August 2019

QUANTUM MECHANICS TEST BANK

1. Kinematics of Quanta [500 level]
   An electron moving along the positive z-axis with the huge γ factor of $\gamma_e = 10^6$, or the energy $E_e = 511$ GeV, hits a very soft CMB photon with an energy $E_\gamma = 10^{-3}$ eV moving in the opposite direction (along the negative z-axis). After the collision the direction of the three-momentum of the incoming photon is reversed, while the electron continues to move along the positive z-xis.
   (a) Derive the appropriate inverse Compton scattering formula for the energy E'_γ of the scattered photon and compute its numerical value. Use the leading term in the asymptotic expansion in the γ-factor of the incoming high energy electron.
   (b) Compute the approximate numerical value of the energy E'_γ of the scattered photon when the initial photon energy is $E_\gamma = 1$ eV.
   (c) Can you use the same approximate formulas when $4\frac{E_e E_\gamma}{(mc^2)^2}$ is larger than 1? Find the appropriate approximate formula in this case.

2. Unknown Potential [500 level]
   A one-dimensional system has the ground state wave function $\psi(x) = Ne^{-x^4/a^4}$, where N is a normalization constant.
   (a) Write an integral expression for the normalization constant N.
   (b) Find the potential if the ground state energy is E_0.

3. One-Dimensional Potential [500 level]
   A particle of mass m moves in one dimension, subject to a δ-function potential $V(x) = -K\delta(x)$.
   (a) Find the matching condition for the wave function across the δ-function at $x = 0$.
   (b) Show that this potential has exactly one bound state.
   (c) Find the bound state energy.

4. Tunneling [700 level]
   Consider a particle in a one-dimensional double-well potential

$$V(x) = V_0 \left[-\frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{4}\left(\frac{x}{a}\right)^4 \right].$$

   (a) Show that the eigenstates of the Hamiltonian are also eigenstates of the parity operator. Sketch the potential and the two lowest energy eigenstates.

(b) At time $t = 0$, the particle is localized on one side of the double well, in the state

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}[\psi_+(x) + \psi_-(x)],$$

where the $\psi_\pm$ are the eigenstates with energies $E_+ < E_-$. Write down the time-dependent wave function $\psi(x, t)$. What is the period of oscillation of the particle?

(c) Using the WKB approximation, estimate the energy difference $E_- - E_+$.

5. One-Dimensional Scattering [700 level]
   A particle of mass m moves in a potential $V(x) = \alpha[\delta(x) + \delta(x - a)]$.
   (a) Write down the matching conditions for the wave function across the δ-function singularities in the potential.
   (b) Determine the time-independent wave function for a particle with wave number k.
   (c) At what energies E will the particle pass through the potential without any reflection?

6. Time-Dependent Potential [700 level]
   At times $t < 0$, a particle is in the ground state of the one-dimensional potential

$$V(x) = \begin{cases} 0, & |x| < a \\ V_0 & |x| \geq a \end{cases},$$

   where V_0 is so large ($\gg \hbar^2/2ma^2$) as to be effectively infinite.
   (a) Find this initial wave function.
   (b) At $t = 0$, the potential instantly changes to $V = 0$ everywhere. Find the wave function for times $t > 0$.

7. Baker-Campbell-Hausdorff Lemma [500 level]
   (a) Demonstrate the Baker-Campbell-Hausdorff Lemma: If $[A, [A, B]] = [B, [A, B]] = 0$, then $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$.
   (b) Show that for the operators x and p,

$$e^{i\alpha x} e^{i\beta p} = e^{i\phi} e^{i\beta p} e^{i\alpha x}$$

   for some phase ϕ, and find ϕ.
   (c) What are the conditions for $e^{i\alpha x}$ and $e^{i\beta p}$ to commute?

8. Two-State System [500 level]
   Consider a system that has only two linearly independent states,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose the Hamiltonian of the system in matrix form is

$$H = \begin{bmatrix} J & 2\sqrt{2}J \\ 2\sqrt{2}J & 3J \end{bmatrix},$$

where J is a constant with the dimensions of energy.
(a) Find the two energy eigenvalues of this system.
(b) Find the normalized eigenstates corresponding to the two eigenvalues.
(c) If the system begins in the state $|\psi(t=0)\rangle = |1\rangle$, find the time behavior of the system $|\psi(t)\rangle$.
(d) What is the probability of finding the system in state $|2\rangle$ at time t?

9. Operator Equations [700 level]
(a) Use the commutation relations between the momentum p and position x (in the Heisenberg picture) to obtain the equations of motion for the expectation values $\langle p\rangle$ and $\langle x\rangle$ when the Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m(\omega^2 x^2 + Ax + B). \tag{1}$$

(b) Find the general solutions of the resulting equations of motion.

10. Wave Packet Spreading [500 level]
(a) Show that for a free particle of mass m, the time dependence of the operator $x(t)^2$ in the Heisenberg picture is

$$x(t)^2 = x(0)^2 + \frac{t}{m}\left[x(0)p(0) + p(0)x(0)\right] + \left(\frac{t}{m}\right)^2 p(0)^2.$$

(b) At time $t = 0$, the particle is prepared in the Gaussian wave packet

$$\psi(x) = \frac{1}{(2\pi\Delta^2)^{1/4}} e^{-x^2/4\Delta^2}.$$

Show that the uncertainty in the position of the particle as a function of time is $[\Delta x(t)]^2 = \Delta^2 + (\hbar t/2m\Delta)^2$.

11. Squeezing Operators [700 level]
This problem concerns operators that produce what are known as "squeezed states" of the harmonic oscillator. Consider the operator

$$U(\eta) = \exp\{\eta[a^2 - (a^\dagger)^2]\},$$

where η is a real-valued parameter (not an operator).
(a) Show that U is unitary.

(b) Define another operator $g_+(\eta) = U(\eta)(a+a^\dagger)U(\eta)^\dagger$. Show that g_+ satisfies the differential equation

$$\frac{dg_+}{d\eta} = 2g_+(\eta).$$

(c) Find $dg_-/d\eta$ for the operator $g_-(\eta) = U(\eta)(a - a^\dagger)U(\eta)^\dagger$.
(d) Use the result from parts (b) and (c) to show that

$$U(a \pm a^\dagger)U^\dagger = e^{\pm 2\eta}(a \pm a^\dagger).$$

12. Commutators [500 level]
    Calculate the following commutators involving the angular momentum operators:
    (a) $[L_x, x]$
    (b) $[L_x, y]$
    (c) $[L_y, z]$
    (d) $[L_z, x]$
    (e) $[L_y, zx]$

13. Commutators [500 level]
    Calculate the following commutators involving the angular momentum operators:
    (a) $[L_x, p_x]$
    (b) $[L_y, p_x]$
    (c) $[L_z, p_y]$
    (d) $[L_x, p_z]$
    (e) $[L_y, p_z]$

14. Angular Momentum [500 level]
    Evaluate the action of the angular momentum operator in the following expressions, involving $r = \sqrt{x^2 + y^2 + z^2}$ and the azimuthal angle ϕ:
    (a) $L_z\, kr$
    (b) $L_z\, \sin kr$
    (c) $L_z\, e^{i\phi}$

15. Angular Momentum [500 level]
    Consider a particle in a state with total angular momentum $\ell(\ell+1)\hbar^2$ and z projection $m\hbar$.
    (a) Using L_+ and L_-, show that $\langle L_x \rangle = \langle L_y \rangle = 0$.
    (b) Using L^2, show that $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{1}{2}[\ell(\ell+1) - m^2]\hbar^2$.

