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“COMPACTLY” SUPPORTED FRAMES FOR SPACES OF DISTRIBUTIONS ON THE BALL

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Abstract. Frames are constructed on the unit ball B^d in $\mathbb{R}^d$ consisting of smooth functions with small shrinking supports. The new frames are designed so that they can be used for decomposition of weighted Triebel Lizorkin and Besov spaces on B^d with weight $w_\mu(x) := (1-|x|^2)^{\mu-1/2}$, μ half integer, $\mu \geq 0$.

1. Introduction

Bases and frames for spaces of functions or distributions are valuable for various theoretical and practical reasons. In this article we focus on the problem for construction of multiscale frames on the unit ball B^d in $\mathbb{R}^d$ consisting of C^∞ functions with small supports which shrink at higher scales. More precisely, our purpose is to construct a frame of the form $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, where $\mathcal{X} = \cup_j \mathcal{X}_j$ is a multilevel index set ($\mathcal{X}_j \subset B^d$), and each jth level frame element θ_ξ ($\xi \in \mathcal{X}_j$) is supported on $B(\xi, c2^{-j})$ the ball centered at $\xi \in B^d$ of radius $c2^{-j}$ with respect to the distance

$$d(x,y) := \arccos\Big\{\langle x,y\rangle + \sqrt{1-|x|^2}\sqrt{1-|y|^2}\Big\} \quad \text{on } B^d. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the Euclidean inner product and norm on $\mathbb{R}^d$, and hence this is just the geodesic distance between the lifted images of $x, y \in B^d$ to the upper unit hemisphere in $\mathbb{R}^{d+1}$. In fact, the set $\mathcal{X}_j$ consisting of the “centers” of the jth level frame elements will be a $c2^{-j}$-net on B^d. The frame $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ to be constructed is reminiscent of compactly supported wavelets on $\mathbb{R}$.

The quality of this tool will be guaranteed by the fact that, as will be shown, $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ can be used for decomposition of weighted Triebel Lizorkin and Besov spaces on B^d with weight

$$w_\mu(x) := (1-|x|^2)^{\mu-1/2}, \tag{1.2}$$

where $\mu \geq 0$ is a half integer (2μ is integer).

The construction of $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ will rely on the general scheme for construction of frames from [3] and the frames (called needlets) for weighted Triebel Lizorkin and Besov spaces on B^d developed in [4, 7]. The overall undertaking hinges on weighted orthogonal polynomials on the ball and related techniques. The gist of our method is in connecting orthogonal polynomials on the ball to the trigonometric system through (i) representation of the orthogonal polynomial projectors by Gegenbauer polynomials (see (2.1)) and (ii) the connection of Gegenbauer polynomials with the trigonometric system via the Dirichlet-Mehler formula (see Lemma 3.4 and (4.5)).

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Observe that a similar construction of frames on the sphere has already been developed in [3]. The inhomogeneity of the current setting on the ball, however, requires more sophisticated tools and techniques than in the case of the sphere. It is an open problem to establish the results of this article in the case when 2μ is not integer.

The paper is organized as follows: In §2 we give all needed prerequisites, which include (i) the weighted Triebel Lizorkin and Besov spaces and frames (needlets) on B^d developed in [7, 4] and (ii) a description of the general method for construction of frames from [3]. In §3 we present the construction of the new frames with small supports and our main results. Section 4 is an appendix, where we give the proofs of some results from §3.

Some useful notation: $L^p = L^p(w_\mu)$ will stand for the weighted space $L^p(B^d, w_\mu)$. We shall denote by $B(\xi, r)$ the ball centered at $\xi \in B^d$ of radius $r > 0$ with respect to the distance $d(\cdot,\cdot)$ in (1.1), i.e. $B(\xi, r) = \{x \in B^d : d(x, \xi) < r\}$. For a measurable set $E \subset B^d$ we shall denote $|E| := \int_E w_\mu(x)\,dx$, $\mathbb{1}_E$ will be the characteristic function of E, and $\tilde{\mathbb{1}} := |E|^{-1/2}\mathbb{1}_E$. Positive constants will be denoted by c, c_1, c_2, ... and they will be allowed to vary at every occurrence; $a \sim b$ will mean $c_1 \le b/a \le c_2$.

2. Background material

In this section we summarize the main results on weighted Triebel Lizorkin and Besov spaces on B^d and frames (needlets) from [7, 4] and review the general method for construction of frames from [3].

2.1. Weighted Triebel Lizorkin and Besov spaces on B^d. We let Π_n denote the space of all algebraic polynomials of degree n in d variables and let V_n be the subspace consisting of all polynomials in Π_n which are orthogonal to lower degree polynomials in $L^2(w_\mu)$. It is shown in [9] that the orthogonal projector $\mathrm{Proj}_n : L^2(w_\mu) \mapsto V_n$ can be written as

$$(\mathrm{Proj}_n f)(x) = \int_{B^d} f(y) P_n(x,y) w_\mu(y) dy, \tag{2.1}$$

where for $\mu > 0$ the kernel $P_n(x, y)$ has the representation

(2.2)
$$P_n(x,y) = b_d^\mu b_1^{\mu-\frac{1}{2}} \frac{n+\lambda}{\lambda} \int_{-1}^{1} C_n^\lambda\left(\langle x,y\rangle + u\sqrt{1-|x|^2}\sqrt{1-|y|^2}\right)(1-u^2)^{\mu-1} du.$$

Here C_n^λ is the n-th degree Gegenbauer polynomial, $\lambda := \mu + \frac{d-1}{2}$, and the constants b_d^μ, $b_1^{\mu-\frac{1}{2}}$ are defined by $(b_d^\gamma)^{-1} := \int_{B^d}(1-|x|^2)^{\gamma-1/2}dx$. For a representation of $P_n(x,y)$ in the limiting case $\mu = 0$, see (4.2) in [7].

It is straightforward to see that [7, Lemma 5.3] for $r < c$, $\xi \in B^d$,

$$|B(\xi,r)| := \int_{B(\xi,r)} w_\mu(x)\,dx \sim r^d(r + \sqrt{1-|\xi|^2})^{2\mu} \sim r^d(r + d(\xi, \partial B^d))^{2\mu}, \tag{2.3}$$

where ∂B^d is the boundary of B^d, i.e. the unit sphere in $\mathbb{R}^d$.

Weighted Triebel-Lizorkin (F-spaces) and Besov spaces (B-spaces) on B^d are naturally defined via orthogonal polynomial decompositions [4]. To be specific, we let $\mathcal{D}$ denote the set of test functions on B^d consisting of all C^∞ complex valued functions on B^d such that

$$\|\phi\|_{W^k_\infty} := \sum_{|\alpha|\le k} \|\partial^\alpha \phi\|_{L^\infty} < \infty \quad \text{for } k = 0, 1, \dots. \tag{2.4}$$

The topology in $\mathcal{D}$ is defined by these norms and as is shown in [4] it can be equivalently defined by the semi-norms

$$\mathcal{N}_k(\phi) := \sup_{n\ge 0} (n+1)^k \|\operatorname{Proj}_n \phi\|_{L^2}, \quad k = 0, 1, \dots. \tag{2.5}$$

The space $\mathcal{D}' = \mathcal{D}'(B^d)$ of distributions on B^d is defined as the set of all continuous linear functionals on $\mathcal{D}$. The pairing of $f \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ will be denoted by $\langle f, \phi\rangle := f(\overline{\phi})$ and as is shown in [4] it is consistent with the inner product $\langle f, g\rangle := \int_{B^d} f(x)\overline{g(x)} w_\mu(x)dx$ in $L^2(w_\mu)$.

If $f \in \mathcal{D}'$ and $\Phi : B^d \times B^d \mapsto \mathbb{C}$ is such that $\Phi(x, \cdot) \in \mathcal{D}$ for all $x \in B^d$, then we let $(\Phi * f)(x) := \langle f, \overline{\Phi(x, \cdot)}\rangle$, where on the right f acts on $\Phi(x, y)$ as a function of y.

Let

$$\Phi_0(x, y) := P_0(x, y) \quad \text{and} \quad \Phi_j(x, y) := \sum_{\nu=0}^{\infty} \hat{a}\Big(\frac{\nu}{2^{j-1}}\Big) P_\nu(x, y), \quad j \ge 1,$$

where $P_\nu(\cdot, \cdot)$ is from (2.2) and $\hat{a}$ satisfies the conditions

$$\begin{aligned} &(i) \quad \hat{a} \in C^\infty[0, \infty), \quad \operatorname{supp} \hat{a} \subset [1/2, 2], \\ &(ii) \quad |\hat{a}(t)| > c > 0 \quad \text{if } t \in [3/5, 5/3]. \end{aligned} \tag{2.6}$$

Definition 2.1. *Let $s, \rho \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. The Tribel-Lizorkin space $F^{s\rho}_{pq}$ is defined as the set of all $f \in \mathcal{D}'$ such that*

$$\|f\|_{F^{s\rho}_{pq}} := \Big\| \Big(\sum_{j=0}^{\infty} (2^{(s-\rho)j} |B(\cdot, 2^{-j})|^{-\rho/d} |\Phi_j * f(\cdot)|)^q \Big)^{1/q} \Big\|_{L^p} < \infty, \tag{2.7}$$

Definition 2.2. *Let $s, \rho \in \mathbb{R}$, $0 < p, q \le \infty$. The Besov space $B^{s\rho}_{pq}$ is defined as the set of all $f \in \mathcal{D}'$ such that*

$$\|f\|_{B^{s\rho}_{pq}} := \Big(\sum_{j=0}^{\infty} \Big(2^{(s-\rho)j} \| |B(\cdot, 2^{-j})|^{-\rho/d} |\Phi_j * f(\cdot)| \|_{L^p} \Big)^q \Big)^{1/q}. \tag{2.8}$$

There is a change of notation in the above definitions compared to [4], which we think makes them more transparent, namely, in [4] the quantities $|B(\cdot, 2^{-j})|$ above are replaced by $2^{-jd} W_\mu(2^j; \cdot)$, where $W_\mu(2^j; x) := \Big(\sqrt{1 - |x|^2} + 2^{-j}\Big)^{2\mu}$. By (2.3), however, $|B(\cdot, 2^{-j})| \sim 2^{-jd} W_\mu(2^j; \cdot)$ and hence these are equivalent norms.

