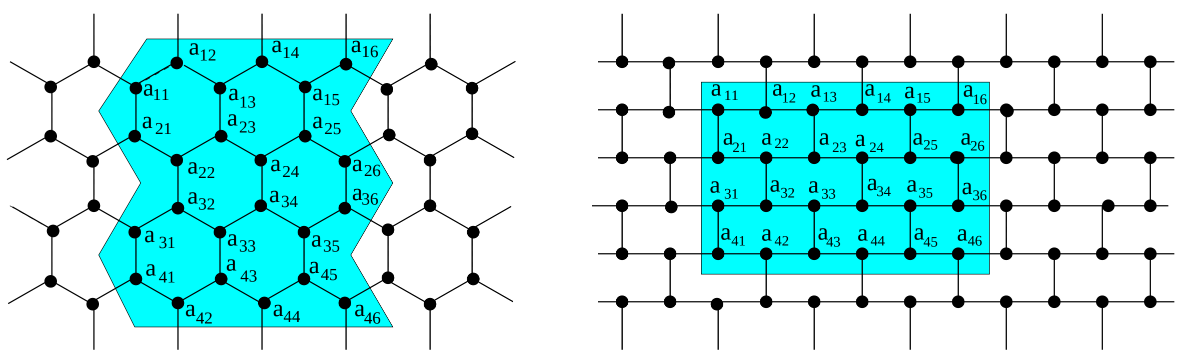


Figure 12: The Doubly Periodic Labeling by Matrix A

Proposition 8.1. *For $0 \leq k \leq \frac{1}{2}$, we have $\lambda(\Gamma_H; k, 1) \leq k + 2$.*

Proof: We use the labeling matrix below, also shown in Figure 13, with the values a, b, c taken to be $k, k+1, k+2$, respectively:

$$A = \begin{bmatrix} 0 & a & 1 & b & 2 & c \\ b & 2 & c & 0 & a & 1 \end{bmatrix}$$

Incidentally, this labeling was obtained by doing a first-fit labeling on one row, then on the next row, and so on. ■

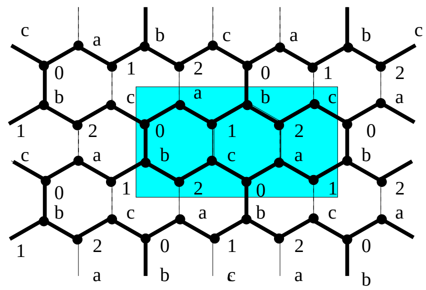


Figure 13: Optimal $L(k,1)$-labeling of Γ_H for $0 \le k \le \frac{1}{2}$ or $k \ge 3$.

We have $\lambda(\Gamma_H; \frac{1}{2}, 1) \le \frac{5}{2}$. By Lemma 2.2, it follows that:

Proposition 8.2. *For $\frac{1}{2} \le k \le \frac{3}{5}$, we have $\lambda(\Gamma_H; k, 1) \le 5k$.* ■

Next we consider $k = 1$:

Proposition 8.3. *We have $\lambda(\Gamma_H; 1, 1) \le 3$. Hence, $\lambda(\Gamma_H; k, 1) \le \begin{cases} 3 & \text{if } \frac{3}{5} \le k \le 1 \\ 3k & \text{if } 1 \le k \le \frac{5}{3} \end{cases}$*

Proof: Because of Lemma 2.2, it is enough to prove the upper bound at $k = 1$.

We will prove $\lambda(\Gamma_H; 1, 1) \le 3$ by using either of the following labeling matrices. Each was obtained by a first-fit labeling process, doing one row at a time. (See Figure 14.)