16. Addition of Angular Momentum [700 level]
What are the possible values of the total orbital angular momentum ℓ for the following combinations of electrons?
(a) Four p electrons
(b) Three p and one f electron

17. Molecular Hamiltonian [500 level]
A rigid diatomic molecule, free to rotate around its center of mass, is given by

$$H = \frac{L^2 - L_z^2}{2I_1} + \frac{L_z^2}{2I_3}.$$

(a) What are the eigenfunctions (expressed in terms of the standard basis of $Y_{\ell m}$ spherical harmonics) and corresponding eigenenergies for this system?
(b) What are the degeneracies of the energy spectrum, and how are they related to the symmetry of this system?
(c) The momenta of the inertia of the molecule obey this relation,

$$\frac{2}{I_1} = \frac{I_3 - I_1}{I_1 I_3} = \frac{1}{8ma_0^2},$$

for electron mass m and Bohr radius a_0. Starting in the ground state, what energy is necessary to excite the system into the *second* excited state? Express the answer in terms of the Rydberg energy.

18. Operator Algebra [500 level]
Consider the vector operator $\mathbf{\Theta} = \mathbf{L} \times \mathbf{r} - i\hbar\mathbf{r}$.
(a) Show that $\mathbf{\Theta}$ may also be written $\mathbf{\Theta} = i\hbar\mathbf{r} - \mathbf{L} \times \mathbf{r}$.
(b) Show that this operator is Hermitian.
(c) Show that $[L^2, \mathbf{r}] = -2i\hbar\mathbf{\Theta}$.

19. Symmetries [700 level]
Consider the following matrix elements. Some of them can be shown to be zero. State which ones are zero and give a brief reason for each answer. Each state has the form $\phi^{(j)}{}_m$, where the superscript and subscript denote the total angular momentum and its z-projection quantum numbers.
(a) $\langle \psi^{(3)}{}_2 | f(r)\mathbf{L} \cdot \mathbf{S} | \phi^{(2)}{}_2 \rangle$
(b) $\langle \psi^{(3)}{}_2 | J_+ | \phi^{(2)}{}_1 \rangle$
(c) $\langle \psi^{(3)}{}_2 | \mu_x + i\mu_y | \phi^{(2)}{}_1 \rangle$
(d) $\langle \psi^{(3)}{}_2 | V(r) | \phi^{(3)}{}_2 \rangle$
(e) $\langle \psi^{(3)}{}_1 | Q_{zz} | \phi^{(1)}{}_0 \rangle$

20. Time Operator [500 level]
Suppose that there is a time operator T canonically conjugate to the Hamiltonian, so that $[T, H] = i\hbar$.

(a) Consider the unitary operator $U = e^{i\alpha T}$. Determine the action of this operator on an energy eigenstate $|\psi_n\rangle$.
(b) Show that spectrum of the Hamiltonian is unbounded from below.
(c) Is there a time operator in quantum mechanics?

21. Fermi Gas [500 points]
A system of N noninteracting, identical, spin-$\frac{1}{2}$ fermions of mass m is constrained to a cubical region of volume V.
(a) Write down the nonrelativistic Schrödinger wave function for a single particle in an energy eigenstate, in terms of the quantum numbers.
(b) For what values of the quantum numbers will relativistic effects start to become important?
(c) Calculate the Fermi energy, using the full relativistic energy-momentum relation.

22. Magnetic Field [700 level]
A spin-$\frac{1}{2}$ particle with magnetic moment μ has Hamiltonian $H = -\mu\sigma\cdot\mathbf{B}$. At time $t = 0$, the magnetic field is $\mathbf{B} = B_0\hat{\mathbf{z}}$, and the particle is in the ground state. (You may take the wave function to be real at $t = 0$.)
(a) After a time T, the magnetic field switches instantaneously to $\mathbf{B} = B_0\hat{\mathbf{x}}$. What is the state of the system at $t = 2T$?
(b) Approximately how fast does the field need to rotate for its change to be effectively instantaneous?
(c) What would happen to the state, qualitatively, if the magnetic field instead rotated very slowly?

23. Total Spin [500 level]
A system of two spin-$\frac{1}{2}$ particles with spin vectors $\mathbf{S}^1$ and $\mathbf{S}^2$ is prepared in a total spin $S = 1$ state. We will consider all three possible m_S substates of this triplet. After the initial preparation of the $S = 1$ state, the the first spin $\mathbf{S}^1$ is rotated 180° around the x-axis; this is implemented through operator $U = \exp\left(i\frac{\pi}{2}\sigma_x^1\right)$.
(a) Simplify the exponential to get an explicit form for the operator U.
(b) Find the probability that the system is in an $S = 0$ state after the rotation, for each possible initial value of m_S:
(i) $m_S = +1$
(ii) $m_S = 0$
(iii) $m_S = -1$

24. Stern-Gerlach Experiments [500 level]
A beam of spin-$\frac{1}{2}$ particles is prepared in a pure $\sigma_y = +1$ state.
(a) The beam is sent through a Stern-Gerlach apparatus oriented along the

x-axis. If the beams are measured at this point, what are the probabilities of finding the spins up and down along the x-axis?
(b) If the x-axis spin up beam is sent through a y-axis Stern-Gerlach apparatus, what are the probabilities for the particles emerging to be spin up and spin down along y?
(c) If, instead, the two beams from the x-axis Stern-Gerlach apparatus are recombined without being measured, and the recombined beam is sent through the y-axis apparatus, what are the probabilities of the particles coming out being spin up or spin down along y?

25. Spin Sum [500 level]
Consider a quantum system of spin $\frac{1}{2}$. The spin operator is $\mathbf{S} = S_x\hat{\mathbf{i}}+S_y\hat{\mathbf{j}}+S_z\hat{\mathbf{k}}$.
(a) What are the eigenvalues and eigenvectors of the operator $S_x + S_y$?
(b) Suppose a measurement of $S_x + S_y$ is made, and the system is found to be in the eigenstate corresponding to the larger eigenvalue. What is the probability that a subsequent measurement of S_z yields $\frac{\hbar}{2}$?
(c) Alternatively, starting from the same $S_x + S_y$ eigenstate, what is the probability that a measurement of S_y yields $\frac{\hbar}{2}$?