Note that as shown in [4] the above definitions of Triebel-Lizorkin and Besov spaces are independent of the choice of $\hat{a}$ provided conditions (2.6) are satisfied.

Two types of weighted Triebel-Lizorkin and Besov spaces are of main interest: F^{s0}_{pq}, B^{s0}_{pq} and F^{ss}_{pq}, B^{ss}_{pq}. For instance, as is shown in [4] Besov spaces of the form $B^{ss}_{\tau\tau}$ are the natural spaces associated with nonlinear n-term approximation from localized frames (needlets) in $L^p(w_\mu)$, while the approximation spaces of linear approximation from algebraic polynomials in $L^p(w_\mu)$ are of the form B^{s0}_{pq}. The forth parameter ρ above was introduced in [4] in order to unify these spaces and handle them simultaneously. We refer the reader to [4] for more details on the subject.

2.2. **Frames (needlets) on B^d.** In this part we describe the construction of frames in [4], called needlets. These are smooth well localized but global functions on B^d. We shall defer from [4] in two ways: (i) We shall deal here with a single frame, which is a particular case of the construction in [4] where pairs of dual frames are used, and (ii) In the definition of the frame here there will be an insignificant shift in the indices. Let $\hat{a}$ satisfy the conditions

$$\begin{aligned} &(i) &&\hat{a} \in C^\infty[0,\infty), \quad \hat{a} \ge 0, \quad \operatorname{supp} \hat{a} \subset [1/2, 2], \\ &(ii) &&\hat{a}(t) > c > 0, \quad \text{if } t \in [3/5, 5/3], \\ &(iii) &&\hat{a}^2(t) + \hat{a}^2(2t) = 1, \quad \text{if } t \in [1/2, 1] \end{aligned} \tag{2.9}$$

and hence,

$$\sum_{j=0}^{\infty} \hat{a}^2(2^{-j}t) = 1, \quad t \in [1,\infty). \tag{2.10}$$

Choose $j_0 \ge -1$ so that $2^{j_0} \le \lambda < 2^{j_0+1}$. (Recall that $\lambda := \mu + \frac{d-1}{2} \ge 1/2$ and λ is half integer.) We define the kernels $\{\Psi_j\}$ by

$$\Psi_j := \sum_{\nu=0}^{\infty} \hat{a}\Big(\frac{\nu+\lambda}{2^j}\Big) P_\nu, \quad j \ge j_0, \tag{2.11}$$

where in the case $\lambda = 1/2$ we set $\Psi_{j_0} := P_0$. From (2.10)-(2.11) it follows that for any $f \in \mathcal{D}'$

$$f = \sum_{j=j_0}^{\infty} \Psi_j * \Psi_j * f \quad \text{in } \mathcal{D}'. \tag{2.12}$$

This identity also holds in $L^p(w_\mu)$ if $f \in L^p(w_\mu)$, $1 \le p \le \infty$. It is reminiscent of the classical Calderon reproducing formula on $\mathbb{R}^d$ and is further descritized in [4] (see also [7]) by using appropriate cubature formulas. In particular, as is shown in [7] there exists a set $\mathcal{X}_j \subset B^d$ and weights $\{\lambda_\xi\}_{\xi\in\mathcal{X}_j}$ such that the cubature formula

$$\int_{B^d} f(x) w_\mu(x)dx \sim \sum_{\xi\in\mathcal{X}_j} \lambda_\xi f(\xi) \tag{2.13}$$

is exact for all polynomials of degree $\le 2^{j+2}$ in d variables. Furthermore, there is a disjoint partition $\{R_\xi\}_{\xi\in\mathcal{X}_j}$ of B^d ($\cup_{\xi\in\mathcal{X}_j} R_\xi = B^d$) such that R_ξ is "centered" at ξ and the points in $\mathcal{X}_j$ are almost uniformly distributed, i.e. there exist constants $c^*, c^\diamond > 0$ such that

$$B(\xi, c^* 2^{-j}) \subset R_\xi \subset B(\xi, c^\diamond 2^{-j}), \quad \xi \in \mathcal{X}_j. \tag{2.14}$$

In addition,

$$\lambda_\xi \sim |B(\xi, 2^{-j})|, \quad \xi \in \mathcal{X}_j, \tag{2.15}$$

with constants of equivalence depending only on μ and d.

The jth level needlets are defined by

$$\psi_\xi(x) := \lambda_\xi^{1/2} \Psi_j(\xi, x), \quad \xi \in \mathcal{X}_j, \tag{2.16}$$

and the entire needlet system by

$$\Psi := \{\psi_\xi\}_{\xi\in\mathcal{X}}, \quad \text{where} \quad \mathcal{X} := \cup_{j=j_0}^{\infty} \mathcal{X}_j. \tag{2.17}$$

Here equal points from different levels $\mathcal{X}_j$ are regarded as distinct points of the index set $\mathcal{X}$.

The discretization of (2.12) by using cubature formulas (2.13) entails the following representation result: For any $f \in \mathcal{D}'$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } \mathcal{D}'. \tag{2.18}$$

Also, it is easy to show [7] that Ψ is a tight frame for $L^2(w_\mu)$, i.e. for $f \in L^2(w_\mu)$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } L^2(w_\mu) \quad \text{and} \quad \|f\|_{L^2(w_\mu)} = \|(\langle f, \psi_\xi\rangle)\|_{\ell^2(\mathcal{X})}. \tag{2.19}$$

We next define the sequence spaces $f^{s\rho}_{pq}$ and $b^{s\rho}_{pq}$ associated with the spaces $F^{s\rho}_{pq}$ and $B^{s\rho}_{pq}$, respectively.

Definition 2.3. *Suppose $s, \rho \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$. Then $f^{s\rho}_{pq}$ is defined as the space of all complex-valued sequences $h := \{h_\xi\}_{\xi\in\mathcal{X}}$ such that*

$$\|h\|_{f^{s\rho}_{pq}} := \Big\| \Big(\sum_{j=j_0}^{\infty} 2^{(s-\rho)jq} \sum_{\xi\in\mathcal{X}_j} [|h_\xi| |B(\xi, 2^{-j})|^{-\rho/d} \tilde{\mathbb{1}}_{R_\xi}(\cdot)]^q \Big)^{1/q} \Big\|_{L^p} < \infty \tag{2.20}$$

with the usual modification for $q = \infty$. Recall the notation $\tilde{\mathbb{1}}_{R_\xi} := |R_\xi|^{-1/2}\mathbb{1}_{R_\xi}$ with $\mathbb{1}_{R_\xi}$ being the characteristic function of R_ξ.

Definition 2.4. *Let $s, \rho \in \mathbb{R}$ and $0 < p, q \le \infty$. Then $b^{s\rho}_{pq}$ is defined as the space of all complex-valued sequences $h := \{h_\xi\}_{\xi\in\mathcal{X}}$ such that*

$$\|h\|_{b^{s\rho}_{pq}} := \Big(\sum_{j=j_0}^{\infty} 2^{(s-\rho)jq} \Big[\sum_{\xi\in\mathcal{X}_j} \Big(|B(\xi, 2^{-j})|^{-\rho/d+1/p-1/2} |h_\xi| \Big)^p \Big]^{q/p} \Big)^{1/q} \tag{2.21}$$

is finite, with the usual modification for $p = \infty$ or $q = \infty$.

The main result in [4] asserts that Ψ is a frame for Triebel-Lizorkin and Besov spaces on B^d in the sense of the following theorem.

Theorem 2.5. [4] *Let $s \in \mathbb{R}$ and $0 < p, q < \infty$.*

(a) If $f \in \mathcal{D}'$, then $f \in F^{s\rho}_{pq}$ if and only if $(\langle f, \psi_\xi\rangle)_{\xi\in\mathcal{X}} \in f^{s\rho}_{pq}$. Moreover, if $f \in F^{s\rho}_{pq}$, then

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \textit{and} \quad \|f\|_{F^{s\rho}_{pq}} \sim \|(\langle f, \psi_\xi\rangle)\|_{f^{s\rho}_{pq}}. \tag{2.22}$$

(b) If $f \in \mathcal{D}'$, then $f \in B^{s\rho}_{pq}$ if and only if $(\langle f, \psi_\xi\rangle)_{\xi\in\mathcal{X}} \in b^{s\rho}_{pq}$. Moreover, if $f \in B^{s\rho}_{pq}$, then

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \textit{and} \quad \|f\|_{B^{s\rho}_{pq}} \sim \|(\langle f, \psi_\xi\rangle)\|_{B^{s\rho}_{pq}}. \tag{2.23}$$

The convergence in (2.22) and (2.23) is unconditional in $F^{s\rho}_{pq}$ and $B^{s\rho}_{pq}$, respectively.

2.3. **General scheme for construction of frames.** Here we describe the method for construction of frames developed in [3]. Assume that H is a separable complex Hilbert space (of functions) and $\mathcal{S} \subset H$ is a linear subspace (of test functions) furnished with a locally convex topology induced by a sequence of norms or semi-norms. Let $\mathcal{S}'$ be the dual of $\mathcal{S}$ consisting of all continuous linear functionals on $\mathcal{S}$ and assume that $H \subset \mathcal{S}'$.