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 2 \end{bmatrix} \qquad \text{or} \qquad A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \quad ■$$

Proposition 8.4. *For $2 \le k \le 3$, we have $\lambda(\Gamma_H; k, 1) \le 2k + 1$. For $\frac{5}{3} \le k \le 2$, we have $\lambda(\Gamma_H; k, 1) \le 5$.*

Proof: The second statement follows immediately from the first at $k = 2$. For $2 \le k \le 3$, one can prove $\lambda(\Gamma_H; k, 1) \le 2k + 1$ by the matrix labeling with entries shown in Figure 15 (left). A simpler construction can be obtained by adapting a construction of Calamoneri [7], originally given for the corresponding integer labeling. We take the following matrix labeling, shown in Figure 15 (right):

$$A = \begin{bmatrix} 1 & k+1 & 2k+1 & 1 & k+1 & 2k+1 \\ 2k & 0 & k & 2k & 0 & k \end{bmatrix}. \quad ■$$

Next we treat large k:

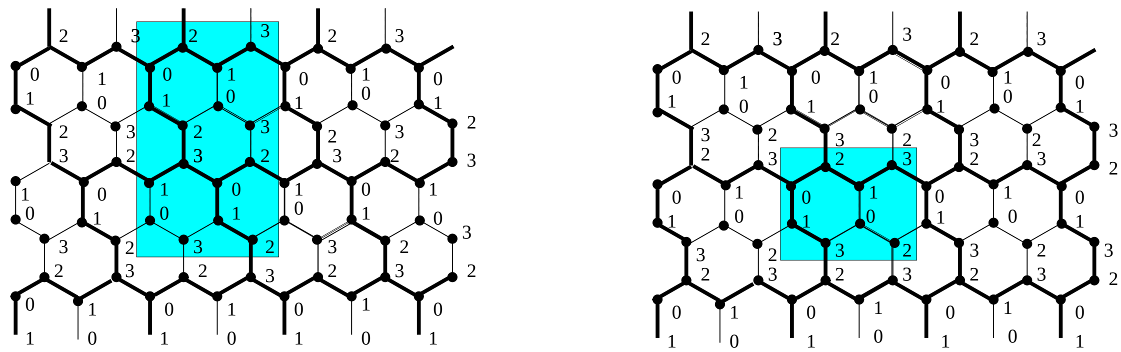


Figure 14: Optimal $L(1,1)$-labeling of Γ_H

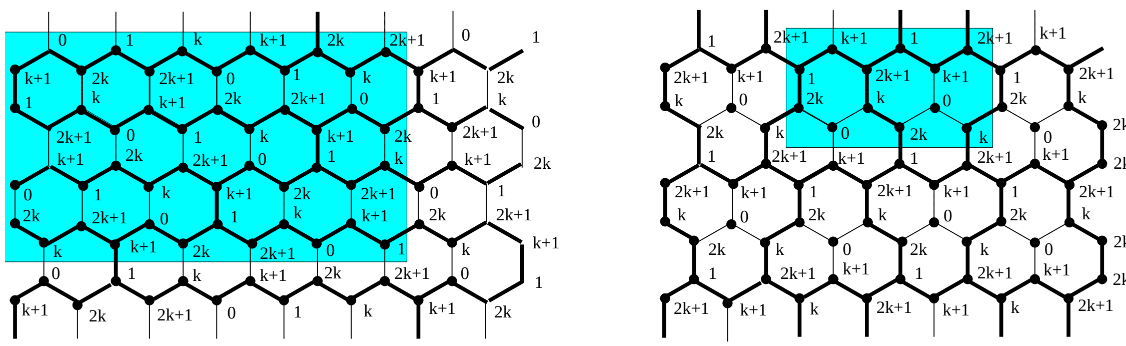


Figure 15: Optimal $L(k,1)$-labelings of Γ_H for $2 \le k \le 3$.