26. Total Angular Momentum [700 level]
A ${}^3P_0$ atomic state hs spin and orbital angular momenta $s = \ell = 1$ but total angular momentum $j = 0$.
(a) Although there are nine basis states $|m_L\rangle \otimes |m_S\rangle$, show by application of $J_z = S_z + L_z$ that only three of them can possibly be part of the ${}^3P_0$ state.
(b) By similar application of J_x and J_y, determine the expansion of the ${}^3P_0$ state in the $|m_L\rangle \otimes |m_S\rangle$ basis.

27. Variational Principle [500 level points]
Consider a particle moving in the one-dimensional potential $V(x) = K|x|$.
(a) Find an upper limit on the ground state energy using a variational wave function $\psi_1(x) = C_1 \exp(-|x|/a)$, where C_1 is the appropriate normalizaiton factor.
(b) Find another upper limit using a different wave function $\psi_2(x) = C_2 \exp(-x^2/b^2)$.
(c) Which function gives a tighter bound on the ground state energy?

28. Variational Principle [700 level]
Consider a system described by the Hamiltonian H. Let ψ_n $(n = 0, 1, 2, \ldots)$ be the normalized eigenstates, $H\psi_n = E_n\psi_n$, $\langle\psi_n|\psi_n\rangle = 1$. Let a normalized wave function $|\phi\rangle$ be expanded as

$$|\phi\rangle = \sum_{n=0}^{\infty} c_n|\phi_n\rangle.$$

(a) Find the condition on the c_n coefficients imposed by the normalization of $|\phi\rangle$.
(b) With the help of the expansion, prove $\langle\phi|H|\phi\rangle \geq E_0$ for any $|\phi\rangle$.
(c) Suppose we have calculated $\langle\phi|H|\phi\rangle$ for various choices of $|\phi\rangle$ and obtained $\langle H\rangle_{\rm min}$ as the minimum value of $\langle\phi|H|\phi\rangle$. Since $\langle H\rangle_{\rm min} \geq E_0$, we may get a rather reliable estimate of the ground state energy as $E_0 \approx \langle H\rangle_{\rm min}$.

Now, consider the harmonic oscillator Hamiltonian

$$H = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

and a family of normalized trial wave functions $\phi_\beta(x) = N\exp(-\beta x^2)$. Find the normalization constant N.
(d) Calculate $f(\beta) = \langle\phi_\beta|H|\phi_\beta\rangle$.
(e) Find an estimate of he ground state energy E_0 by minimizing $f(\beta)$.

You may find the following integrals useful in your calculations:

$$\begin{aligned}\int_{-\infty}^{\infty} dy\, \exp(-\gamma y^2) &= \sqrt{\frac{\pi}{\gamma}} \\ \int_{-\infty}^{\infty} dy\, y^2 \exp(-\gamma y^2) &= \frac{1}{2}\sqrt{\frac{\pi}{\gamma^3}}\end{aligned}$$

29. Identical Particles [500 level]
Three identical bosons with spin $s = 1$ are placed into the same orbital state $\phi(\mathbf{r})$.
(a) Count the number of possible states of the three bosons.
(b) Among the states, which ones have total z spin projection $m_S = 0$?
(c) What are the possible values of the total spin of the three-boson system?

30. Identical Particles [700 level]
The normalization for a two-particle wave function $\Psi(x_1, x_2)$ is given by

$$\int dx_1 \int dx_2\, |\Psi(x_1, x_2)|^2 = 1.$$

Let $\psi_\alpha(x)$ and $\psi_\beta(x)$ be normalized one-particle wave functions,

$$\int dx\, |\psi_\alpha(x)|^2 = \int dx\, |\psi_\beta(x)|^2 = 1.$$

We allow for the possibility that ψ_α and ψ_β are not orthogonal.
(a) If the particles, 1 and 2, are distinguishable, what is the expression for the normalized wave function $\Psi(x_1, x_2)$ for the two-particle wave function?

(b) Find the probability density for observing particle 1 at $x_1 = a$, regardless of the position of the distinguishable particle 2.
(c) If the particles are indistinguishable bosons, the two-particle wave function must be

$$\Psi(x_1, x_2) = N[\psi_\alpha(x_1)\psi_\beta(x_2) + \psi_\beta(x_1)\psi_\alpha(x_2)].$$

Determine the normalization constant N for the general case in which $\langle\psi_\alpha|\psi_\beta\rangle \neq 0$.
(d) For identical bosons, find the probability density of observing one particle at $x = a$, regardless of the position of the other particle.
(e) Suppose the wave function ψ_α is confined in a laboratory somewhere on Earth, while ϕ_β is localized on the moon. We would naively expect that we can ignore the symmetrization requirement between particles 1 and 2 if one is located on Earth and the other on the moon. To justify this expectation, discuss the behavior of the probability density for one particle to be found in the terrestrial laboratory and show that the results are the same as for distinguishable particles.

31. Harmonic Oscillator [500 level]
Consider a simple harmonic oscillator with Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2$ and the lowering operator $a = \sqrt{\frac{m\omega_0}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega_0\hbar}}p$.
(a) Use the commutator $[a, H]$ to find the time dependence of the operator a in the Heisenberg representation.
(b) Find the normalization constant N such that the coherent state

$$|\alpha\rangle = Ne^{\alpha a^\dagger}|0\rangle$$

satisfies $\langle\alpha|\alpha\rangle = 1$.
(c) Show that in the Schrödinger representation, the state $|\alpha\rangle$ evolves in time as $|\alpha(t)\rangle = |\alpha(0)e^{-i\omega_0 t}\rangle$.

32. Harmonic Oscillator [500 level]
A one-dimensional harmonic oscillator has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega.$$

(a) Starting from an initial state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right),$$

find the expectation values of $\langle x(t)\rangle$ and $\langle p(t)\rangle$.
(b) Show that these expectation values obey the classical equations of motion.

33. Atomic Units [500 level]
In atomic physics, it is natural to use units that match the scales in atoms. Mass is measured in units of the electron mass m_e. Energy is measured in the Hartree unit, $m_e e^4/\hbar^2$ (in Gaussian units); this is the magnitude scale of the kinetic and potential energies in a hydrogen atom. Length is measured in Bohr radii, $a_0 = \hbar^2/m_e e^2$. Angular momentum is measured in units of $\hbar$. Magnetic moments are in units of Bohr magnetons, $\mu_B = e\hbar/m_e c$. This problem concerns the appropriate atomic units of electric and magnetic fields. All dimensionless factors of order unity may be neglected.
(a) The natural unit of $\mathbf{E}$ is the field strength at a distance a_0 from a hydrogen nucleus. Express this E_0 in terms of fundamental constants.
(b) There are two possible ways of formulating a unit of $\mathbf{B}$. One possibility is the magnetic field at the nucleus from an electron in the first Bohr orbit. Find this field B_N.
(c) There is also the field that splits the spin-up and spin-down states of a particle with magnetic moment μ_B by 1 Hartree. Find this field B_H.
(d) Find the relationships between these two magnetic field units and E_0, in terms of the dimensionless fine structure constant $\alpha = e^2/\hbar c$.
(e) Which of the units for $\mathbf{B}$ is related to the natural scale of magnetic energy shifts in atoms?