Assume further that $L \subset \mathcal{S}'$ with norm $\|\cdot\|_L$ is a quasi-Banach space of distributions, which is continuously embedded in $\mathcal{S}'$, $\mathcal{S} \subset H \cap L$ and $\mathcal{S}$ is dense in H and L.

We also assume that $\ell(\mathcal{X})$ with norm $\|\cdot\|_{\ell(\mathcal{X})}$ is an associated to L quasi-Banach space of complex-valued sequences with domain a countable index set $\mathcal{X}$. Coupled with a frame Ψ the sequence space $\ell(\mathcal{X})$ will be used for characterization of the space L. In addition to being a quasi-norm we assume that $\|\cdot\|_{\ell(\mathcal{X})}$ obeys the conditions:

(i) For any sequence $(h_\eta)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ one has $\|(h_\xi)\|_{\ell(\mathcal{X})} = \|(|h_\xi|)\|_{\ell(\mathcal{X})}$ and $|h_\xi| \le c\|h\|_{\ell(\mathcal{X})}$ for $\xi \in \mathcal{X}$.

(ii) If the sequences $(h_\xi)_{\xi\in\mathcal{X}}, (g_\xi)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ and $|h_\xi| \le |g_\xi|$ for $\xi \in \mathcal{X}$, then $\|(h_\xi)\|_{\ell(\mathcal{X})} \le c\|(g_\xi)\|_{\ell(\mathcal{X})}$.

(iii) Compactly supported sequences are dense in $\ell(\mathcal{X})$.

The existing (old) frame. Our next assumption is that $\Psi := \{\psi_\xi\}_{\xi\in\mathcal{X}} \subset \mathcal{S}$, where $\mathcal{X}$ is a countable index set, is a frame for H, that is, for any $f \in H$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } H \quad \text{and} \quad \|f\|_H \sim \|(\langle f, \psi_\xi\rangle)\|_{\ell^2(\mathcal{X})}. \tag{2.24}$$

More importantly, we assume that Ψ is a frame for L in the following sense:

A1. For any $f \in L$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } \; L. \tag{2.25}$$

A2. For any $f \in L$, $(\langle f, \psi_\xi\rangle)_\xi \in \ell(\mathcal{X})$, and

$$c_1\|f\|_L \le \|(\langle f, \psi_\xi\rangle)\|_{\ell(\mathcal{X})} \le c_2\|f\|_L. \tag{2.26}$$

The goal is by "small perturbation" of the elements of the existing frame Ψ to construct a new system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\}$ with some prescribed features, which is a frame for L in the following sense:

Definition 2.6. *We say that $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset H$ is a frame for the space L with associated sequence space $\ell(\mathcal{X})$ if the following conditions are obeyed:*

B1. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|_L \le \|(\langle f, \theta_\xi\rangle)\|_{\ell(\mathcal{X})} \le c_2\|f\|_L \quad \text{for } f \in L, \tag{2.27}$$

where $\langle f, \theta_\xi\rangle$ is defined by $\langle f, \theta_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta\rangle\langle\psi_\eta, \theta_\xi\rangle$.

B2. *The frame operator $S : L \mapsto L$ defined by*

$$Sf = \sum_{\xi\in\mathcal{X}} \langle f, \theta_\xi\rangle \theta_\xi$$

is bounded and invertible on L; S^{-1} is also bounded on L and

$$S^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle S^{-1}\theta_\xi \quad \text{in } \; L.$$

B3. *There exist constants* $c_3, c_4 > 0$ *such that*

$$c_3\|f\|_L \le \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{\ell(\mathcal{X})} \le c_4\|f\|_L \quad \textit{for } f \in L, \tag{2.28}$$

where as above by definition $\langle f, S^{-1}\theta_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta\rangle\langle\psi_\eta, S^{-1}\theta_\xi\rangle$.

B4. *For any* $f \in L$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi = \sum_{\xi\in\mathcal{X}} \langle f, \theta_\xi\rangle S^{-1}\theta_\xi \quad \textit{in } L. \tag{2.29}$$

Above "in H" or "in L" means that the convergence is unconditional in H or in L.

Construction of a new frame. The key idea of the method from [3] for constructing a new frame $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\}$ for L (as described above) is to build $\{\theta_\xi\}$ with appropriate localization and approximation properties with respect to the given tight frame Ψ. The localization of Θ is measured in terms of the size of the inner products $\langle\psi_\xi, \psi_\eta\rangle$, $\langle\theta_\eta, \psi_\xi\rangle$, $\langle\psi_\xi, \theta_\eta\rangle$. More precisely, we construct $\{\theta_\xi\}$ so that the operators with matrices

$$\begin{aligned} \mathbf{A} &:= (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & a_{\xi,\eta} &:= \langle\psi_\eta, \psi_\xi\rangle, \\ \mathbf{B} &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & b_{\xi,\eta} &:= \langle\theta_\eta, \psi_\xi\rangle, \\ \mathbf{C} &:= (c_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & c_{\xi,\eta} &:= \langle\psi_\eta, \theta_\xi\rangle, \end{aligned} \tag{2.30}$$

are bounded on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$. The approximation property of Θ is measured in terms of the size of the inner products $\langle\psi_\eta, \psi_\xi - \theta_\xi\rangle$, $\langle\psi_\eta - \theta_\eta, \psi_\xi\rangle$. Namely, we construct $\{\theta_\xi\}$ so that the operators with matrices

$$\begin{aligned} \mathbf{D} &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & d_{\xi,\eta} &:= \langle\psi_\eta, \psi_\xi - \theta_\xi\rangle, \\ \mathbf{E} &:= (e_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & e_{\xi,\eta} &:= \langle\psi_\eta - \theta_\eta, \psi_\xi\rangle, \end{aligned} \tag{2.31}$$

are bounded on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$ and for a sufficiently small $\varepsilon > 0$

$$\|\mathbf{D}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \le \varepsilon, \quad \|\mathbf{E}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \le \varepsilon, \tag{2.32}$$

$$\|\mathbf{D}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \le \varepsilon, \quad \|\mathbf{E}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \le \varepsilon. \tag{2.33}$$

Notice that $\mathbf{C} = \mathbf{B}^*$ the adjoint of $\mathbf{B}$ and $\mathbf{E} = \mathbf{D}^*$.

We shall utilize the following results from [3].

Theorem 2.7. *Let* $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ *be a frame for* H *and* L *as described above. Suppose the system* $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset H$ *is constructed so that the operators with matrices* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ *from* (2.30)-(2.31) *are bounded on* $\ell(\mathcal{X})$ *and* $\mathbf{C}$, $\mathbf{D}$ *are bounded on* $\ell^2(\mathcal{X})$ *as well. Then if for a sufficiently small* $\varepsilon > 0$ *the matrices* $\mathbf{D}$, $\mathbf{E}$ *obey* (2.32)-(2.33)*, the sequence* Θ *is a frame for* L *in the sense of Definition 2.6.*

Most importantly, if $f \in \mathcal{S}'$*, then* $f \in L$ *if and only if* $(\langle f, S^{-1}\theta_\xi\rangle) \in \ell(\mathcal{X})$*, and for* $f \in L$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \textit{in } L \quad \textit{and} \quad \|f\|_L \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{\ell(\mathcal{X})}. \tag{2.34}$$

3. New frame with elements of small shrinking supports on B^d

In this section we present our construction of the desired new frame on the ball and show that the new frame can be used for decomposition of weighted Triebel Lizorkin and Besov spaces on B^d. For convenience we shall divide the proof of our main result into several parts.

3.1. **Construction of the new frame.** To construct our new frame with elements of small support on B^d will utilize the results form §2.3, where $H := L^2(B^d)$; $\mathcal{S} := \mathcal{D}$ and $\mathcal{S}' := \mathcal{D}'$ are the classes of test functions and distributions from §2.1, $L := F^{s\rho}_{pq}$ or $L := B^{s\rho}_{pq}$, the $F-$ or $B-$ spaces from §2.1. The role of of the old frame will be played by the needlet frame Ψ described in §2.2 and the sequence space $\ell(\mathcal{X}) := f^{s\rho}_{pq}$ or $\ell(\mathcal{X}) := b^{s\rho}_{pq}$, the $f-$ or $b-$ spaces from §2.1. It is readily seen that these spaces and the frame Ψ satisfy all the requirements from §2.3.

As suggested by Theorem 2.7 the new frame $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$ should be constructed to be well "localized" and sufficiently "close" to the needlet system Ψ.

Combining (2.16) with (2.11) shows that the needlets $\{\psi_\xi\}$ have the representation

$$\psi_\xi(x) := \lambda_\xi^{1/2}\sum_{\nu=0}^{\infty}\hat{a}\Big(\frac{\nu+\lambda}{2^j}\Big)P_\nu(\xi,x), \quad \xi\in\mathcal{X}_j,\ j>j_0,$$

where $\hat{a}$ is from (2.9). Denote again by $\hat{a}$ the even extension of $\hat{a}$ to $\mathbb{R}$, i.e. $\hat{a}(-t) = \hat{a}(t)$. We shall use the following definition of the Fourier transform $\hat{f}$ of a function f on $\mathbb{R}$: $\hat{f}(\xi) := \int_{\mathbb{R}} f(y)e^{-i\xi y}dy$. Then the inverse Fourier transform a of $\hat{a}$ is real valued, even, and belongs to the Schwartz class $\mathcal{S}$ of rapidly decaying C^∞ functions on $\mathbb{R}$.

Recall our assumption that $\mu\ge 0$ and $2\mu\in\mathbb{N}_0$.