Proposition 8.5. *For $k \ge 3$, we have $\lambda(\Gamma_H; k, 1) \le k + 4$.*

Proof: Following the construction in [7] for the corresponding integer labeling, we again have the matrix labeling as in Figure 13, where this time $a = k+4, b = k+3, c = k+2$:

$$A = \begin{bmatrix} 0 & a & 1 & b & 2 & c \\ b & 2 & c & 0 & a & 1 \end{bmatrix}. \quad \blacksquare$$

We next verify the lower bounds. By Theorem 2.7 we get that for all $k \ge 0$, $\lambda(\Gamma_\square; k, 1) \ge \lambda(T_3; k, 1)$. From the values of $\lambda(T_3; k, 1)$ presented in Theorems 2.3 and 2.5, we obtain the claimed values of $\lambda(\Gamma_\square; k, 1)$ for all k outside the interval $(\frac{3}{2}, 2)$. In this remaining interval, we must improve the lower bound on $\lambda(\Gamma_\square; k, 1)$. In view of Lemma 2.2, all we need to do to complete the proof is to establish the lower bound at $k = 5/3$:

Proposition 8.6. *We have $\lambda(\Gamma_H; 5, 3) \ge 15$. Hence, $\lambda(\Gamma_H; k, 1) \le \begin{cases} 3k & \text{if } 1 \le k \le \frac{5}{3} \\ 5 & \text{if } \frac{5}{3} \le k \le 2 \end{cases}$*

Proof: It suffices to prove the first statement, due to Lemma 2.2. We will show $\lambda(\Gamma_H; 5, 3) \ge 15$.

Assume otherwise, $\lambda(\Gamma_H;5,3) < 15$. Then there exists a $L(5,3)$-labeling f with all labels in the set $\{0,\ldots,14\}$.
Claim 1. *The labeling f cannot use label 4 or 10.*
Proof: Assume $f(v) = 4$ for some $v \in V(\Gamma_H)$. The three distinct labels around v are $\geq f(v) + 5 = 9$. Suppose they are labels $x_1 < x_2 < x_3$. Since any pair of the three labels differ by at least 3 (because they are at distance two each other), one of them is ≥ 15, a contradiction. By the Symmetry Argument, f cannot use label $14 - 4 = 10$. □
Claim 2. *The labeling f cannot use label 5 or 9.*
Proof: Assume $f(v) = 5$ for some $v \in V(\Gamma_H)$. The three labels around v are $\leq f(v)-5 = 0$ or $\geq f(v) + 5 = 10$. But 10 is excluded by the previous Claim. Since any pair of the three labels differ by at least 3 it must be that the three labels used are 0, 11, 14. Then the three neighbors of the label 11 are each $\leq 11-5 = 6$ and any two are at least 3 apart, so they need to be 0, 3, and 6. But this is a contradiction since one of them is $f(v) = 5$. By symmetry, we must also exclude label 9. □
Claim 3. *The labeling f cannot use label 7.*
Proof: Assume $f(v) = 7$ for some $v \in V(\Gamma_H)$. The three labels around v are each $\leq f(v) - 5 = 2$ or $\geq f(v) + 5 = 12$. Each is at most 14. But they have to be at least three apart from each other, and this cannot be done. a contradiction. □
Claim 4. *The labeling f cannot use labels $1, 2, 12, or 13$.*
Proof: Assume $f(v) = 1$ or 2 for some $v \in V(\Gamma_H)$. The three distinct labels around v are $\geq f(v) + 5 \geq 6$. Suppose they are labels $x_1 < x_2 < x_3$. Since any pair of the three labels differ by at least 3, $6 \leq x_1 \leq 8, x_2 = 11, x_3 = 14$. The three neighbors of the label 11 are $\leq 11 - 5 = 6$, where the difference of any pair of them is at least 3, a contradiction with one of them being $f(v) = 1$ or 2. The labels 12, 13 are then excluded by symmetry.□

Now all labels of f belong to $\{0, 3, 6, 8, 11, 14\}$, call this set L.
Claim 5. *The labeling f cannot use label 3 or 11.*
Proof: Assume $f(v) = 3$ for some $v \in V(\Gamma_H)$. The three labels around v are $\geq f(v)+5 = 3 + 5 = 8$. They are $8, 11, 14$ as in Figure 16.

The three neighbors of the label 11 are ≤ 6, with one of them $f(v) = 3$ and the others are $0, 6$. By the separation conditions and set L, we have $c, d \in \{0, 14\}$. We have two cases.

Case 1. $a = 0, b = 6$.

Since $a = 0$, then $c = 14$. We cannot find a feasible label g in L, a contradiction.