34. Hydrogen Atom [500 level]
The ground state wave function for an electron in a hydrogen atom has the form

$$\psi(\mathbf{r}) = N e^{-r/a_0},$$

where N is a normalization and constant and $a_0 = \hbar^2/me^2$.
(a) Find the average value of the electron kinetic energy $\langle T \rangle$.
(b) Find the average value of the potential energy $\langle V \rangle$.

35. Variational Principle in a Helium-Like Atom [700 level]
A helium-like ion with two electrons has Hamiltonian

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$

Take as an *ansatz* for the ground state wave function

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{Z_{\text{eff}}^3}{\pi a_0^3} e^{-Z_{\text{eff}}(r_1+r_2)/a_0},$$

where Z_{eff} is an unknown “effective” nuclear charge. If the two electrons did not repel, this (with $Z_{\text{eff}} = Z$) would be the exact wave function.

(a) Find the expectation value of the energy for the proposed state. You will find the integral

$$\int d^3r_1 \int d^3r_2 \frac{e^{-\lambda(r_1+r_2)}}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{20\pi^2}{\lambda^5}$$

useful.
(b) Find the value of Z_{eff} that minimizes the energy.
(c) For this kind of position wave function, what must the spin state of the two electrons be?

36. Stark Effect [500 level]
An electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$ is applied to a hydrogen atom.
(a) Find the correct basis for the perturbation acting on the degenerate $n = 2$ states with wave functions

$$\begin{aligned} \psi_{2S}(\mathbf{r}) &= \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0} \\ \psi_{2P_0}(\mathbf{r}) &= \frac{\cos\theta}{4\sqrt{2\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}. \end{aligned}$$

(b) Find the first-order Stark energy shifts for the basis of states found in part (a).
(c) What are the first-order energy shifts for the remaining

$$\begin{aligned} \psi_{2P_{+1}}(\mathbf{r}) &= -\frac{e^{i\phi}\sin\theta}{8\sqrt{\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0} \\ \psi_{2P_{-1}}(\mathbf{r}) &= \frac{e^{-i\phi}\sin\theta}{8\sqrt{\pi}a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}. \end{aligned}$$

states?

37. Spherical Potential [700 level]
A particle of mass m is confined to the interior of a hollow sphere of radius R.
(a) What are the ground state wave function of the particle and the corresponding energy?
(b) Calculate the pressure the particle exerts on the sphere. The pressure may be defined as $p = -\partial E/\partial V$.

38. Spherical Potential [700 level]
A particle of mass m moves in an attractive spherical potential

$$V(\mathbf{r}) = \begin{cases} -V_0, & r < a \\ 0, & r \geq a \end{cases},$$

where $V_0 > 0$ is a constant.
(a) Write down the radial Schrödinger equation for a state of angular momentum ℓ.
(b) Find the smallest value of V_0 for which there is a bound state with zero energy and zero angular momentum.

39. Relativistic Potentials [700 level]
Two different relativistic effects (*Zitterbewegung* and interactions with vacuum fluctuations of the electric field) each lead to an effective smearing out of an electron's position. If the smearing extends over a characteristic distance λ, the electron effectively interacts with a potential that is averaged over the smearing region. If the true potential is $V(\mathbf{r})$, the effective potential that appears in the effective Schrödinger equation is

$$V_{\text{eff}}(\mathbf{r}) \approx V(\mathbf{r}) + \frac{1}{8}\lambda^2 \vec{\nabla}^2 V(\mathbf{r}).$$

(a) In a hydrogen atom, with $V(\mathbf{r}) = -\frac{\alpha}{r}$, which atomic states are most affected by the modified V_{eff}?
(b) Find the leading order energy shift due to this effect for the hydrogen ground state $\psi(\mathbf{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$.

40. Perturbation Theory [500 level]
The unperturbed Hamiltonian for a theory can be written

$$H_0 = \begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 \\ 0 & 0 & 200 & 0 \\ 0 & 0 & 0 & 300 \end{bmatrix}.$$

(a) Find the unperturbed eigenvalues of the energy.
(b) A perturbation

$$H_1 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

is addded to H. Find the energy eigenvalues for first order in H_1.
(b) Find the eigenstates to first order.
(c) Find the eigenvalues to second order.

41. Perturbation Theory [500 level]
A two-level quantum system with non-degenerate level energies $E_1^{(0)}$ and $E_2^{(0)}$

experiences a perturbation V with matrix elements as shown:

$$H_0 = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{bmatrix}.$$

(a) Find the corrections to the energies up to second order in perturbation theory.
(b) Find the exact expressions for the eigenenergies.
(c) Show that when the exact expression is expanded to second order, it agrees with the result from part (a).

42. Perturbation Theory [700 level]
A system of two coupled harmonic oscillators has Hamiltonian

$$\begin{aligned} H &= H_0 + H_1 \\ H_0 &= \frac{p_1^2}{2m} + \frac{1}{2}m\omega_1^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_2^2 x_2^2 \\ H_1 &= \lambda x_1 x_2, \end{aligned}$$

with $\omega_1 \neq \omega_2$.
(a) Find the energy of the perturbed number state $|n_1, n_2\rangle$ to first order in λ. Hint: Remember, $x_i = \sqrt{\frac{\hbar}{2m_i\omega_i}}(a_i + a_i^\dagger)$.
(b) Find the second-order energy shift for the $|n_1, n_2\rangle$ state.
(c) What is the condition on λ for the validity of perturbation theory?

43. Perturbation Theory [700 level points]
Consider an electron in a one-dimensional harmonic oscillator potential $\frac{1}{2}m\omega^2 x^2$, placed in an electric field $\mathcal{E}$ pointing in the x-direction.
(a) Write the Hamiltonian H in terms of raising and lowering operators $a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\omega\hbar}}p$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2m\omega\hbar}}p$.
(b) Calculate the first-order energy shift of the state $|n\rangle$ due to $\mathcal{E}$.
(c) Calculate the second-order energy shift of $|n\rangle$.
(d) Find the exact energy levels of this system, and compare the results to the first- and second-order perturbative shifts.

44. Perturbation Theory [700 level points]
A particle of mass m moves in a one-dimensional square well potential with a δ-function in the center of the well,

$$V(x) = \begin{cases} \infty, & |x| > a \\ \lambda\delta(x), & |x| < a \end{cases}.$$

(a) Find the energy eigenvalues for $\lambda = 0$.
(b) Find the energy levels to first order in $\lambda \neq 0$, using perturbation theory.