We shall construct the new frame Θ of the form $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$, where $\mathcal{X} := \{\mathcal{X}_j\}_{j\ge j_0}$ is the index set of the needlet system Ψ and $\operatorname{supp}\theta_\xi\subset B(\xi, c2^{-j})$. We proceed in two steps:

Step 1: Given $M>1$, an integer $N\ge 1$, and $\varepsilon>0$, we construct $g\in C^\infty(\mathbb{R})$ so that g is even and obeys the following conditions:

$$\begin{aligned}
&(i) && \operatorname{supp} g\subset[-R,R] \ \text{ for some } R>0,\\
&(ii) && |a^{(r)}(t)-g^{(r)}(t)|\le\varepsilon(1+|t|)^{-M} \quad \text{for } 0\le r\le N+2\mu+d-1,\\
&(iii) && \int_{\mathbb{R}} t^r g(t)\,dt = 0 \quad \text{for } 0\le r\le N+2\mu+d-2.
\end{aligned}\tag{3.1}$$

Note that the Fourier transform $\hat{g}$ of g is even and belongs to $\mathcal{S}$. A function g of this sort has already been constructed and used for the development of frames on the sphere in [3]. For the reader's convenience we sketch the somewhat simplified construction of g in comparison with [3] in the appendix.

Step 2: For any $\xi\in\mathcal{X}_j$ $(j>j_0)$ we define θ_ξ by

$$\theta_\xi(x) := \lambda_\xi^{1/2}\sum_{\nu=0}^{\infty}\hat{g}\Big(\frac{\nu+\lambda}{2^j}\Big)P_\nu(\xi,x)\tag{3.2}$$

and set $\theta_\xi := \psi_\xi$ if $\xi\in\mathcal{X}_{j_0}$. Then $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$ is our new system on B^d.

With the next theorem we show that for appropriately selected parameters M, N, and ε the new system Θ has the claimed support property and is a frame for the F- and B- spaces.

In the following we shall use the notation $\mathcal{J} := (d+2\mu)/\min\{1,p,q\}$ in the case of F-spaces and $\mathcal{J} := (d+2\mu)/\min\{1,p\}$ for B-spaces.

Theorem 3.1. *Suppose $\mu \in 2\mathbb{N}_0$, $s \in \mathbb{R}$, $0 < p, q < \infty$ and let $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$ be the system constructed above, where*

$$M > \mathcal{J} + 2\mu|\rho/d + 1/2| \quad \text{and} \quad N > \max\{s, \mathcal{J} - d - s, 1\} + (4\mu + 2d)|\rho/d + 1/2|.$$

Then for a sufficiently small $\varepsilon > 0$ the system Θ is a frame for the spaces $L^2(B^d)$, $F^{s\rho}_{pq}$, and $B^{s\rho}_{pq}$ in the sense of Definition 2.6 with the above selection of the spaces H, L, $\ell(\mathcal{X})$. In particular, we have

(a) The operator

$$Sf := \sum_{\xi\in\mathcal{X}} \langle f, \theta_\xi\rangle \theta_\xi, \tag{3.3}$$

where $\langle f, \theta_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta\rangle\langle\psi_\eta, \theta_\xi\rangle$, is bounded and invertible on $L^2(B^d)$, $F^{s\rho}_{pq}$, $B^{s\rho}_{pq}$, and S^{-1} is also bounded on $L^2(B^d)$, $F^{s\rho}_{pq}$, $B^{s\rho}_{pq}$, and

$$S^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle S^{-1}\theta_\xi. \tag{3.4}$$

(b) If $f \in \mathcal{D}'$, then $f \in F^{s\rho}_{pq}$ if and only if $(\langle f, S^{-1}\theta_\xi\rangle) \in f^{s\rho}_{pq}$, and for $f \in F^{s\rho}_{pq}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \text{and} \quad \|f\|_{F^{s\rho}_{pq}} \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{f^{s\rho}_{pq}}. \tag{3.5}$$

(c) If $f \in \mathcal{D}'$, then $f \in B^{sr}_{pq}$ if and only if $(\langle f, S^{-1}\theta_\xi\rangle) \in b^{s\rho}_{pq}$, and for $f \in B^{s\rho}_{pq}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \text{and} \quad \|f\|_{B^{sr}_{pq}} \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{b^{sr}_{pq}}. \tag{3.6}$$

The convergence in (3.3)-(3.6) is unconditional in the respective space L^2, $F^{s\rho}_{pq}$, or $B^{s\rho}_{pq}$. Above, (b) and (c) also hold with the roles of θ_ξ and $S^{-1}\theta_\xi$ interchanged.

Moreover, for any $\xi \in \mathcal{X}_j$, $j \geq j_0$, the frame element θ_ξ is supported on the ball $B(\xi, c_*2^{-j}) \subset B^d$, where $c_* = \pi R/2$ with $R > 0$ the constant from (3.1).

3.2. **Almost diagonal matrices.** By Theorem 2.7 it readily follows that the new system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\}$ will be a frame for $F^{s\rho}_{pq}$ (or $B^{s\rho}_{pq}$) if the operators with matrices

$$\begin{aligned}
\mathbf{A} &:= (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & a_{\xi,\eta} &:= \langle\psi_\eta, \psi_\xi\rangle,\\
\mathbf{B} &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & b_{\xi,\eta} &:= \langle\theta_\eta, \psi_\xi\rangle,\\
\mathbf{C} &:= (c_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & c_{\xi,\eta} &:= \langle\psi_\eta, \theta_\xi\rangle\\
\mathbf{D} &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & d_{\xi,\eta} &:= \langle\psi_\eta, \psi_\xi - \theta_\xi\rangle,\\
\mathbf{E} &:= (e_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, & e_{\xi,\eta} &:= \langle\psi_\eta - \theta_\eta, \psi_\xi\rangle,
\end{aligned} \tag{3.7}$$

are bounded on $f^{s\rho}_{pq}$ (or $b^{s\rho}_{pq}$), and $\|\mathbf{D}\|_{f^{s\rho}_{pq}\mapsto f^{s\rho}_{pq}} \leq \varepsilon$, $\|\mathbf{E}\|_{f^{s\rho}_{pq}\mapsto f^{s\rho}_{pq}} \leq \varepsilon$ (or we have $\|\mathbf{D}\|_{b^{s\rho}_{pq}\mapsto b^{s\rho}_{pq}} \leq \varepsilon$, $\|\mathbf{E}\|_{b^{s\rho}_{pq}\mapsto b^{s\rho}_{pq}} \leq \varepsilon$) for sufficiently small ε.

In analogy with the classical case on $\mathbb{R}^n$ (see [2]), we shall show the boundedness of the above operators by using the machinery of the almost diagonal operators.

To avoid complicated indices we shall use the notation

$$r(\xi) := 2^{-j} \quad \text{if } \xi \in \mathcal{X}_j, \tag{3.8}$$

i.e. $r(\xi)$ is the radius of $B(\xi, 2^{-j})$, and

$$B_\xi := B(\xi, 2^{-j}) \quad \text{if} \;\; \xi \in \mathcal{X}_j. \tag{3.9}$$

Definition 3.2. *Let $\mathbf{A}$ be a linear operator acting on sequences of the form $\{h_\xi\}_{\xi\in\mathcal{X}}$ with associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$. We say that $\mathbf{A}$ is almost diagonal if there exists $\delta > 0$ such that*

$$\sup_{\xi,\eta\in\mathcal{X}} \frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)} < \infty,$$

where

$$\begin{aligned}\omega_\delta(\xi,\eta) \;\; := \;\; & \Big(\frac{r(\xi)}{r(\eta)}\Big)^{s-\rho-d/2}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\Big(1+\frac{d(\xi,\eta)}{\max\{r(\xi),r(\eta)\}}\Big)^{-\mathcal{J}-\delta} \\ \times \;\; & \min\Big\{\Big(\frac{r(\xi)}{r(\eta)}\Big)^{(d+\delta)/2}, \Big(\frac{r(\eta)}{r(\xi)}\Big)^{(d+\delta)/2+\mathcal{J}-d}\Big\},\end{aligned}$$

with $\mathcal{J} := (d+2\mu)/\min\{1,p,q\}$ for $f^{s\rho}_{pq}$ and $\mathcal{J} := (d+2\mu)/\min\{1,p\}$ for $b^{s\rho}_{pq}$.

We shall show that the almost diagonal operators are bounded on $f^{s\rho}_{pq}$ and $b^{s\rho}_{pq}$. More precisely, with the notation

$$\|\mathbf{A}\|_\delta := \sup_{\xi,\eta\in\mathcal{X}} \frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)} \tag{3.10}$$

the following result holds:

Theorem 3.3. *Suppose $s\in\mathbb{R}$, $0<q\le\infty$, and $0<p<\infty$ ($0<p\le\infty$ in the case of b-spaces) and let $\|\mathbf{A}\|_\delta<\infty$ (in the sense of Definition 3.2) for some $\delta>0$. Then there exists a constant $c>0$ such that for any sequence $h := \{h_\xi\}_{\xi\in\mathcal{X}} \in f^{sr}_{pq}$*

$$\|\mathbf{A}h\|_{f^{s\rho}_{pq}} \le c\|\mathbf{A}\|_\delta \|h\|_{f^{s\rho}_{pq}}, \tag{3.11}$$

and for any sequence $h := \{h_\xi\}_{\xi\in\mathcal{X}} \in b^{sr}_{pq}$

$$\|\mathbf{A}h\|_{b^{s\rho}_{pq}} \le c\|\mathbf{A}\|_\delta \|h\|_{b^{s\rho}_{pq}}. \tag{3.12}$$

For the proof of this theorem we shall use the idea of the proof of Theorem 3.3 in [2]. We give it in the appendix.