Case 2. $a = 6, b = 0$.

Since $b = 0$, then $e \in \{6, 8\}, f = 0$, so that $d = 14$. We cannot find a feasible label h in L, a contradiction. □

Now all labels of f belong to $\{0, 6, 8, 14\}$. We cannot label the induced subgraph $K_{1,3}$, a contradiction. ■

This completes the proof of the span formulas for Γ_H.

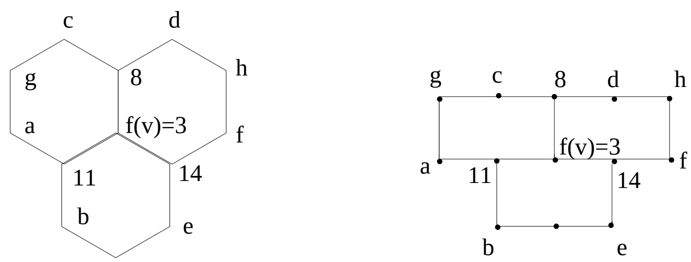


Figure 16: The $L(5,3)$-labeling of a Subgraph of Γ_H

9 Further Research

Despite considerable effort, we have failed so far to complete the determination of all values $\lambda(\Gamma_\Delta; k, 1)$ for $0 < k < 1$. We believe that our upper bounds are the correct values for $k \in [1/3, 1/2]$, while the line $5k + 2$ should be correct for $k \in [1/2, 4/5]$.

We continue to ponder the general properties of λ numbers of graphs. As noted early on in the paper, we have shown in another paper for general graphs G of bounded maximum degree that $\lambda(k, 1)(G)$ is piecewise linear as a function of $k \geq 0$ with only finitely many pieces. The graphs for G being one of the three regular lattices, described in this paper, show that the function, though nondecreasing and continuous, is not in general either concave up or concave down.

To the contrary, we observe that the slopes of successive pieces in these examples exhibit a curious oscillation, alternately increasing and decreasing. Indeed, they all show what we call the *left-right property*: Starting from the first piece at $k = 0$, successive pieces turn left (increasing slope) and right (decreasing slope), ending with a right turn. This left-right property holds for most other graphs we checked as well, but not in general– it appears to fail for wheels consisting of an n-cycle and a vertex adjacent to all of its vertices, $n \geq 5$.

Having determined the lambda numbers of the three lattices with conditions at distance two (almost!), it is natural to extend the investigation to conditions at distance three. This more general question is almost wide open. One definite result in this direction is given by Bertossi, Pinotti, and Tan [4], who determined that $\lambda(\Gamma_\Delta; 2, 1, 1) = 11$. Their construction that achieves the optimal value, 11, can be described by a labeling matrix:

$$A = \begin{bmatrix} 0 & 10 & 2 & 6 & 1 & 9 \\ 4 & 8 & 5 & 11 & 3 & 7 \\ 2 & 6 & 1 & 9 & 0 & 10 \\ 5 & 11 & 3 & 7 & 4 & 8 \\ 1 & 9 & 0 & 10 & 2 & 6 \\ 3 & 7 & 4 & 8 & 5 & 11 \end{bmatrix}.$$

We note that this value of 11 actually holds for $\lambda(\Gamma_\Delta; k, 1, 1)$ for $1 \leq k \leq 2$: Considering the sublattice B_{12} in Figure 17, we see that its twelve vertices must receive distinct labels in an optimal integer labeling, so that $\lambda(\Gamma_\Delta; 1, 1, 1) \geq 11$.

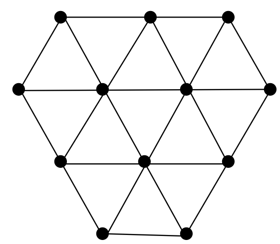


Figure 17: The Subgraph B_{12} of the Triangular Lattice.

We are continuing this project by seeking to describe *all* optimal $L(k, 1)$-labelings of the three regular lattices, and by searching for optimal labelings with nice symmetry properties, such as being periodic or doubly periodic.

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