(c) Find a transcendental equation for the exact eigenvalues, and show that expanding this equation to first order in λ gives the same result as part (b).

45. Time-Dependent Harmonic Oscillator [700 level points]
A one-dimensional harmonic oscillator with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{kx^2}{2}$$

has energy eigenvalues $E_n^{(0)} = (n + 1/2)\hbar\omega_0$, where $\omega_0 = \sqrt{k/m}$. Suppose the spring constant is changed according to $k \to k' = k(1+\epsilon)$, with $\epsilon \ll 1$.
(a) Find the exact new eigenvalues, and expand the result up to second order in ϵ.
(b) Now calculate the perturbed energies, treating the Hamiltonian as $H = H_0 + \epsilon V$, where $\epsilon V = \epsilon k x^2/2$ is a small perturbation. Work to first order in ϵ.
(c) Calculate the perturbed energy for the ground state to second order in λ, and show that the result agrees with the result from part (a).

46. Unstable Potential [500 level]
Consider the Schrödinger equation with potential

$$V(r) = -\frac{K}{r^2},$$

with a positive constant K.
(a) Show that if $\psi(\mathbf{r})$ is a solution of the time-independent Schrödinger equation, so is the wave function $\psi(c\mathbf{r})$ with coordinates rescaled by a $c > 0$, although the energies of the two eigenfunctions are different.
(b) Show that if $\psi(\mathbf{r})$ is a normalizable wavefunction, so is $\psi(c\mathbf{r})$.
(c) Show that this potential has no lowest-energy state. (This phenomenon is known as "falling to the center.")

47. Carbon Atom [500 level points]
The electron configuration of a neutral carbon atom is $1s^2 2s^2 2p^2$.
(a) Taking the approximation in which L^2, S^2, J^2, and J_z are good quantum numbers, what possible values can they take in the ground state manifold?
(b) Which of the quantum numbers are still good with the full relativistic Hamiltonian?
(c) What values can the true good quantum numbers take in the ground state?

48. Spins [500 level points]
The spin components of a beam of spin-$\frac{1}{2}$ atoms prepared in an initial state $|\psi_0\rangle$ are measured, and the following probabilities are obtained: $P(S_z = +) = \frac{1}{2}$, $P(S_z = -) = \frac{1}{2}$, $P(S_x = +) = \frac{3}{4}$, $P(S_x = -) = \frac{1}{4}$, $P(S_y = +) = 0.067$, and $P(S_y = -) = 0.933$. From the experimental data, determine the input state.

49. Spin-1 Probabilities [500 level points]
A beam of spin-1 particles is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{29}}|1_y\rangle + i\frac{3}{\sqrt{29}}|0_y\rangle - \frac{4}{\sqrt{29}}|-1_y\rangle$$

in the y-quantization basis.
(a) What are the possible results of a measurement of the spin component S_y and with what probabilities would they occur?
(b) What are the possible results of a measurement of the spin component S_z and with what probabilities?
(c) Plot a histogram of the results of parts (a) and (b), and calculate the expectation values for both measurements.

50. Spin-1 Probabilities [500 level points]
A spin-1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ 5i \end{bmatrix}$$

in the S_z basis. (a) What are the possible results of a measurement of the spin component S_z, and with what probabilities do they occur?
(b) Calculate the expectation value of the spin component S_z.
(c) Calculate the expectation value of the spin component S_x.

51. Wavefunction Overlap [500 level points]
A spin-1 particle is prepared in the state

$$|\psi_i\rangle = \sqrt{\frac{1}{6}}|1_z\rangle - \sqrt{\frac{2}{6}}|0_z\rangle + i\sqrt{\frac{3}{6}}|-1_z\rangle$$

in the z basis.
(a) Find the probability that the system is measured to be

$$|\psi_f\rangle = \frac{1+i}{\sqrt{7}}|1_y\rangle + \frac{2}{\sqrt{7}}|0_y\rangle - i\frac{1}{\sqrt{7}}|-1_y\rangle.$$

(b) What is the probability of $|\psi_f\rangle$ being found if the system is first measured and found to have spin projection $S_x = 1$?

52. Spin Projections [500 level points]
Consider the projection operators P_+ and P_- that project out the spin-up and spin-down states (of a spin-$\frac{1}{2}$ particle) in the z-direction. Determine whether these operators are Hermitian, and explain why.

53. Time-Dependent Observables [500 level points]
(a) Show that the probability of a measurement of the energy is time independent for a general state $|\psi(t)\rangle = \sum_n c_n(t)|E_n\rangle$ that evolves due to a time-independent Hamiltonian.
(b) Show that probability of measurements for other observables are also independent of time if those observables commute with the Hamiltonian.

54. Two-State Hamiltonian [500 level points]
Consider a Hamiltonian

$$H = \frac{\hbar}{2}\begin{bmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{bmatrix}.$$

(a) Diagonalize this Hamiltonian. Find the eigenvalues and eigenvectors.
(b) Find time dependence of the initial state $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

55. Magnetic Precession [500 level points]
Consider a spin-$\frac{1}{2}$ particle with a magnetic moment μ. At time $t = 0$, the state of the particle is $|\psi(t=0)\rangle = |+_{\hat{\mathbf{n}}}\rangle$—up in the $\hat{n}$ direction, with $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The system is allowed to evolve in a uniform magnetic field $\mathbf{B} = B_0(\hat{\mathbf{x}} + \hat{\mathbf{z}})/\sqrt{2}$. What is the probability that the particle will be measured to have spin up in the y-direction after a time t?

56. Observable Operator [500 level points]
(a) Consider a two-state system with a Hamiltonian

$$H = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}.$$

Another physical observable A is described by the operator

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix},$$

where a is real and positive. Let the initial state of the system be $|\psi(0)\rangle = |a_1\rangle$, where $|a_1\rangle$ is the eigenstate corresponding to the larger of the two eigenvalues of of A. Find this eigenstate.
(b) What is the frequency of oscillation of the expectation value of A?

57. Three-State System [500 level points]
Let the matrix representation of the three-state system be

$$H = \begin{bmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{bmatrix}.$$

using the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.
(a) If the state of the system at time $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is the probability that the system is in the state $|2\rangle$ at time t?
(b) If, instead, the state of the system at time $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is the probability that the system is in the state $|3\rangle$ at time t?

58. Composite Spin State [500 level points]
Consider a state of two $s = \frac{1}{2}$ spins,

$$|\psi_a\rangle = \frac{1}{\sqrt{2}} \left(|+_z\rangle_1 |-_z\rangle_2 + |-_z\rangle_1 |+_z\rangle_2 \right).$$

(a) Transform each of the spins into the x basis and find the resulting two-particle spin wavefunction.
(b) Transform each of the spins into the y basis and find the resulting two-particle spin wavefunction.