3.3. **Estimation of supp $\boldsymbol{\theta_\xi}$ and localization of kernels.** We shall have to deal with kernels of the form

$$\Lambda_n(x,y) := \sum_{\nu\ge 0} \hat\sigma\Big(\frac{\nu+\lambda}{n}\Big) P_\nu(x,y), \tag{3.13}$$

where the function σ has a certain decay and smoothness properties. The explicit representation of $P_\nu(x,y)$ in (2.2) leads to

$$\Lambda_n(x,y) = b_d^\mu b_1^{\mu-\frac12}\int_{-1}^{1} Q_n\big(\langle x,y\rangle + u\sqrt{1-|x|^2}\sqrt{1-|y|^2}\big)(1-u^2)^{\mu-1}du, \tag{3.14}$$

where

$$Q_n(x) := \sum_{\nu\ge 0}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)\frac{\nu+\lambda}{\lambda} C^\lambda_\nu(x). \tag{3.15}$$

Lemma 3.4. *Let $2\mu \in \mathbb{N}_0$ and $\lambda = \mu + \frac{d-1}{2}$. Then for any even function $\sigma \in \mathcal{S}$ the kernel Q_n from above has the representation*

$$Q_n(\cos\alpha) = c(\sin\alpha)^{1-2\lambda}\int_\alpha^\pi (\cos\alpha - \cos\varphi)^{\lambda-1}K_n(\varphi)d\varphi,\ \ 0 \le \alpha \le \pi, \tag{3.16}$$

where

$$K_n(\alpha) = (\pi/2)n\sum_{\nu\in\mathbb{Z}}(-1)^{\nu(2\mu+d-1)}S\Big(\frac{d}{d\alpha}\Big)\sigma(n(\alpha+2\pi\nu)) \tag{3.17}$$

with

$$S(z) := \prod_{r=1}^{\lfloor \mu+\frac{d-1}{2}\rfloor}(-z^2-(\lambda-r)^2) \times \begin{cases} -z\sin\lambda\pi, & 2\mu+d \text{ even} \\ \cos\lambda\pi, & 2\mu+d \text{ odd}\end{cases} \tag{3.18}$$

and $c > 0$ depends only on d and μ.

This key lemma is quite similar to Proposition 3.2 in [5] and Lemma 3.11 in [3]. For the reader's convenience we give its proof in the appendix.

We next use the above lemma to establish localization estimates, first, for Q_n from (3.15) and, second, for the kernels Λ_n from (3.13).

Lemma 3.5. *If $\sigma \in \mathcal{S}$ (the Schwartz class) is even and*

$$|\sigma^{(m)}(t)| \le \frac{A}{(1+|t|)^M},\quad t\in\mathbb{R},\ 0\le m\le 2\mu+d-1, \tag{3.19}$$

for some constants $M > 1$ and $A > 0$, then

$$|Q_n(\cos\alpha)| \le \frac{c_1An^{2\mu+d}}{(1+n\alpha)^M},\quad 0\le\alpha\le\pi, \tag{3.20}$$

and

$$|\Lambda_n(x,y)| \le \frac{c_2A}{\sqrt{|B(x,n^{-1})|}\sqrt{|B(y,n^{-1})|}(1+n\,d(x,y))^M},\quad x,y\in B^d. \tag{3.21}$$

Here $c_1, c_2 > 0$ depend only on M, μ, and d.

Proof. Representation (3.17) and the fact that $S(z)$ from (3.18) is a polynomial of degree $2\mu + d - 1$ readily imply

$$|K_n(\alpha)| \le cAn\sum_{\nu\in\mathbb{Z}}\frac{n^{2\mu+d-1}}{(1+n|\alpha+2\pi\nu|)^M} \le \frac{cAn^{2\mu+d}}{(1+n\alpha)^M}.$$

We use this in (3.16) precisely as in the proof of Lemma 2 in [6] to obtain (3.20). Finally, we use (3.20) in (3.14) as in the proof of Theorem 4.2 in [7] to obtain (3.21). We omit the details. □

Lemma 3.6. *We have*

$$\operatorname{supp}\theta_\xi \subset B(\xi,\pi R2^{-j-1})\quad \text{for } \xi\in\mathcal{X}_j,\ j>j_0. \tag{3.22}$$

Proof. Here we shall use the kernels Q_n from (3.15) and Λ_n from (3.13) with $\sigma = g$, where g is from the definition of θ_ξ in (3.2).

Assuming that $\xi \in \mathcal{X}_j$, $j > j_0$, we have by the definition of θ_ξ in (3.2) and (2.2)

$$\theta_\xi(x) = \lambda_\xi^{1/2}b_d^\mu b_1^{\mu-\frac12}\int_{-1}^1 Q_{2^j}\big(\langle x,\xi\rangle + u\sqrt{1-|x|^2}\sqrt{1-|\xi|^2}\big)(1-u^2)^{\mu-1}du, \tag{3.23}$$

and by Lemma 3.4

$$Q_{2^j}(\cos\alpha) = c(\sin\alpha)^{1-2\lambda}\int_\alpha^\pi (\cos\alpha - \cos\varphi)^{\lambda-1}K_{2^j}(\varphi)d\varphi, \quad 0 \le \alpha \le \pi,$$

where

$$K_{2^j}(\alpha) = (\pi/2)n\sum_{\nu\in\mathbb{Z}}(-1)^{\nu(2\mu+d-1)}S\Big(\frac{d}{d\alpha}\Big)g(2^j(\alpha+2\pi\nu)).$$

By (3.1) we have $\operatorname{supp} g \subset [-R, R]$ that readily implies $\operatorname{supp} K_{2^j} \subset [-R2^{-j}, R2^{-j}]$ if $R2^{-j} < \pi$. Therefore, $Q_{2^j}(\cos\alpha) = 0$ if $\alpha \ge R/2^j$. We set $t = \cos\alpha$ and use that $1 - \cos\alpha = 2\sin^2(\alpha/2) \le \alpha^2/2$ to obtain

$$Q_{2^j}(t) = 0 \quad \text{if } t \in [-1, 1] \text{ and } 1 - t \ge R^2/2^{2j+1}. \tag{3.24}$$

Denote briefly $t := \langle x, \xi\rangle + u\sqrt{1-x^2}\sqrt{1-y^2}$. Then using (**??**) we get as in [7]

$$\begin{aligned}
1 - t &= 1 - \langle x, \xi\rangle - u\sqrt{1-x^2}\sqrt{1-y^2}\\
&= 1 - \langle x, \xi\rangle - \sqrt{1-x^2}\sqrt{1-y^2} + (1-u)\sqrt{1-x^2}\sqrt{1-y^2}\\
&\ge 1 - \langle x, \xi\rangle - \sqrt{1-x^2}\sqrt{1-y^2} = 1 - \cos d(x,\xi) = 2\sin^2\frac{d(x,y)}{2} \ge \frac{2}{\pi^2}d(x,\xi)^2.
\end{aligned}$$

From this and (3.24) it follows that $Q_{2^j}(t) = 0$ (with t from above) if $d(x,\xi) \ge \frac{\pi R}{2^{j+1}}$. Consequently, on account of (3.23), $\theta_\xi(x) = 0$ if $d(x,\xi) \ge \pi R 2^{-j-1}$, which proofs (3.22) in this case. The case when $R2^{-j} \ge \pi$ is trivial. □

3.4. **Estimation of inner products.** For the proof of our main result - Theorem 3.1 we need to study the localization properties of inner products $\langle U_\xi, V_\eta\rangle$, where

$$U_\xi(x) := \lambda_\xi^{1/2}\sum_{\nu=0}^\infty \hat{u}\Big(\frac{\nu+\lambda}{2^j}\Big)P_\nu(x,\xi),\ V_\eta(x) := \lambda_\eta^{1/2}\sum_{\nu=0}^\infty \hat{v}\Big(\frac{\nu+\lambda}{2^k}\Big)P_\nu(x,\eta). \tag{3.25}$$

For a given function u on $\mathbb{R}$ we denote $u_j(t) := 2^j u(2^j t)$. We start with a well know lemma:

Lemma 3.7. *Suppose the functions $u \in C^N(\mathbb{R})$ and $v \in C(\mathbb{R})$ satisfy the conditions:*

$$|u^{(r)}(t)| \le \frac{A_1}{(1+|t|)^{M_1}}, \quad 0 \le r \le N, \quad |v(t)| \le \frac{A_2}{(1+|t|)^{M_2}},$$

and

$$\int_{\mathbb{R}} t^r v(t)dt = 0 \quad \textit{for } 0 \le r \le N-1,$$

where $N \ge 1$, $M_2 \ge M_1 > 1$, $M_2 > N+1$, and $A_1, A_2 > 0$. Then for $k \ge j$

$$|u_j * v_k(t)| \le cA_1A_2 2^{-(k-j)N}\frac{2^j}{(1+2^j|t|)^{M_1}},$$

where $c > 0$ depends only on M_1, M_2, and N.

This lemma is quite similar to Lemma B.1 in [2]. We omit its proof.

We now turn to the estimation of inner products of functions U_ξ, V_η as above.

Lemma 3.8. *Suppose $u, v \in \mathcal{S}$ are both even and real valued,*

$$|u^{(m)}(t)| \le \frac{A_1}{(1+|t|)^M} \quad \text{and} \quad |v^{(m)}(t)| \le \frac{A_2}{(1+|t|)^M},\ 0 \le m \le N+2\mu+d-1, \tag{3.26}$$

and

$$\int_{\mathbb{R}} t^r u(t)dt = \int_{\mathbb{R}} t^r v(t)dt = 0, \quad 0 \le m \le N-1, \tag{3.27}$$

where $N > 1$ and $M > N+1$. Then for $\xi \in \mathcal{X}_j$ and $\eta \in \mathcal{X}_k$

$$|\langle U_\xi, V_\eta\rangle| \le cA_1A_2 2^{-|k-j|(N+d/2)}\big(1+2^{\min\{k,j\}}d(\xi,\eta)\big)^{-M}. \tag{3.28}$$

Proof. Because of the symmetry in (3.28) we may assume that $k \ge j$. Since $P_n(\cdot,\cdot)$ is the kernel of the orthogonal projector $\mathrm{Proj}_n : L^2(w_\mu) \mapsto V_n$ we have

$$\int_{B^d} P_m(x,\xi)P_\ell(x,\eta)w_\mu(x)dx = \delta_{m,\ell}P_m(\xi,\eta).$$

Using this and the fact that $\lambda_\xi \sim |B(\xi, 2^{-j})|$ for $\xi \in \mathcal{X}_j$ we obtain for $\xi \in \mathcal{X}_j$ and $\eta \in \mathcal{X}_k$

$$\langle U_\xi, V_\eta\rangle \sim |B(\xi,2^{-j})|^{1/2}|B(\eta,2^{-k})|^{1/2}\sum_{\nu=0}^{\infty}\hat{u}\Big(\frac{\nu+\lambda}{2^j}\Big)\hat{v}\Big(\frac{\nu+\lambda}{2^k}\Big)P_\nu(\xi,\eta).$$

It is readily seen that

$$\hat{u}\Big(\frac{\nu+\lambda}{2^j}\Big)\hat{v}\Big(\frac{\nu+\lambda}{2^k}\Big) = (u_j * v_k)^\wedge(\nu+\lambda) = (u * v_{k-j})^\wedge\Big(\frac{\nu+\lambda}{2^j}\Big).$$

Evidently,

$$(u * v_{k-j})^{(m)}(t) = (u^{(m)} * v_{k-j})(t)$$

and therefore, by Lemma 3.7,

$$|(u * v_{k-j})^{(m)}(t)| \le \frac{cA_1A_2 2^{-(k-j)N}}{(1+|t|)^M}, \quad 0 \le m \le 2\mu+d-1.$$

Observe that since u, v are even, then $u * v_{k-j}$ is also even. We now use Lemma 3.5 to obtain

$$\begin{aligned}|\langle U_\xi, V_\eta\rangle| &\le cA_1A_2\frac{|B(\eta,2^{-k})|^{1/2}}{|B(\eta,2^{-j})|^{1/2}}\frac{2^{-(k-j)N}}{(1+2^j\,d(\xi,\eta))^M}\\ &\le cA_1A_2 2^{-(k-j)(N+d/2)}(1+2^j\,d(\xi,\eta))^{-M},\end{aligned}$$

where in the last inequality we used that $|B(\eta,2^{-k})| \le c2^{-(k-j)d}|B(\eta,2^{-j})|$. □

We shall need the following useful inequality:

Lemma 3.9. *For any $x, y \in B^d$, $j,k \ge 0$, and $\gamma \in \mathbb{R}$ we have*

$$|B(x,2^{-j})|^\gamma \le c|B(y,2^{-k})|^\gamma\big(1+2^{\min\{j,k\}}\,d(x,y)\big)^{2\mu|\gamma|}2^{|j-k|(4\mu+d)|\gamma|}, \tag{3.29}$$

where the constant $c > 0$ is independent of x, y, j, k.

Proof. The following simple estimate is established in [7] (see estimate (4.23) in [7]):

$$W_\mu(n;x) \le 2^\mu W_\mu(n;y)(1+nd(x,y))^{2\mu}, \quad x,y \in B^d,\ n \ge 1, \tag{3.30}$$

where $W_\mu(n;x) := \Big(\sqrt{1-|x|^2}+n^{-1}\Big)^{2\mu}$. On the other hand, by (2.3) we have $W_\mu(n;x) \sim n^d|B(x,n^{-1})|$. Using this and (3.30) one routinely derives (3.29). □

3.5. **Completion of the proof of Theorem 3.1.** Note that Theorem 3.1 follows by Theorem 2.7 if the matrices defined in (3.7) are almost diagonal and $\|\mathbf{D}\|_\delta < \varepsilon$, $\|\mathbf{E}\|_\delta < \varepsilon$ for some $\delta > 0$ and sufficiently small ε.

We shall only prove that $\|\mathbf{E}\|_\delta < \varepsilon$. The proof of $\|\mathbf{D}\|_\delta < \varepsilon$ is the same. By the definition of the needlets $\{\psi_\xi\}$ we have

$$\psi_\xi(x) := \lambda_\xi^{1/2} \sum_{\nu=0}^\infty \hat{a}\Big(\frac{\nu+\lambda}{2^j}\Big) P_\nu(\xi, x), \quad \xi \in \mathcal{X}_j,$$

and by the definition of θ_ξ in (3.2) it follows that

$$\psi_\eta(x) - \theta_\eta(x) = \lambda_\eta^{1/2} \sum_{\nu=0}^\infty (a-g)^\wedge\Big(\frac{\nu+\lambda}{2^k}\Big) P_\nu(\eta, x), \quad \eta \in \mathcal{X}_k.$$

The function $\hat{a}$ has already been extended as an even function on $\mathbb{R}$ in §3.1. Then by (2.9) it readily follows that there exists a constant $A_1 > 0$ such that

$$|a^{(r)}(t)| \le A_1(1+|t|)^{-M}, \quad 0 \le r \le N+2\mu+d-1, \quad \text{and} \quad \int_{\mathbb{R}} t^r a(t)dt = 0, \quad r \ge 0.$$

On the other hand, by construction g is even,

$$\begin{aligned} &|(a-g)^{(r)}(t)| \le \varepsilon(1+|t|)^{-M}, \quad 0 \le r \le N+2\mu+d-1, \quad \text{and} \\ &\int_{\mathbb{R}} t^r (a-g)(t)dt = 0, \quad 0 \le r \le N-1. \end{aligned}$$

We now apply Lemma 3.8 with $u = a$ and $v = a - g$ to obtain

$$|\langle \psi_\eta - \theta_\eta, \psi_\xi \rangle| \le cA_1\varepsilon \min\Big\{\frac{r(\xi)}{r(\eta)}, \frac{r(\eta)}{r(\xi)}\Big\}^{N+\frac{d}{2}} \Big(1 + \frac{d(\xi,\eta)}{\max\{r(\xi), r(\eta)\}}\Big)^{-M}$$

We claim that since $M > \mathcal{J} + 2\mu|\rho/d + 1/2|$ and $N > \mathcal{J} - d - s + (4\mu + 2d)|\rho/d + 1/2|$

$$|e_{\xi,\eta}| := |\langle \psi_\eta - \theta_\eta, \psi_\xi \rangle| \le cA_1\varepsilon\omega_\delta(\xi,\eta) \tag{3.31}$$

and hence $\|\mathbf{E}\|_\delta < cA_1\varepsilon$. However, ε is independent of c, A_1, M, and N. Therefore, $cA_1\varepsilon$ above can be replaced by ε.

For the proof of (3.31) consider the case when $r(\xi) \ge r(\eta)$, i.e. $\xi \in \mathcal{X}_j, \eta \in \mathcal{X}_k$ and $k \ge j$. From Lemma 3.9 we get

$$\Big(\frac{|B(\xi, 2^{-j})|}{|B(\eta, 2^{-k})|}\Big)^{\rho/d+1/2} \ge c\big(1 + 2^{\min\{j,k\}}\, d(x,y)\big)^{-2\mu|\rho/d+1/2|} 2^{-|j-k|(4\mu+d)|\rho/d+1/2|}$$

and hence, for sufficiently small $\delta > 0$,

$$\begin{aligned} |e_{\xi,\eta}| &\le cA_1\varepsilon 2^{-|j-k|(N+d/2)}(1 + 2^j d(\xi,\eta))^{-M} \\ &\le cA_1\varepsilon \Big(\frac{|B(\xi, 2^{-j})|}{|B(\eta, 2^{-k})|}\Big)^{\rho/d+1/2} \frac{2^{-|j-k|(N+d/2-(4\mu+d)|\rho/d+1/2|)}}{(1 + 2^j d(\xi,\eta))^{M-2\mu|\rho/d+1/2|}} \\ &\le cA_1\varepsilon\omega_\delta(\xi,\eta), \end{aligned}$$

where in the last inequality we used that by assumption $M > \mathcal{J} + 2\mu|\rho/d + 1/2|$ and $N > \mathcal{J} - d - s + (4\mu + 2d)|\rho/d + 1/2|$.

The proof of (3.31) in the case $r(\xi) < r(\eta)$ is the same and will be omit it. $\square$

4. Appendix

4.1. **Construction of the function g from §3.1.** Here we sketch the construction of the function g in *Step 1* of the development of the new frame $\Theta = \{\theta_\xi\}_{\xi\in\mathcal{X}}$ in §3.1. The construction will be carried out in two steps.

One first shows that for any $\varepsilon > 0$, $M > 0$, a positive integer N and an even function h in the Schwartz class $\mathcal{S}$ on $\mathbb{R}$ there is an even compactly supported function $\varphi \in C^\infty$ such that

$$|h^{(r)}(t) - \varphi^{(r)}(t)| \le \varepsilon(1+|t|)^M, \quad t\in\mathbb{R}, \ \ r = 0,1,\dots,N. \tag{4.1}$$

To this end, choose an even function $\phi \in C^\infty$ such that $\operatorname{supp}\phi \subset [-1,1]$ and $\int_{\mathbb{R}}\phi = 1$ and define $\varphi_k := h * \phi_k$, where $\phi_k(t) := k\phi(kt)$. Evidently

$$h^{(r)}(t) - \varphi_k^{(r)}(t) = \int_{\mathbb{R}} [h^{(r)}(t) - h^{(r)}(t-y)]\phi_k(y)dy.$$

It is easy to see that for sufficiently large $k > 0$ the function $\varphi := \varphi_k$ will satisfy (4.1) with ε replaced by $\varepsilon/2$ on the right and hence for sufficiently large $L > 0$ the function $\varphi(t) := \int_L^L h(y)\phi_k(t-y)dy$ is even, compactly supported, $\varphi \in C^\infty$ and φ satisfies (4.1).