59. Infinite Square Well [500 level points]
Consider an infinite square well of width L, centered at $x = 0$.
(a) Find the wavefunction for the n-th eigenstate.
(b) Find the expectation values of x and p as functions of n.
(c) Find the uncertainties Δx and Δp as functions of n.

60. Square Well States [500 level points]
Consider an infinite square well of width L, extending from $x = 0$ to L. Find the probability that a particle lies in the region $3L/4 < x < L$ for the three lowest eigenstates of the Hamiltonian.

61. Square Well [500 level points]
A particle at $t = 0$ is known to be in right half of an infinite square well of width L, with a probability density that is uniform in the right half of the well.
(a) What is the initial wavefunction of the particle?
(b) Calculate the expectation value of the energy.
(c) Find that the probability that the energy takes on its n-the eigenvalue.

62. Half-Infinite Square Well [500 level points]
Consider a potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0 & 0 < x < L \\ V_0 & x > L \end{cases}.$$

(a) Find a transcendental equation for the energy of the ground state of the system.
(b) Find the ground state energy, to the lowest nonvanishing order in $1/V_0$.

63. Coherent State [500 level points]
Consider a harmonic oscillator with frequency ω. Suppose the system is prepared in a state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\alpha^2/2}}{\sqrt{n!}} |n\rangle,$$

where α is a positive real number.
(a) Plot a histogram of the possible measured energies.
(b) Find the expectation value of the energy.
(c) Find the uncertainty of the energy.

64. Wavefunctions [500 level points]
Consider these three wavefunctions:
(i) $\psi(x) = Ae^{-x^2/3}$,
(ii) $\psi(x) = B/(x^2 + 2)$,
(iii) $\psi(x) = C\text{sech}\,(x/5)$.
(a) Calculate the normalization constant for each wavefunction.
(b) For each wavefunction, find the probability that a particle is to be found in the range $0 < x < 1$.

65. Finite Wave Train [700 level points]
A beam of particles of mass m is prepared in a momentum state $|p_0\rangle$. The beam is directed to a shutter that is open for a finite time τ.
(a) Find the position-space wavefunction of the system immediate after passing through the shutter.
(b) Find the momentum probability distribution of the beam after the shutter.

66. Free Particle Operators [500 level points]
(a) Show that the momentum and Hamiltonian operators for a free particle in three dimensions commute, using the explicit position-space representations of these operators.
(b) Show that the two operators commute using the canonical commutation relations.
(c) Calculate the expectation value of the angular momentum component L_z for a plane wave.

67. Wave Packet Spreading [700 level points]
Consider a wavefunction that is initially

$$\psi(x, 0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \phi(p) e^{ipx/\hbar},$$

where the momentum-space wavefunction is $\phi(p) = (2\pi\beta^2)^{-1/4} e^{-(p-p_0)^2/4\beta^2}$.
(a) Write down an integral expression for the time-evolved wavefunction $\psi(x, t)$.

(b) Evaluate the resulting integral.
(c) Find the position uncertainty $\Delta x(t)$ as a function of t.

68. Gaussian Wavefunctions [500 level points]
(a) Show that a wave packet that is Gaussian in position space is also Gaussian in momentum space.
(b) Calculate $\Delta x \Delta p$ for a Gaussian packet.
(c) Explain why, for a free particle, Δx depends on time, but Δp does not.

69. Sinusiodal Wavefunction [500 level points]
Consider a particle whose wavefunction is $\psi(x) = A\sin(p_0 x/\hbar)$. (If you wish, you make take the wave function to be defined on a space of length L, with periodic boundary conditions.)
(a) Is this an eigenstate of momentum? Find the expectation value of p.
(b) Calculate the uncertainty Δp. What are the possible results of a measurement of the momentum?

70. Uncertainty Estimate [500 level points]
Use the uncertainty principle to estimate the ground-state energy of a particle of mass m bound in the harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$. How does this compare to the true ground state energy?

71. Uncertainty Estimate [500 level points]
Use the uncertainty principle to estimate the ground-state energy of a particle of mass m bound in the harmonic oscillator potential $V(x) = a|x|$.

72. Uncertainty Estimate [500 level points]
Use the uncertainty principle to estimate the ground-state energy of a particle of mass m bound in the harmonic oscillator potential $V(x) = bx^4$.

73. Position Uncertainty [500 level points]
Calculate the position uncertainty for a particle bound in an infinite square well of width L if
(a) the particle is in the ground state;
(b) the wave function is uniform across the well.

74. Beam Wavefunction [700 level points]
A beam of particles is described by the wavefunction

$$\psi(x) = \begin{cases} Ae^{ip_0 x/\hbar}(b - |x|), & |x| < b \\ 0, & |x| > b \end{cases}.$$

(a) Normalize the wavefunction, using an appropriate convention for continuum states.

(b) Sketch the wavefunction.
(c) Calculate and sketch a plot of the momentum probability distribution.

75. Uncertainty Estimate [500 level points]
Starting with the definition of the angular momentum, show that the orbital angular momentum operators in spherical coordinates are

$$\begin{aligned} L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ L_z &= i\hbar\frac{\partial}{\partial\phi} \end{aligned}$$

76. Uncertainty Estimate [500 level points]
Consider the normalized state $|\psi\rangle$ for a particle of mass μ constrained to move ona circle of radius r_0, given by

$$|\psi\rangle = \frac{N}{2+\cos(3\phi)}.$$

(a) Find the normalization constant N.
(b) Sketch the wavefunction.
(c) What is the expectation value of L_z in this state?

77. Uncertainty Estimate [500 level points]
Consider the normalized state of a particle of mass m on a sphere given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{i}{\sqrt{6}}|00\rangle$$

in the $|\ell m\rangle$ angular momentum basis.
(a) What are the probabilities that a measurement of L_z will yield $2\hbar$, $-\hbar$, or $0\hbar$?
(b) What is the expectation value of L_z?
(c) What is the expectation value of $\mathbf{L}^2$?
(d) What are the Hamiltonian and the expectation value of the energy in this state?
(e) What is the expectation value of L_y?

78. Hydrogenic Wavefunction [500 level points]
(a) Calculate the normalization constant N, for the radial hydrogenic wavefunction $R_{10}(r) = Ne^{-Zr/a_0}$.
(b) Calculate the expectation value for $\langle r\rangle$.

79. Hydrogenic Wavefunctions [500 level points]
(a) Calculate the probability that the electron is measured to be within one Bohr radius of the nucleus for the wavefunction

$$\psi_{200}(x) = 2(Z/2a_0)^{3/2}(1 - Zr/2a_0)e^{-Zr/2a_0}Y_{00}(\theta, \phi).$$

(b) Calculate the probability that the electron is measured to be within one Bohr radius of the nucleus for the wavefunction

$$\psi_{211}(x) = \frac{1}{\sqrt{3}}(Z/2a_0)^{3/2}(Zr/a_0)e^{-Zr/2a_0}Y_{11}(\theta, \phi).$$

(c) Explain the difference between the $\ell = 0$ and $\ell = 1$ states.

80. Hydrogenic Wavefunctions [500 level points]
(a) Calculate the probability that the electron is measured to be in the classical forbidden region for the wavefunction

$$\psi_{200}(x) = 2(Z/2a_0)^{3/2}(1 - Zr/2a_0)e^{-Zr/2a_0}Y_{00}(\theta, \phi).$$

(b) Calculate the probability that the electron is measured to be in the classical forbidden region for the wavefunction

$$\psi_{211}(x) = \frac{1}{\sqrt{3}}(Z/2a_0)^{3/2}(Zr/a_0)e^{-Zr/2a_0}Y_{11}(\theta, \phi).$$

(c) Explain the difference between the $\ell = 0$ and $\ell = 1$ states.

81. Hydrogenic Wavefunctions [500 level points]
(a) Calculate the probability that the electron is measured to be inside the nucleus for the ground state wavefunction

$$\psi_{100}(x) = 2(Z/a_0)^{3/2}e^{-Zr/a_0}Y_{00}(\theta, \phi).$$

A nucleus with A nucleons (meaning Z protons and $A - Z$ neutrons) has an approximate radius of $r_N \approx 1.2A^{1/3}$ fm.
(b) Evaluate this probability for ordinary hydrogen and U-238.

82. Tritium Decay [500 level points]
Tritium is an isotope of hydrogen, with a nucleus consisting of one proton and two neutrons. The tritium nucleus is radioactive, decaying by β-decay to the He-3 nucleus. An electron is initially in the ground state

$$\psi_{100}(x) = 2(Z/a_0)^{3/2}e^{-Zr/a_0}Y_{00}(\theta, \phi).$$

of the tritium atom. After the (essentially instantaneous) β-decay, what is the probability that the electron is in the ground state of the new atom?

83. Three-Dimensional Square Well [700 level points]
Consider a particle of mass m bound in an infinite square well in three dimensions

$$V(x,y,z) = \begin{cases} 0 & 0 < x < L,\ 0 < y < L\, 0 < z < L \\ \infty & \text{otherwise} \end{cases} .$$

(a) Use separation of variables to find the energy eigenvalues and eigenstates.
(b) Find the degeneracies of the first six energy levels.

84. Time-Dependent States [500 level points]
A particle in a harmonic oscillator potential is initially been in the state

$$|\psi(t=0)\rangle = N\left[|0\rangle + 2e^{i\pi/2}|1\rangle\right] .$$

(a) Find the normalization constant N.
(b) Find the time-evolved state $|\psi(t)\rangle$.
(c) Calculate $\langle x\rangle$ as a function of time.
(d) Calculate $\langle p\rangle$ as a function of time.
(e) Verify that Ehrenfest's Theorem holds.

85. Time-Dependent Harmonic Oscillator [700 level points]
A particle is in the ground state of the harmonic oscillator potential $V_1(x) = \frac{1}{2}m\omega_1^2x^2$, when the potential suddenly changes to $V_2(x) = \frac{1}{2}m\omega_2^2x^2$ essentially instantaneously.
(a) What is the probability that a measurement of the particle energy yields $\frac{1}{2}\hbar\omega_2$ after the change?
(a) What is the probability that a measurement of the particle energy yields $\frac{3}{2}\hbar\omega_2$?
(b) Evaluate the results of parts a and b for the case $\omega_2 = 1.7\omega_1$.

86. Half Harmonic Oscillator [500 level points]
A particle of mass m moves in the one-dimensional potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2, & x < 0 \\ \infty, & x > 0 \end{cases} .$$

(a) Find the allowed energy eigenvalues.
(b) Find the normalized ground-state wavefunction.

87. Perturbed Harmonic Oscillator [700 level points]
Consider a particle bound in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. A perturbation $H' = \gamma x^3$ is applied to the system.
(a) Calculate the first-order corrections to the eigenstate energies.
(b) Calculate the second-order corrections to the energies for the first three

energy levels.
(c) Calculate the the first-order corrections the wavefunctions for the first three energy levels.

88. Perturbed Potential [500 level points]
Consider a particle moving in an infinite square well that extends over the region $0 < x < L$ (with the potential vanishing in this region). A "ramp" perturbation $H' = \beta x$ is added to the system.
(a) Find the first-order energy shift for the ground state of the system.
(b) Find the first-order energy-shift for the first excited state of the system.

89. Perturbed Potential [700 level points]
Consider a particle moving in an infinite square well that extends over the region $0 < x < L$ (with the potential vanishing in this region). A perturbation $H' = V_0 \sin(\pi x/L)$ is added to the system.
(a) Find the first-order energy shift for the ground state of the system.
(b) Find the first-order energy-shift for an arbitrary excited state of the system.

90. Perturbed Potential [700 level points]
Consider a particle moving in an infinite square well that extends over the region $0 < x < L$ (with the potential vanishing in this region). A perturbation $H' = \gamma x(L - x)$ is added to the system.
(a) Find the first-order energy shift for the ground state of the system.
(b) Find the first-order energy-shift for an arbitrary excited state of the system.

91. Addition of Angular Momenta [700 level points]
Consider two electrons, each with orbital angular momentum $\ell_i = 1$.
(a) What are the possible values for the quantum number ℓ corresponding to the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$?
(b) What are the possible values for the quantum number s corresponding to the total angular momentum $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$?
(c) Using the results from parts a and b, what are the possible values for the quantum number j corresponding to the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$?
(d) What are the possible values for the quantum number j_1 for elctron 1, corresponding to the total angular momentum $\mathbf{J}_1 = \mathbf{L}_1 + \mathbf{S}_1$?
(e) Using the result from part d, determine the possible values of j again, and show that your results agree with part c.

92. Particle Statistics [700 level points]
Consider two noninteracting particles of mass m in an infinite square well of width L. For the case of one particle in a state $|n\rangle$ and the other particle in the state $|k\rangle$ (with $n \neq k$). Calculate the value of the squared interparticle spacing $\langle (x_1 - x_2)^2 \rangle$, assuming:

(a) the particles are distinguishable;
(b) the particles are identical spin-0 bosons;
(c) the particles are identical spin-$\frac{1}{2}$ fermions in a spin triplet state.

93. Particle Statistics [700 level points]
Consider two noninteracting particles of mass m in an infinite square well of width L. For the case of one particle in the ground state and the other particle in the first excited state. calculate the probability density for the interparticle separation $P(x_1 - x_x)$, assuming:
(a) the particles are distinguishable;
(b) the particles are indistinguishable in a symmetric state;
(c) the particles are indistinguishable in an antisymmetric state.