The second step uses the result of the first step. Consider the shift operator $T_\delta f(t) := f(t+\delta)$. Then $\Delta_\delta^s f := (T_\delta - T_{-\delta})^s f$ is the sth centered difference of f and $(\Delta_\delta^s f)^\wedge(\xi) = (2i\sin\delta\xi)^s \hat{f}(\xi)$ is its Fourier transform. Choose $s := 2N$ and $0 < \delta \le 1/s$, and define the function h from the identity $\hat{h}(\xi) := \frac{\hat{a}(\xi)}{(2i\sin\delta\xi)^s}$, where $\hat{a}$ is from (2.9). Since $\hat{a}(\xi) = 0$ for $\xi \in [-1/2, 1/2]$, then $\hat{h} \in \mathcal{S}$ and hence $h \in \mathcal{S}$. Further, $\hat{h}$ and h are even since $\hat{a}$ and s are even. Moreover, by the construction $a = \Delta_\delta^s h$. Now one uses the result of the first step to construct an even compactly supported C^∞ function φ which satisfies (4.1) with h from above.

After this preparation, g is defined by $g := \Delta_\delta^s \varphi$. We claim that g has the desired properties. Indeed, evidently $a^{(r)} - g^{(r)} = \Delta_\delta^s(h^{(r)} - \varphi^{(r)})$ and by (4.1)

$$|a^{(r)}(t) - g^{(r)}(t)| \le \varepsilon 2^{s+M}(1+|t|)^{-M}, \quad r = 0,1,\dots,N,$$

and also

$$\int_{\mathbb{R}} t^r g(t)dt = \int_{\mathbb{R}} t^r \Delta_\delta^s \varphi(t)dt = (-1)^s \int_{\mathbb{R}} \varphi(t)\Delta_\delta^s t^r dt = 0, \quad r = 0,1,\dots,s-1.$$

By choosing ε and N appropriately this completes the construction.

4.2. **Proof of Theorem 3.3.** We shall need the maximal operator $\mathcal{M}_t$ $(t > 0)$ defined by

$$\mathcal{M}_t f(x) := \sup_{B\ni x}\left(\frac{1}{|B|}\int_B |f(y)|^t w_\mu(y)\,dy\right)^{1/t}, \quad x\in B^d, \tag{4.2}$$

where the sup is over all balls (with respect to $d(\cdot,\cdot)$) $B \subset B^d$ containing x.

By (2.3) it follows that $|B(x,2r)| \le c|B(x,r)|$ for $x \in B^d$ and $r > 0$, which means that $|E| := \int_E w_\mu(x)\,dx$ is a doubling measure on B^d. Therefore, the general theory of maximal operators applies and the Fefferman-Stein vector-valued maximal inequality holds (see [8]): If $0 < p < \infty, 0 < q \le \infty$, and $0 < t < \min\{p,q\}$ then for any sequence of functions $\{f_\nu\}_\nu$ on B^d

$$\Big\|\Big(\sum_{\nu=1}^{\infty}|\mathcal{M}_t f_\nu(\cdot)|^q\Big)^{1/q}\Big\|_{L^p} \le c\Big\|\Big(\sum_{\nu=1}^{\infty}|f_\nu(\cdot)|^q\Big)^{1/q}\Big\|_{L^p}. \tag{4.3}$$

We shall need the following lemma:

Lemma 4.1. *Let $0 < t \le 1$ and $M > (d+2\mu)/t$. Then for any sequence of complex numbers $\{h_\eta\}_{\eta\in\mathcal{X}_m}$, $m \ge j_0$, we have for $x \in R_\xi$, $\xi \in \mathcal{X}$,*

$$\sum_{\eta\in\mathcal{X}_m}|h_\eta|\left(1+\frac{d(\xi,\eta)}{\max\{r(\xi),r(\eta)\}}\right)^{-M} \le c\max\left\{2^{(m-j)(d+2\mu)/t},1\right\} M_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{R_\eta}\Big)(x).$$

Proof. Consider the case $r(\xi) \ge r(\eta)$. The proof in the case $r(\xi) < r(\eta)$ is similar and will be omitted. Fix $\xi \in \mathcal{X}_j$ $(j \le m)$ and set $\Omega_0 := \{\eta \in \mathcal{X}_m : d(\eta,\xi) \le c^\diamond 2^{-j}\}$ and

$$\Omega_\nu := \{\eta \in \mathcal{X}_m : c^\diamond 2^{\nu-1} < 2^j d(\eta,\xi) \le c^\diamond 2^\nu\}, \quad \nu \ge 1,$$

where $c^\diamond$ is the constant from (2.14). For $\nu \ge 0$ we set

$$B_\nu := B(\xi, c^\diamond 2^{-m}(1+2^{\nu-j+m})).$$

Evidently $R_\eta \subset B_\nu$ if $\eta \in \Omega_\nu$.

By (2.3) we have $|B(x,r)| \sim r^d(r+d(r,\partial B^d))$ and by (2.14) $|R_\eta| \sim |B(\eta,2^{-m})|$ for $\eta \in \mathcal{X}_m$. Observe also that

$$d(\xi,\partial B^d) \le d(\xi,\eta) + d(\eta,\partial B^d) \le c^\diamond 2^{\nu-j} + d(\eta,\partial B^d), \quad \eta \in \Omega_\nu.$$

Using the above we get for $\eta \in \Omega_\nu$

$$\frac{|B_\nu|}{|R_\eta|} \le c2^{(\nu-j+m)d}\Big(\frac{2^{-m}(1+2^{\nu-j+m})+d(\xi,\partial B^d)}{2^{-m}+d(\eta,\partial B^d)}\Big)^{2\mu} \le c2^{(\nu-j+m)(d+2\mu)}. \tag{4.4}$$

Since $0 < t \le 1$ we have

$$\sum_{\eta\in\mathcal{X}_m}|h_\eta|\big(1+2^j d(\xi,\eta)\big)^{-M} \le \sum_{\nu\ge0}2^{-\nu M}\sum_{\eta\in\Omega_\nu}|h_\eta| \le \sum_{\nu\ge0}2^{-\nu M}\big(\sum_{\eta\in\Omega_\nu}|h_\eta|^t\big)^{1/t}.$$

We now use this and (4.4) to obtain for $x \in R_\xi$

$$\begin{aligned}
\sum_{\eta\in\Omega_\nu}|h_\eta|^t &= \int_{B^d}\Big(\sum_{\eta\in\Omega_\nu}|h_\eta||R_\eta|^{-1/t}\mathbb{1}_{R_\eta}(y)\Big)^t w_\mu(y)\,dy\\
&= \frac{1}{|B_\nu|}\int_{B^d}\Big(\sum_{\eta\in\Omega_\nu}|h_\eta|\Big(\frac{|B_\nu|}{|R_\eta|}\Big)^{1/t}\mathbb{1}_{R_\eta}(y)\Big)^t w_\mu(y)\,dy\\
&\le c2^{(\nu-j+m)(d+2\mu)}\frac{1}{|B_\nu|}\int_{B^d}\Big(\sum_{\eta\in\Omega_\nu}|h_\eta|\mathbb{1}_{R_\eta}(y)\Big)^t w_\mu(y)\,dy\\
&\le c2^{(\nu-j+m)(d+2\mu)}\Big[\mathcal{M}_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{R_\eta}\big)(x)\Big]^t.
\end{aligned}$$

Therefore, since $M > (d+2\mu)/t$ we get for $x \in R_\xi$

$$\begin{aligned}
\sum_{\eta\in\mathcal{X}_m}|h_\eta|\big(1+2^j d(\xi,\eta)\big)^{-M} &\le \sum_{\nu\ge0}c2^{-\nu M}2^{(\nu-j+m)(d+2\mu)/t}\mathcal{M}_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{R_\eta}\big)(x)\\
&\le c2^{(m-j)(d+2\mu)/t}\mathcal{M}_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{R_\eta}\big)(x),
\end{aligned}$$

which completes the proof. $\square$

We now proceed with the proof of estimate (3.11). The proof of (3.12) is similar and will be omitted. Let $\mathbf{A}$ be an almost diagonal operator on $f^{s\rho}_{pq}$ in the sense of Definition 3.2 with associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$ and let $h\in f^{s\rho}_{pq}$. Then we have $(\mathbf{A}h)_\xi=\sum_{\eta\in\mathcal{X}}a_{\xi\eta}h_\eta$, where the series converges absolutely (see proof below). Using this in the definition of $f^{s\rho}_{pq}$, we have

$$\begin{aligned}\|\mathbf{A}h\|_{f^{s\rho}_{pq}} &:= \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[r(\xi)^{-(s-\rho)}|B_\xi|^{-\rho/d}|(\mathbf{A}h)_\xi|\tilde{\mathbb{1}}_{R_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}\\ &\le c\Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[r(\xi)^{-(s-\rho)}|B_\xi|^{-\rho/d}\sum_{\eta\in\mathcal{X}}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{R_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}\le c(\Sigma_1+\Sigma_2),\end{aligned}$$

where

$$\begin{aligned}\Sigma_1 &:= \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[r(\xi)^{-s+\rho}|B_\xi|^{-\rho/d}\sum_{r(\eta)\le r(\xi)}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{R_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}\quad\text{and}\\ \Sigma_2 &:= \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[r(\xi)^{-s+\rho}|B_\xi|^{-\rho/d}\sum_{r(\eta)> r(\xi)}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{R_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

To estimate Σ_1 we shall use that $\|\mathbf{A}\|_\delta<\infty$. Thus whenever $r(\eta)\le r(\xi)$

$$|a_{\xi\eta}|\le c\|\mathbf{A}\|_\delta\Big(\frac{r(\eta)}{r(\xi)}\Big)^{\mathcal{J}-s+\rho+\delta/2}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\Big(1+\frac{d(\xi,\eta)}{r(\xi)}\Big)^{-\mathcal{J}-\delta}.$$

Set $\lambda_\xi:=r(\xi)^{-s+\rho}|B_\xi|^{-\rho/d-1/2}\mathbb{1}_{R_\xi}(\cdot)$ and choose $0<t<\min\{1,p,q\}$ so that $\mathcal{J}+\frac{\delta}{2}-(d+2\mu)/t)>0$. Then we have

$$\begin{aligned}\frac{\Sigma_1}{\|\mathbf{A}\|_\delta} &\le c\Big\|\Big(\sum_{\xi\in\mathcal{X}}\Big[\sum_{r(\eta)\le r(\xi)}\Big(\frac{r(\eta)}{r(\xi)}\Big)^{\mathcal{J}-s+\rho+\frac{\delta}{2}}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\\ &\qquad\times\Big(1+\frac{d(\xi,\eta)}{r(\xi)}\Big)^{-\mathcal{J}-\delta}|h_\eta|\lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p}\\ &= c\Big\|\Big(\sum_{j\ge j_0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m\ge j}2^{(j-m)(\mathcal{J}-s+\rho+\frac{\delta}{2})}\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\\ &\qquad\times|h_\eta|\big(1+2^jd(\xi,\eta)\big)^{-\mathcal{J}-\delta}\lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p}.\end{aligned}$$

We now apply Lemma 4.1 and the maximal inequality (4.3) to obtain

$$\begin{aligned}\frac{\Sigma_1}{\|\mathbf{A}\|_\delta} &\le c\Big\|\Big(\sum_{j\ge j_0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m\ge j}2^{(j-m)(\mathcal{J}-s+\rho+\frac{\delta}{2}-(d+2\mu)/t)}\\ &\qquad\times M_t\Big(\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}|h_\eta|\mathbb{1}_{R_\eta}\Big)(\cdot)\lambda_\xi(\cdot)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\\ &\le c\Big\|\Big(\sum_{j\ge j_0}\Big[\sum_{m\ge j}2^{(j-m)(\mathcal{J}+\frac{\delta}{2}-(d+2\mu)/t))}M_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\lambda_\eta\big)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\\ &\le c\Big\|\Big(\sum_{j\ge j_0}\Big(M_t\big(\sum_{\xi\in\mathcal{X}_j}|h_\xi|\lambda_\xi\big)\Big)^q\Big)^{\frac1q}\Big\|_{L^p}\le c\|h\|_{f^{s\rho}_{pq}}.\end{aligned}$$

To estimate Σ_2 we again use that $\|\mathbf{A}\|_\delta<\infty$. Then if $r(\eta)>r(\xi)$ we have

$$|a_{\xi\eta}|\le c\|\mathbf{A}\|_\delta\Big(\frac{r(\xi)}{r(\eta)}\Big)^{s-\rho+\delta/2}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\Big(1+\frac{d(\xi,\eta)}{r(\eta)}\Big)^{-\mathcal{J}-\delta}.$$

Therefore, setting again $\lambda_\xi:=r(\xi)^{-s+\rho}|B_\xi|^{-\rho/d-1/2}\mathbb{1}_{R_\xi}(\cdot)$ we have

$$\begin{aligned}\frac{\Sigma_2}{\|\mathbf{A}\|_\delta} &\le c\Big\|\Big(\sum_{\xi\in\mathcal{X}}\Big[\sum_{r(\eta)>r(\xi)}\Big(\frac{r(\xi)}{r(\eta)}\Big)^{s-\rho+\frac{\delta}{2}}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\\ &\qquad\times\Big(1+\frac{d(\xi,\eta)}{r(\eta)}\Big)^{-\mathcal{J}-\delta}|h_\eta|\lambda_\xi(\cdot)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\\ &= c\Big\|\Big(\sum_{j\ge j_0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m<j}2^{(m-j)(s-\rho+\frac{\delta}{2})}\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}\\ &\qquad\times|h_\eta|\big(1+2^md(\xi,\eta)\big)^{-\mathcal{J}-\delta}\lambda_\xi(\cdot)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}.\end{aligned}$$

Employing again Lemma 4.1 and the maximal inequality (4.3) we obtain

$$\begin{aligned}\frac{\Sigma_2}{\|\mathbf{A}\|_\delta} &\le c\Big\|\Big(\sum_{j\ge j_0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m<j}2^{(m-j)(s-\rho+\frac{\delta}{2})}\\ &\qquad\times M_t\Big(\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|B_\xi|}{|B_\eta|}\Big)^{\rho/d+1/2}|h_\eta|\mathbb{1}_{R_\eta}\Big)\lambda_\xi(\cdot)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\\ &\le c\Big\|\Big(\sum_{j\ge j_0}\Big[\sum_{m<j}2^{(m-j)(\delta/2)}M_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\lambda_\eta\Big)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\\ &\le c\Big\|\Big(\sum_{j\ge j_0}\Big[M_t\big(\sum_{\xi\in\mathcal{X}_j}|h_\xi|\lambda_\xi\big)\Big]^q\Big)^{\frac1q}\Big\|_{L^p}\le c\|h\|_{f^{s\rho}_{pq}}.\end{aligned}$$

The above estimates for Σ_1 and Σ_2 imply (3.11). $\square$

4.3. **Proof of Lemma 3.4.** We shall need the Dirichlet-Mehler integral representation of Gegenbauer polynomials [1, p. 177]

$$C^\lambda_\nu(\cos\alpha)=\frac{2^\lambda\Gamma(\lambda+\frac12)\Gamma(\nu+2\lambda)}{\sqrt{\pi}\nu!\Gamma(\lambda)\Gamma(2\lambda)(\sin\alpha)^{2\lambda-1}}\int_\alpha^\pi\frac{\cos\big((\nu+\lambda)\varphi-\lambda\pi\big)}{(\cos\alpha-\cos\varphi)^{1-\lambda}}\,d\varphi. \tag{4.5}$$

This and (3.15) imply that (3.16) holds with

$$K_n(\alpha)=\sum_{\nu=0}^{\infty}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)\frac{(\nu+\lambda)(\nu+2\mu+d-2)!}{\nu!}\times\begin{cases}\sin\lambda\pi\sin(\nu+\lambda)\alpha, & 2\mu+d \text{ even}\\ \cos\lambda\pi\cos(\nu+\lambda)\alpha, & 2\mu+d \text{ odd.}\end{cases}$$

Since $\frac{(\nu+\lambda)(\nu+2\mu+d-2)!}{\nu!}=(\nu+\lambda)(\nu+2\mu+d-2)\dots(\nu+1)$ we have

$$\frac{(\nu+\lambda)(\nu+2\mu+d-2)!}{\nu!}=\prod_{r=1}^{\lfloor\mu+\frac{d-1}{2}\rfloor}\big((\nu+\lambda)^2-(\lambda-r)^2\big)\times\begin{cases}\nu+\lambda, & 2\mu+d \text{ even}\\ 1, & 2\mu+d \text{ odd,}\end{cases}$$

and setting

$$F(z):=\prod_{r=1}^{\lfloor\mu+\frac{d-1}{2}\rfloor}\big(z^2-(\lambda-r)^2\big)\times\begin{cases}z\sin\lambda\pi, & 2\mu+d \text{ even}\\ \cos\lambda\pi, & 2\mu+d \text{ odd.}\end{cases}$$

we arrive at

$$K_n(\alpha)=\sum_{\nu=0}^{\infty}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)F(\nu+\lambda)\times\begin{cases}\sin(\nu+\lambda)\alpha, & 2\mu+d \text{ even}\\ \cos(\nu+\lambda)\alpha, & 2\mu+d \text{ odd.}\end{cases}$$

It is readily seen that that $F(-z)=(-1)^{2\mu+d-1}F(z)$ and F has zeros at the points $\pm(\lambda-r)$, $r=1,\dots,\lfloor\mu+\frac{d-1}{2}\rfloor$. Most importantly, since $\hat\sigma$ is even and because of the symmetry and zeros of F

$$K_n(\alpha)=(1/2)\sum_{\nu\in\mathbb{Z}}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)F(\nu+\lambda)\times\begin{cases}\sin(\nu+\lambda)\alpha, & 2\mu+d \text{ even}\\ \cos(\nu+\lambda)\alpha, & 2\mu+d \text{ odd.}\end{cases}\tag{4.6}$$

Set

$$S(z):=\prod_{r=1}^{\lfloor\mu+\frac{d-1}{2}\rfloor}\big(-z^2-(\lambda-r)^2\big)\times\begin{cases}-z\sin\lambda\pi, & 2\mu+d \text{ even}\\ \cos\lambda\pi, & 2\mu+d \text{ odd}\end{cases}$$

which is a polynomial of degree $2\mu+d-1$ (related to F). Then (4.6) can be rewritten in the form

$$\begin{aligned}K_n(\alpha)&=(1/2)S\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)\cos(\nu+\lambda)\alpha\\ &=(1/4)S\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}}\hat\sigma\Big(\frac{\nu+\lambda}{n}\Big)e^{i(\nu+\lambda)\alpha}.\end{aligned}\tag{4.7}$$

Now, set $\hat f(\xi):=\hat\sigma(\frac{\xi+\lambda}{n})e^{i(\xi+\lambda)\alpha}$. It is easy to see that this is the Fourier transform of $f(y)=ne^{-i\lambda y}\sigma\left(n(y+\alpha)\right)$. We now invoke the Poisson summation formula:

$$\sum_{\nu\in\mathbb{Z}}f(2\pi\nu)=(2\pi)^{-1}\sum_{\nu\in\mathbb{Z}}\hat f(\nu)$$

and put everything together in (4.7) to obtain

$$K_n(\alpha)=(\pi/2)nS\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}}e^{-2\pi i\nu\lambda}\sigma\left(n(\alpha+2\pi\nu)\right).\tag{4.8}$$

This implies (3.17). $\square$

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