94. Multiparticle Perturbations [700 level points]
Consider two indistinguishable spin-$\frac{1}{2}$ particles in the one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The two particles interact with each other through a perturbing potential $H' = \frac{1}{2}\alpha(x_1 - x_x)^2$, where the constant α is small ($\alpha \ll m\omega^2$).
(a) For the unperturbed two-particle system, find the energy eigenvalues and degeneracies for the ground state and the first excited energy level.
(b) Discuss qualitatively how the energies from part a are perturbed by H'. Draw an energy level diagram showing the unperturbed and perturbed energy levels.

95. Time-Dependent Potential Well [700 level]
A particle of mass m bounces elastically between two infinite plane walls separated by a distance D. The particle is in the lowest possible energy state.
(a) What is the energy of this state?
(b) The separation between the walls is slowly (i.e. adiabatically) increased to $2D$.
(i) How does the expectation value of the energy change?
(ii) Compare this energy with the result obtained classically from the mean force exerted on a wall by the bouncing particle.
(c) Now assume that the separation between the walls is increased rapidly, with one wall moving at a speed $v \gg \sqrt{E/m}$. Classically, there is no change in the particle's energy, since the wall is moving faster than the particle and cannot be struck by the particle while the wall is moving.
(i) What happens to the expectation value of the energy quantum mechanically?
(ii) Compute the probability that the particle is left in the lowest possible energy state.

96. Crossed Fields [500 level]
Consider a particle with charge e and mass m in constant, crossed $\mathbf{E}$ and $\mathbf{B}$ fields,

$$\begin{aligned}\mathbf{E} &= (0, 0, E) \\ \mathbf{E} &= (0, B, 0),\end{aligned}$$

in $\mathbf{r} = (x, y, z)$ coordinates.
(a) Write down the Schrödinger equation (in a convenient gauge).
(b) Separate variables and reduce it to a one-dimensional problem.
(c) Calculate the expectation value of the velocity in the x-direction in any energy eigenstate (sometimes called the "drift velocity").

97. Pion Scattering [700 level]
Pion-nucleon scattering at low energies can be qualitatively described by an effective interaction potential of the form

$$V = \frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}\mathbf{I}^{(\pi)} \cdot \mathbf{I}^{(N)}.$$

Here, g and μ are constants, r is the relative pion-nucleon distance coordinate, and $\mathbf{I}^{(\pi)}$ and $\mathbf{I}^{(N)}$ are the pion and nucleon isospin operators.
(a) Calculate the ratio of the scattering cross sections with total isospin $I = 3/2$ and $I = 1/2$.
(b) Calculate, in the Born approximation, the low-energy total cross sections for the reactions

$$\begin{aligned}\pi^+ + p &\rightarrow \pi^+ + p \\ \pi^- + p &\rightarrow \pi^- + p \\ \pi^- + p &\rightarrow \pi^0 + n.\end{aligned}$$

Note: If you are not familiar with isospin, you may consider the two particles to have (ordinary) spin 1 and spin 1/2 with spin-spin interactions and initial and final states which are eigenstates of S_z for each particle. The corresponding S_z values are

$$\begin{aligned}\begin{bmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ \begin{bmatrix} p \\ n \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.\end{aligned}$$

98. One-Dimensional Scattering [500 level]
A particle of total energy $E = \hbar^2\alpha^2/2m$ moves in a series of contiguous one-dimensional regions. The potential in the n-th region is

$$V_n = -(n^2 - 1)E,$$

where $n = 1, 2, \ldots, N$. All of the regions of of equal width π/α except for the first and the last, which are of effectively infinite extent. Calculate the two transmission coefficients for a particle incident from either end.

99. Zeeman Effect [500 level]
The electron and positron have the same (absolute) magnetic moment, but opposite g-factors. Show that the "ground state" manifold of the e^+e^- atom positronium—which consists of a doublet of ${}^1S_0$ and ${}^3S_1$ states—cannot have a linear Zeeman effect if this is true. Argue in terms of the total magnetic moment operator.

100. Nuclear Quantum Numbers [500 level]
Experiments (deuteron "stripping") show that the ground state of the ^{17}O is formed from that of ^{16}O only by acceptance of a neutron of orbital angular momentum $\ell = 2$. The first excited state is formed by the acceptance of a neutron with angular momentum $\ell = 0$.
(a) What can you conclude about the spin and parity of the ground state of ^{17}O?
(b) What can you conclude about the spin and parity of the first excited state?

101. Anticommuting Operator [500 level]
Let B and C be two anticommuting operators, i.e.

$$\{B, C\} \equiv BC + CB = 0.$$

(a) Let $|\psi\rangle$ be an eigenstate of both B and C. What can be said about the corresponding eigenvalues?
(b) For B = baryon number and C = charge conjugation, the relations $\{B, C\} = 0$ and $C^2 = 1$ hold. What does the result from part (a) imply about this case?

102. Operator Algebra [500 level]
(a) Simplify the operator $\Lambda_{jk} = [x_j, [L^2, x_k]]$, where $j, k = 1, 2, 3$, and $L^2 = (\mathbf{r} \times \mathbf{p})^2$.
(b) Find all the eigenvalues of Λ_{jk}.

103. Bohr-Sommerfeld Quantization [500 level]
Use the Bohr-Sommerfeld quantization rule to find approximate values for

the allowed energy levels of a ball which is bouncing elastically in a vertical direction.

104. Zero-Point Pressure [500 level]
An electron is contained inside a sphere of radius R.
(a) What is the pressure P exerted on the surface of the sphere, if the electron is in the lowest S energy state?
(b) What is the pressure P exerted on the surface of the sphere, if the electron is in the lowest P state?

105. Variational Principle [700 level]
Using the variational principle, and taking a trial wave function of the form $xe^{-\alpha x}$, estimate the ground state energy of a particle in the potential

$$V(x) = \left\{ \begin{array}{ll} \infty, & x < 0 \\ Cx, & x > 0 \end{array} \right. .$$

106. Angular Variables [500 level]
Two identical plane rotors with coordinates θ_1 and θ_2 are coupled according to the Hamiltonian

$$H = A\left(p_{\theta_1}^2 + p_{\theta_2}^2\right) - B\cos(\theta_1 - \theta_2),$$

where A and B are positive constants. (Note that $\theta_i + 2\pi$ is describes the same state of a rotor as does θ_i.) From the Schrödinger equation, determine the energy eigenvalues and eigenfunctions when the following conditions hold:
(a) In the case $B \ll A\hbar^2$, discussing only terms linear in B. Watch out for degeneracies.
(b) In the case $B \gg A\hbar^2$, by reducing the problem to an oscillator problem (with small oscillations).

107. Spin Precession [500 level]
Consider an electron in a uniform magnetic field in the positive z-direction. The result of a measurement has shown that the electron spin is along the positive x-direction at $t = 0$. For $t > 0$, compute the probability for finding the electron in the spin states:
(a) $S_x = \frac{\hbar}{2}$
(b) $S_x = -\frac{\hbar}{2}$
(c) $S_z = \frac{\hbar}{2}$.